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SOME ASPECTS OF THE THEORY OF BARRELED SPACES

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Summary. This is an expository article in which certain aspects of the theory of barreled spaces are discussed. Recent results concerning the problem about conditions on spaces assuring continuity of mappings between them which have the closed graph are presented.

Further, if a metrizable barreled space is the union of a nondecreasing sequence of closed absolutely convex sets then at least one of them is a neighbourhood of the origin. Recent research related to this property is analysed as well as the question of preservation of barreledness in projective tensor products.

Keywords: Barrel space, B_r complete space, Baire space.

This is an expository article and contains a somewhat expanded version of the content of the talk given by the author at Český Krumlov in the 14th Seminar in Functional Analysis in May 83.

Our purpose is to survey certain aspects of the structural theory of barreled spaces, mainly to discuss the incidence of Amemiya-Komura's property in recent research, and to study the preservation of barreledness in projective tensor products. Our view of the problems considered here is highly influenced by the type of research which has been done at the Universities of Valencia and, although an effort has been done to include a fair amount of references, we are sure many important cites are still missing; thus no claim of completeness is made.

In Section 1 we give the necessary definitions and study the basic properties of barreled spaces, their permanence properties (tensor products are delayed to section 3) and a short account of Pták's closed graph theorem and the properties of B- and B_r-complete spaces. We focus on recent results which deal with the problem of knowing what kind of properties a space E has to possess if a linear mapping $f: E \to F$ with closed graph should be continuous, when F belongs to different families of Banach spaces.

Section 2 deals with Amemiya-Komura's property and we discuss the main results of the seminal paper [74]. We examine Sason's contributions which triggered the study of many strong barreledness conditions which for a locally convex space are weaker than being a Baire space. (LF)-spaces play a rôle in this situation and a lengthy discussion on metrizable (LF)-spaces is provided. Examples which separate all strong barreledness conditions are given. We discuss also some properties of the space l_0

in this connection and give Marquina-Sanz Serna-Mendoza's characterization of the barreledness of the space of vector-valued sequences $c_0(E)$.

Section 3 is concerned with the preservation of barreledness of the projective tensor product $E \otimes_\pi F$ and its completion for barreled spaces E and F, when E is normed or metrizable.

0. NOTATION

The linear spaces we shall deal with are defined over the field K of the real or complex numbers. Given a subset B of a linear space, $[B]$ stands for its linear hull and $\Gamma(B)$ for its absolutely convex hull. A space means a Hausdorff locally convex topological linear space. If a space E is endowed with its Mackey topology, we say that E is a Mackey space and write $(E, \mu(E, E'))$ or E_μ. E_β stands for a space E endowed with its strong topology $\beta(E, E')$. If F is a linear subspace of a space (E, t), we write (F, t) to denote the space F endowed with the topology induced on F by t. If the topology on E is not specified, we suppose F endowed with the induced topology. If F is a closed subspace of a space (E, t) we write $(E/F, \hat{t})$ or $(E, t)/F$ to denote the quotient space E/F provided with the quotient topology of t. $E^\wedge$ is the completion of E.

A space (E, t) is an (LF)-space if there is a strictly increasing sequence of Fréchet spaces (E_n, t_n) with $E = \bigcup(E_n: n = 1, 2, \ldots)$, the topology t_{n+1} induced on E_n is coarser than t_n and t is the finest locally convex topology inducing on every E_n a topology coarser than t_n. We write $(E, t) = \operatorname{ind}(E_n, t_n)$. If the sequence (E_n, t_n) is constituted by Banach spaces we say that (E, t) is an (LB)-space.

Dropping the word "strictly" in the definition above, Fréchet and Banach spaces can be considered to be (LF) spaces and (LB)-spaces, respectively. Finite products of (LF)-spaces are again (LF)-spaces, but no countable product of (LF)-spaces is an (LF)-space. Barreled and quasi-Baire spaces are defined in the next paragraph. Quasi-Baire spaces are stable with respect to quotients, completions, countable codimensional subspaces and arbitrary products.

