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PARTITION NUMBERS, CONNECTIVITY AND HAMILTONICITY

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Summary. The author studies the connectivity of (n, P)-critical graphs, where P is a k-like property. As a corollary, hamiltonicity of such graphs is obtained.

Keywords: Partition numbers, connectivity, hamiltonicity.

1. INTRODUCTION

In the present paper finite graphs without loops and multiple edges will be considered. Notation and terminology not introduced here follow the book [3] (but we use the terms vertex and edge rather than point and line).

Let a graph G and a property P of graphs be given. Suppose that for a graph G there exists a partition of $V(G)$ into n classes, called P-sets, each of which induces a subgraph with the property P. Such a partition is called an (n, P)-partition.

The minimum n for which there exists an (n, P)-partition is denoted by $\chi_P(G)$. If $\chi_P(G) = n$, while $\chi_P(G - v) < n$ for any vertex v of G, then G is said to be (n, P)-critical with respect to the property P. We call a property P hereditary if, whenever a graph has P, so does each of its subgraphs. Let G, H be graphs with $V(H) \cap V(G) = \{v\}$ which both have a hereditary property P. If the graph $G \cup H$ has the property P, then the property P is called a good property. A hereditary property P is k-like if

(i) the complete graph K_m has the property P for every m, $1 \leqq m \leqq k + 1$, and

(ii) whenever S is a P-set of G and a vertex $v \notin S$ is adjacent to at most k vertices of S, then $S \cup \{v\}$ is also a P-set of G. A graph G is said to be k-degenerate, $k \geqq 0$, if $\delta(H) \leqq k$ for every induced subgraph H of G.

Let $U \subset V(G)$ be a P-set if and only if $\langle U \rangle_G$ is k-degenerate. It is easy to see that P is a k-like hereditary property which would be denoted by D_k, and $\chi_P(G)$ is the vertex-partition number introduced in [4]. This property is good only for $k = 0, 1$, but many properties of graphs are good.

In [2] Dirac has proved the following result: Every (n, D_0)-critical graph, $n \geqq 3$, is 2-connected. McCarthy [5] has extended this result to (n, D_1)-critical graphs.

The aim of this paper is to extend Dirac's result to all (n, P)-critical graphs, where either P is a good 0-like property or P is a k-like hereditary property of G and the number of vertices is bounded from above by a simple expression in n and k. This upper bound is the best possible provided $k = 2, 3$ and $n = 2$. Moreover, our result shows that the assumption of 2-connectedness in McCarthy's theorem 5 concerning the existence of Hamiltonian circuits in (n, D_k)-critical graphs is superfluous.

2. CONNECTIVITY

Theorem 1. *If G, $|V(G)| \geqq 3$, is an (n, P)-critical graph with respect to a good 0-like property P, then G is 2-connected.*

Proof. Suppose that G is not connected. Let $G_1, \ldots, G_r$, $r \geqq 2$, be components of G. If A and B are two P-sets of G contained in two different components, then $A \cup B$ is a P-set of G, too. If it were not so, the property P would not be good 0-like. Therefore, $\chi_P(G) = \max \{\chi_P(G_i): G_i \text{ is a component of } G\}$. Hence, the graph G that is (n, P)-critical should be connected. Suppose G has a cut-vertex v.

Let us denote by $V_1, \ldots, V_s$, $s \geqq 2$, the vertex sets of the components of $G - v$. For each induced subgraph $G_i = \langle V_i \cup \{v\} \rangle$ we have $\chi_P(G_i) < n$. Hence, $V_i \cup \{v\}$ can be partitioned into sets W_{ij}, $j = 1, \ldots, n - 1$, some of them may be empty, and each of them which is non empty is a P-set. Suppose that $v \in W_{i1}$ for $i = 1, \ldots, s$. Since P is a good property, the sets $W_j = W_{1j} \cup \ldots \cup W_{sj}$, $j = 1, \ldots, n - 1$, form P-sets. Thus, $\chi_P(G) < n$, a contradiction.

Now, we need some lemmas.

Lemma 1. *Let P be a k-like property. If G is (n, P)-critical, then $\delta(G) \geqq (k + 1) \cdot (n - 1)$.*

The proof is essentially the same as for (n, D_k)-critical graphs (see [4]) and will be omitted.