A space E is l^∞-barreled if every bounded sequence in $(E', \sigma(E', E)$ is E-equicontinuous. Replacing $\sigma(E', E)$ by $\beta(E', E)$ we have the notion of the l^∞-quasi-barreled space. A space E is $\aleph_0$-barreled if every $\sigma(E', E)$-bounded subset of E' which is the countable union of E-equicontinuous sets is itself E-equicontinuous. Replacing $\sigma(E', E)$ by $\beta(E', E)$ we have the notion of a $\aleph_0$-quasi-barreled space. A $\aleph_0$-quasibarreled space provided with a fundamental sequence of bounded sets is called a DF-space.

$E \otimes_\pi F$ and $E \otimes_\varepsilon F$ denote the tensor product $E \otimes F$ provided with the projective topology and the injective topology, respectively.

Given a space E, E^N and $E^{(N)}$ are the topological product and the topological direct sum of an infinite countable family of spaces equal to E, respectively. A space E

contains $K^{(N)}$ if it contains a linear subspace which is isomorphic to $K^{(N)}$ with the induced topology.

Finally, iff means as usual "if and only if", and for real numbers a and b, $a \vee b$ stands for their maximum.

1. INTRODUCTION: BARRELED SPACES AND SOME OF THEIR PROPERTIES

Banach's classical theorem of the closed graph asserts that, if E and F are Fréchet spaces and if $f: E \to F$ is a linear mapping with closed graph, then f is continuous. This result is equivalent to the so-called open-mapping-theorem which guarantees that a continuous linear mapping g from a Fréchet space F onto a Fréchet space E is open. The proof of the closed graph theorem rests on the following two facts: (i) Let E be a Fréchet space and let F be a space. Then every linear mapping $f: E \to F$ is quasi-continuous. (ii) If E is metrizable and F is a Fréchet space, there every quasi-continuous linear mapping $f: E \to F$ with closed graph is continuous.

Bourbaki classifies (locally convex) spaces according to their behaviour with respect to the basic principles of Functional Analysis. Barreled spaces appear as those spaces which satisfy the uniform boundedness principle and so we have the following

Definition — Theorem 1. *A space E is barreled if every closed, absolutely convex subset of E, which is absorbing in E (i.e., a barrel in E), is a 0-neighbourhood. A space E is barreled iff every linear mapping $f: E \to F$, F being any space, is quasi-continuous. A Mackey space E is barreled iff its weak dual $(E', \sigma(E', E))$ is quasi-complete. A space E is barreled iff every bounded set of $(E', \sigma(E', E))$ is E-continuous.*

Baire spaces are an important class of barreled spaces. The following characterization due to Saxon [60], is of interest.

Theorem 2. *A topological linear space is Baire iff every balanced closed absorbing subset is a neighbourhood of some point.*

Metrizable complete spaces are Baire spaces but a Baire space need not to be metrizable nor complete. Arbitrary products of metrizable complete spaces are Baire as well as arbitrary products of metrizable separable Baire spaces. Quotients of Baire spaces are again Baire and a space containing a Baire dense subspace is also Baire. But the absence of convexity in the former characterization is responsible for their "bad" behaviour with respect to products and finite-codimensional subspaces.

Theorem 3 (Arias [4]). *Every infinite-dimensional separable Banach space has a dense hyperplane of the first category.*

This result was extended by Valdivia by proving

Theorem 4. *Every separable infinite-dimensional Baire space has a dense hyperplane of the first category.*

Whether the assumption "separable" can be omitted in the former result is not known.

Theorem 5 (Arias [5] and Valdivia [85]). *There are two Baire metrizable spaces whose topological product is not Baire.*

$K^{(N)}$ is a barreled space which is not Baire since it is an increasing union of proper closed subspaces, a property which every strict (LF) space shares. In [61] the following definition can be found:

Definition 6. *A space E is quasi-Baire if it is barreled and is not the increasing union of proper closed subspaces.*

In contrast, barreled spaces are stable under the formation of completions, topological products, quotients, direct sums and countable-codimensional subspaces. All closed subspaces of Fréchet spaces, as well as closed subspaces of K^I or $K^{(I)}$, are obviously barreled. There are barreled spaces such that every closed subspace is barreled (e.g., Silva spaces or dense hyperplanes of Fréchet spaces) but, in general, not every closed subspace of barreled space is of this type. In fact, every space can be embedded in a topological product of barreled spaces (and therefore a barreled space) as a closed subspace: following Kōmura, let E be a space and let F be the topological product of its canonical spectrum which as a product of Banach spaces is barreled. Let $(x_i\colon i \in I)$ be a co-basis of E in F. Write H_i to denote the linear hull of E and the elements of the co-basis except x_i. H_i, as an hyperplane of a barreled space, is itself barreled and set $G := \Pi(H_i\colon i \in I)$ which is barreled again. There is a topological isomorphism from E onto a closed subspace of G.

Due to a result of Lindenstrauss-Tzafriri, every non-prehilbertian Banach space E has a closed subspace F which is not complemented. Moreover, if E is separable, there F has a quasi-complement G, i.e. a closed subspace G such that $G \cap F = \{0\}$ and $(G + F)^- = E$. Clearly $G + F$ is dense in E and different from E. If $G + F$ were barreled, the open-mapping theorem applied to the addition mapping $G \times F \to$ $\to G + F$ would imply that $G + F$ is complete, a contradiction. The existence of Banach spaces such that every dense subspace is barreled is an open question due to the following result due to Saxon-Wilansky [64].

Theorem 7. *Let E be an infinite-dimensional Banach space. E has an infinite-dimensional separable quotient iff E has a dense non-barreled subspace.*

As their hereditary properties suggest, barreled spaces are a large class: in particular, (LF)-spaces belong to the class; and sequentially complete DF spaces which are (i) separable or (ii) have metrizable bounded sets or (iii) whose strong dual is weakly compact generated [45], are barreled. The preservation of barreledness in projective tensor products will be studied in the third section.

Now we review the stability of barreled topologies by countable enlargements, and the barreled spaces we shall consider are assumed to be endowed with a topology strictly coarser than the finest locally convex topology. Let (E, t) be a barreled space and set $E' := (E, t)'$. If M is a subspace of E^* such that $M \cap E' = \{0\}$ we call the Mackey topology $\mu(E, E' + M)$ an enlargement of t which is finite, countable or non countable in dependence on the dimension of M. The following result can be attributed to Dieudonne [16] or [14].

Theorem 8. *Let $\mathcal{U}$ be a class of Mackey spaces which is stable under the formation of separated quotients and finite products. Then $\mathcal{U}$ is stable under the formation of finite-codimensional subspaces iff $\mathcal{U}$ is stable by the formation of finite enlargements.*

The following result due to Tweddle-Yeomans [71], ensures the existence of barreled countable enlargements for many barreled spaces.

Theorem 9. *Every barreled space which has a bounded set generating a linear subspace of dimension larger than or equal to c, has a barreled countable enlargement.*

The next theorem, due to S. Dierolf [15], and Roelcke - S. Dierolf [55], is instrumental in the construction of countable enlargements which can be barreled or not barreled.

Theorem 10. *Let (E, t) be a space, L a dense subspace of E and s a locally convex topology on E/L. If r denotes the initial topology on E with respect to the identity $j: E \to (E, t)$ and the canonical surjection $q: E \to (E/L, s)$ then:* (i) *r induces on L the original topology t and $r^\wedge$ coincides with s;* (ii) *if $M := \{f \circ q : f \in (E/L, s)'\}$, then $M \cap E' = \{0\}$;* (iii) *if $(E/L, s)$ and (L, t) are Mackey spaces, then $r = \mu(E, E' + M)$;* (iv) *if $(E/L, s)$ and (L, t) are barreled, then (E, r) is barreled.*

Corollary 10.1 (i) *Every barreled space has a non-barreled countable enlargement;* (ii) *every barreled space which has a dense subspace of codimention larger than or equal to c has a barreled countable enlargement.*

It is not known if every barreled space has a barreled countable enlargement.