From Lemma 1 we directly obtain

Lemma 2. *The unique graph G which is (n, P)-critical with respect to a k-like property P with $|V(G)| \leqq (k + 1)(n - 1) + 1$ is the complete graph of order $(k + 1)(n - 1) + 1$.*

Theorem 2. *If G is (n, P)-critical with respect to a k-like property P and $3 \leqq |V(G)| \leqq 2(k + 1)(n - 1) + 2$, then G is 2-connected.*

Proof. Let v be a cut-vertex of G and let $G_1, \ldots, G_s$, $s \geqq 2$, be the components of $G - v$. For a vertex $u \in V(G_i)$, $d(u) \geqq (k + 1)(n - 1) - 1$. Therefore,

$$|V(G_i)| \geqq (k + 1)(n - 1), \quad i = 1, \ldots, s$$

and

$$|V(G)| \geqq s(k + 1)(n - 1) + 1 .$$

By assumption

(a) $$s(k + 1)(n - 1) + 1 \leqq |V(G)| \leqq 2(k + 1)(n - 1) + 2\,.$$

From this, we have

(b) $$(s - 2)(k + 1)(n - 1) \leqq 1\,.$$

If $k = 0$, $n \geqq 2$, the inequality (b) implies that $2 \leqq s \leqq 3$. If $k \geqq 1$ and $n \geqq 2$, we obtain $s = 2$.

Suppose that $s = 2$. In both cases, by (a) at least one of $G_i' = \langle V(G_i) \cup \{v\} \rangle$ (the subgraph induced by $V(G_i) \cup \{v\}$), $i = 1, 2$, say G_1', has $(k + 1)(n - 1) + 1$ vertices. Since $d(u) \geqq (k + 1)(n - 1)$ for any vertex of G, G_1' is a complete graph. By Lemma 2, G_1' is (n, P)-critical. But G_1' is a proper subgraph of G, a contradiction.

Now, let $k = 0$, $n \geqq 2$ and $s = 3$. By (a), $3n - 2 \leqq |V(G)| \leqq 2n$. Hence, $n = 2$. It implies that G is isomorphic to $K_{1,3}$. In a similar way as before, the complete graph K_2 is a proper subgraph of G, a contradiction. Thus G is 2-connected.

In some cases the order of graphs of Theorem 2 is the best possible. Since D_k is a k-like property, according to [5], let H be a graph obtained from K_{k+2} by subdividing one of its edges by a new vertex v. Let G be the graph obtained by joining $[(k + 2)/2]$ copies of H at the vertex v. For $k \geqq 2$, G is $(2, D_k)$-critical, but v is a cut-vertex of G. For $n = 2$, $k = 2, 3$ G has the smallest order, but the problem is still open whether for every $n, k \geqq 2$ the order of graphs from Theorem 2 is the best possible?

3. HAMILTONICITY

Proposition 1 [1]. *Let G be a graph on p vertices and with all degrees at least m. Then, if G is 2-connected, it contains a circuit of length at least $2m$, or a Hamiltonian circuit.*

Proposition 2 [5]. *If a graph G is 2-connected and (n, D_k)-critical and if $|V(G)| \leqq$ $\leqq 2(k + 1)(n - 1) + 2$, then G is Hamiltonian.*

From Theorems 1, 2, Lemma 1 and both Propositions, we have the following results:

Corollary 1. *If G, $|V(G)| \geqq 3$, is (n, P)-critical with respect to a k-like property P, then G contains a circuit of length at least $2(k + 1)(n - 1)$, or a Hamiltonian circuit.*

Corollary 2. *If a graph G is (n, D_k)-critical and if $3 \leqq |V(G)| \leqq 2(k + 1)\,.$ $.\,(n - 1) + 2$, then G is Hamiltonian.*

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Souhrn

ROZKLADOVÁ ČÍSLA, SOUVISLOST A HAMILTONICITA

MIECZYSLAW BOROWIECKI

Autor vyšetřuje souvislost (n, P)-kritických grafů, kde P je vlastnost splňující jisté podmínky („k-like property"). Jako důsledek dostává hamiltonicitu takových grafů.

Резюме

ЧИСЛА РАЗЛОЖЕНИЙ, СВЯЗНОСТЬ И ГАММИЛЬТОНИЧНОСТЬ

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Автор исследует связность (n, P) - критических графов, где P — свойство, удовлетворяющее некоторым условиям (свойство типа k), и в качестве следствия получает, что эти графы гамильтоновы.

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