Theorem 1.1. *Let (E, t) be a barreled space. Then:* (i) *If H is a subspace of E which admits a finer topology having a barreled countable enlargement, then E has it as well.* (ii) *Let H be a closed subspace which is barreled and such that $(E/H, \hat{t})$ has a barreled countable enlargement. Then (E, t) has it as well.* (iii) *If (E, t) has a barreled countable enlargement and if F is a countable-codimensional subspace of E, then F has a barreled countable enlargement.*

(i) is suggested by [71], Theorem 4, Note 3.2. (ii) easily follows from Theorem 10 and (iii) is contained in [11].

Much of the importance of barreled spaces comes from their connection with the classical closed graph theorem. Pták was the first to go beyond the scope of metrizable spaces in Banach's result by using the techniques of duality theory. Contributions in this research line are mainly due to Pták, Persson, McIntosh, Iyahen, Komura, Adasch, Valdivia and Kalton. A very useful result for spaces used in the applications is due to Baernstein and was extended by Ruess. Following the techniques of Komura, Adasch and Valdivia characterized the optimal range class for a closed graph theorem where the barreled spaces constitute the domain class. The main drawback of this result is that many spaces which are important in applications do not belong to the range class. Another attempt to extend Banach's result, not from the structural point of view but with emphasis on applications, was given by Schwartz. A space F in Suslin if it is a continuous image of a topological separable space such that there is a metric inducing its topology for which the space is complete. Many spaces which appear in connection with the theory of distributions are Suslin spaces and Schwartz proves: "Let E be a Banach space (or an ultrabornological space), let F be a Suslin space and let $f: E \to F$ be a linear mapping with a borelian graph. Then f is continuous." This result does not generalize Banach's result since Suslin spaces are necessarily separable. Martineau improves this result using K-Suslin spaces instead of Suslin spaces and proves: "Let E be a Baire space, let F be a K-Suslin space and $f: E \to F$ a linear mapping with closed graph. Then f is continuous". But this again does not generalize Banach's result although Banach reflexive spaces with their weak topologies are K-Suslin spaces, but not all Banach spaces fall in this category. For a full treatment of those topics see [73].

Grothendieck shows the following

Theorem 12. *Let (E, t) be a Baire space. $(F, U) = \mathrm{ind}(F_n, U_n)$ an* (LF)*-space and $f: E \to F$ a linear mapping with closed graph. Then there is a natural number p such that $f(E)$ is contained in F_p and $f: (E, t) \to (F_p, U_p)$ is continuous.*

After proving his closed graph theorem, Grothendieck conjectures the existence of a large class of spaces containing the Banach spaces and stable under the formation of countable products, countable direct sums, separated quotients and closed subspaces which can serve as a range class for a closed graph theorem where ultrabornological spaces stay in the domain class. Solutions to this conjecture where given by Slowikowski [65], Raikov [54] and De Wilde [13]. De Wilde introduced the class of strict webbed spaces which enjoy the properties required by Grothendieck. A similarnotion to the class of spaces defined by Slowikowski for the strict case can be defined, and Valdivia [84], showed the existence of webbed spaces which do not have a strict web. All different approaches to Grothendiesk's conjecture were seen to be equivalent by Efimova. For further reference, we shall use the following version of De Wilde's theorem.

Theorem 13. *Let E be a Baire space, F the countable reduced inductive limit of strictly webbed spaces (F_n) and $f: E \to F$ a linear mapping with closed graph. Then there is an index p such that $f(E)$ is contained in F_p and $f: E \to F_p$ is continuous.*

As a corollary one gets that a strictly webbed Baire space is a Fréchet space.

A stronger localization theorem for webbed spaces has been obtained recently by Valdivia [83]: If $f: E \to F$ is a linear mapping with closed graph, E being a Baire space and F webbed, there is a subspace H of F and a topology s on H finer that the induced topology of F such that $f(E)$ is contained in H, (H, s) is a Fréchet space and $f: E \to (H, s)$ is continuous.

As corollaries one gets: (i) Let F be a webbed space. If F is the locally convex hull of Baire spaces, then F is ultrabornological. (ii) Let F be a webbed space. If F is the locally convex hull of a sequence of Baire spaces, then F is an (LF)-space.

The webbed spaces enjoy good hereditary properties. Valdivia has shown [87]: "$E \otimes_a F (a = \varepsilon, \pi)$ is webbed iff one of the two following conditions is satisfied: (i) E is webbed and $\dim F \leqq \aleph_0$, (ii) F is webbed and $\dim E \leqq \aleph_0$".

It was an open problem whether De Wilde's closed graph theorem contained Schwartz's closed graph theorem. Since the projective tensor product of Suslin spaces is a Suslin space [87], it is enough to consider Suslin spaces E and F of non countable infinite dimension to conclude that there are Suslin spaces which are not webbed and thus De Wilde's theorem does not contain Schwarz's result [87].

In the category of Banach spaces the closed graph theorem is equivalent to the open-mapping theorem and this is not necessarily true in the category of locally convex spaces. In order to give a closed graph theorem extending Banach's classical result beyond the scope of metrizable spaces, Pták introduced B_r-complete spaces as spaces F such that every quasi-continuous linear mapping $f: E \to F$, E being any space, with closed graph is continuous and therefore one has: "Let E be a barreled space, F a B_r-complete space, then any linear mapping $f: E \to F$ with closed graph is continuous."

The property of being B_r-complete is connected with the Krein-Smul'jan property and with completeness by means of the following characterization: "A space E is B_r-complete iff every dense subspace of its topological dual E' which intersects every closed absolutely convex E-equicontinuous set in weakly closed sets is closed in $(E', \sigma(E', E))$. According to the Krein-Smul'jan theorem every Fréchet space is B_r-complete (thus, the range class for Pták's closed graph theorem contains all Fréchet spaces) and every B_r-complete space is complete due to Pták-Collins's characterization of completeness. In order to give an open-mapping-theorem, Pták introduced B-complete spaces which are characterized by means of "A space E is B-complete iff every subspace of its topological dual E' which intersects every closed absolutely convex E-equicontinuous set in weakly closed sets is closed in $(E', \sigma(E', E))$." If a space is B-complete, every separated quotient is B-complete and every closed subspace is also B-complete. Moreover, clearly every B-complete space is B_r-complete and a space such that every separated quotient is B_r-complete is necessarily B_r-complete. There is a richness of complete non B-complete spaces due to the following result [89]: "Let E be a bornological space. There is a complete semi-Montel space F such that E is a quotient of F." Very recently Valdivia has distinguished between B- and B_r-completeness by constructing on the conjugate of l^∞ a topology s, which is a countable enlargement of $\mu(l^{\infty\prime}, l^\infty)$ with elements of the second conjugate of l^∞, such that $((l^\infty)', s)$ is B_r-complete but not B-complete. Moreover, he showed the existence of B_r-complete spaces with non complete separated quotients [86].

Now we quote examples of B-complete spaces: $K^{(N)}, K^I, (K^N)^{(N)}, (K^{(N)})^N$ and $K^{(N)} \times E$ for every B-complete space E. Fréchet spaces and Mackey duals of Fréchet spaces as well as closed subspaces of B-complete spaces are B-complete. The countable direct sum of quasireflexive (i.e., spaces of finite codimension in its second conjugate) Banach spaces is B-complete [43]. It would be interesting to characterize the B-complete countable direct sums of Banach spaces.

Köthe showed that the product $(K^N)^{(N)} \oplus (K^{(N)})^N$ has a quotient which is not complete and therefore is not B-complete. A general argument due to Grothendieck shows the richness in the existence of non B-complete spaces: "Let E_n be a dense subspace of a space (F_n, t_n), $n = 1, 2, \ldots$, and assume a topology s_n on E_n finer than t_n. Then $(H, r) := \prod_1^\infty (E_n, s_n) \times \bigoplus_1^\infty (F_n, t_n)$ has a non complete quotient." The idea is to consider the mapping $q\colon H \to \prod_1^\infty E_n + \bigoplus_1^r F_n$ which is a subspace of $\prod_1^\infty F_n$ by means of $q((x_n), (y_n)) = (x_n - y_n)$ which is linear, continuous, open and onto and $\ker q = \{((x_n), (x_n)) : x_n \text{ in } \bigoplus_1^\infty E_n\}$. Clearly $\prod_1^\infty E_n + \bigoplus_1^\infty F_n$ is not complete since, if we take elements z_n in F_n but not in E_n, we set $z^n := (z_1, \ldots, z_n, 0, 0, \ldots)$ which is a Cauchy sequence and converges to $(z_1, z_2, \ldots, z_n, z_{n+1}, \ldots)$ which does not belong to $\prod_1^\infty E_n + \bigoplus_1^\infty F_n$.

Other examples can be constructed by looking for non complete quotients. The countable direct sum of copies of c_0 has such a quotient (i.e., the non complete (LB)-space of Köthe [33]) and since l^1 has c_0 is a quotient, the countable direct sum of copies of l^1 also has a non complete quotient. In [69] it is shown that the space of L. Schwartz $D(R)$ has a quotient which is a proper dense subspace of K^N and therefore $D(R)$ as well as $D(\Omega)$, Ω being an open set of the Euclidean space R^n, are not B-complete since they are homeomorphic. Raikov and Harvey-Harvey [25] show that the space of distributions $D'(\Omega)$ is not B-complete. The strong dual of a Fréchet space need not be B-complete (consider the countable product of copies of c_0).

In order to look for non B_r-complete spaces the following lemma is useful [76]: "Let E be a separable space and suppose E is the increasing union of a family of subspaces E_n. If there is a bounded set of E which is not contained in any E_n, then there is a proper dense subspace F of E such that the intersection $F \cap E_n$ is finite dimensional for any n."

This lemma is successfully applied in [76] to show that the space $D'(\Omega)$ is not B_r-complete. The space $D(\Omega)$ is not B_r-complete [78]; the detailed proof of this result can be found in [44].

The lemma above combined with the technique of Grothendieck mentioned before produces richness of examples of non B_r-complete spaces. We compile in a theorem several results of Valdivia (the formulation below is taken from [20] which is an excellent source of information about B- and B_r-completeness as well as about the general theory of the closed graph theorem) [77] and [88].

Theorem 14. *Let (E_n) and (F_n) be infinite-dimensional Fréchet spaces. Then the following spaces are not B_r-complete:*

(i) $\bigoplus_1^\infty E_n \times \prod_1^\infty F_n$ with $F_n \neq K^N$,

(ii) $\bigoplus_1^\infty E'_{n\mu} \times \prod_1^\infty F_n$ with either $F_n \neq K^N$ or F_n having a closed subspace with a continuous norm,

(iii) $\bigoplus_1^\infty E_n \times \prod_1^\infty F'_n$ with F_n having an infinite-dimensional separable quotient,

(iv) $\bigoplus_1^\infty E'_{n\mu} \times F'_{n\mu}$ with $E_n \neq K^N$ or F_n having an infinite-dimensional separable quotient.

Thus the product of two B_r-complete spaces is not necessarily B_r-complete. The strong dual of a Fréchet space need not be B_r-complete: Shavgulidze [28], states (without proof) that $l^1 \times \oplus l^2$ is not B_r-complete (it is the strong dual of $c_0 \times \prod l^2$). It is shown in [82] that if $(E_i: i \in I)$ are Banach spaces of infinite dimension and $\operatorname{card}(I) \geqq 2^{\aleph_0}$, then $\prod(E_i: i \in I)$ is not B_r-complete.

If $\mathscr{B}$ denotes the class of all Banach spaces, a result due to Mahowald [35], assures that the class of barreled spaces is maximal as a domain class for the closed graph theorem, $\mathscr{B}$ being contained in the range class, by showing that given a non barreled space E there is a Banach space F and a non continuous linear mapping $f: E \to F$ with closed graph. On the other hand, the class of B_r-complete spaces is not maximal as a range class for a closed graph theorem where the barreled spaces form the domain class since a Banach space of infinite dimension endowed with its weak topology is not complete (and therefore not B_r-complete) and satisfies the closed graph theorem of Pták.

Kōmura developed a general theory for the study of maximal classes in the range of a closed graph theorem where the domain is a fixed class $\mathscr{F}$.

Let $\mathscr{F}$ be a class of spaces which is stable under the formation of inductive limits and contains the finite dimensional spaces. $\mathscr{F}_r$ denotes the class of such spaces F that if $E \in \mathscr{F}$ and $f: E \to F$ is a linear mapping with closed graph then f is continuous.

Given a space (H, t), the topology t^a associated to t of class $\mathscr{F}$ is the inductive topology of all those topologies on H finer than t for which H is of the class $\mathscr{F}$. For a class $\mathscr{F}$ with the properties stated above, Kōmura characterizes the maximal class in the range $\mathscr{F}_r$ as follows:

Theorem 15. *Let (H, t) be a space. (H, t) is in $\mathscr{F}_r$ iff for every Hausdorff locally convex topology s coarser than t it follows that s^a is finer than t.*

This result leads to the following one for a family $\mathscr{F}$ of Mackey spaces:

Theorem 16. *Let (H, t) be a space. (H, t) is in $\mathscr{F}_r$ iff every dense subspace G of the algebraical dual H^* of H which is $\sigma(H^*, H)$-dense in H^* and such that* (i) *$G \cap H'$ is $\sigma(H', H)$-dense in H',* (ii) *H endowed with the Mackey topology $\mu(H, G)$ belongs to the class $\mathscr{F}$, contains H'.*

Following Valdivia let us call Γ_r-spaces the members of the class $\mathscr{F}_r$, $\mathscr{F}$ being the barreled spaces. It follows from the definitions that every B_r-complete space is a Γ_r-space and every barreled Γ_r-space is B_r-complete. Thus $D(\Omega)$ and $D(\Omega)'$ are not Γ_r-spaces and the need for another closed graph theorem including these spaces in the range class becomes apparent.

Γ_r-spaces need not be complete but they are complete if endowed with their associated barreled topologies (but not necessarily B_r-complete). Their hereditary properties are as "bad" as those of B_r-complete spaces.

Due to the maximality of both the domain and the range class we can either restrict the range class (and then the domain class will grow), or enlarge the range class and look for subclasses of barreled spaces as domain classes. The first path will be pursued now.

Let $\mathscr{R}$ be a class of spaces and let $\mathscr{R}_s$ be the maximal domain class for a closed graph theorem whose range class is $\mathscr{R}$, i.e. the family of spaces E such that if $F \in \mathscr{R}$ then every linear mapping $f: E \to F$ with closed graph is continuous. As mentioned above, if $\mathscr{R}$ denotes the class $\mathscr{B}$ of all Banach spaces, then $\mathscr{R}_s$ is the class of barreled spaces. We shall study the class $\mathscr{R}_s$ for certain important subclasses $\mathscr{R}$ of Banach spaces and we shall show the existence of Banach spaces F such that F does not belong to $\mathscr{R}_{sr}$.

We shall start by showing the existence of classes of Banach spaces $\mathscr{R}$ distinct from $\mathscr{B}$ such that $\mathscr{R}_s$ is still the class of barreled spaces.

Theorem 17. *Let $\mathscr{R}_0$ be the class of all Banach spaces of the form $C(K)$, K being a Hausdorff compact set. Then $\mathscr{R}_{0s}$ is the family of all barreled spaces.*

The former theorem is due to Wilansky [92]. Kalton was the first who exhibited subclasses $\mathscr{R}$ of $\mathscr{B}$ for which $\mathscr{R}_s$ is strictly larger than the class of all barreled spaces:

Theorem 18. *Let (E, t) be a space such that $(E', \sigma,(E', E))$ is sequentially complete and let F be a separable B_r – complete space. Then every linear mapping $f: E \to F$ with closed graph is continuous from $(E, \mu(E, E'))$ into F.*

It is not difficult to exhibit spaces E for which $(E', \sigma(E', E))$ is sequentially complete but not uasi-complete (i.e., barreled spaces): every l^∞-barreled which is not barreled. The preceding result as well as the two which follow can be found in [30].

Theorem 19. *A space E belongs to $\mathscr{R}_s = (c_0)_s$ iff every Cauchy sequence in $(E', \sigma(E', E))$ is E-equicontinuous.*

Theorem 20. *A space E belongs to $\mathscr{R}_s = (C[0, 1])_s$ iff every metrizable bounded subset of $(E', \sigma(E', E))$ is E-equicontinuous. Moreover, if F is a separable B_r-complete space, then $F \in (C[0, 1])_{sr}$.*

It is shown in [34] that if $\mathscr{R}_1 :=$ {all separable B_r-complete spaces}, $\mathscr{R}_2 :=$ {all separable Banach spaces} and $\mathscr{R}_3 :=$ {the space $C[0, 1]$} then $(\mathscr{R}_1)_s = (\mathscr{R}_2)_s = (\mathscr{R}_3)_s$, a class which appears also in [42] were the hereditary properties of the classes are studied. The class $(C[0, 1])_s$ appears also as a particular case of the $G(\alpha)$-barreled spaces [50], which are defined as follows: let $\mathscr{R}(\alpha)$ be the class of all Banach spaces with density character at most α and let E be a space. A barrel U in E is said to be a $G(\alpha)$-barrel if $E/N(U)^\wedge$ belongs to $\mathscr{R}(\alpha)$. E is $G(\alpha)$-barreled if every $G(\alpha)$-barrel is a 0-neighbourhood. The following result can be found in [50]:

Theorem 21. *Let E be a space. E is $G(\alpha)$-barreled iff E belongs to $(\mathscr{R}(\alpha))_s$. $\mathscr{R}(\alpha)$ can be replaced by the class of all B-complete spaces with density character at most α.*

Then $(C[0, 1])_s$ coincides with the $G(\aleph_0)$-barreled spaces and Theorem 20 provides a characterization of $G(\aleph_0)$-barreled spaces through the dual.

For any class of spaces $\mathscr{R}$, $\mathscr{R}_s$ is stable under the formation of finite products and inductive limits. Let $(E_i\colon i \in I)$ be members of $\mathscr{R}_s$. If K^I belongs to $\mathscr{R}_s$ then $\prod(E_i\colon i \in I)$ also belongs to $\mathscr{R}_s$ (see [12], 4, Th. 1). Then arbitrary products of elements in $(C[0, 1])_s$ and in $(c_0)_s$ belong to their respective classes. It is also true that infinite countable codimensional subspaces of spaces in $(C[0, 1])_s$ and $(c_0)_s$ belong to $(C[0, 1])_s$ and $(c_0)_s$, respectively, but the arguments how to show it depend on the following result of Saiflu-Tweddle [57]:

Theorem 22. *Let $\mathscr{R}$ be a class of spaces such that if F belongs to $\mathscr{R}$ then $F \times K^{(N)}$ also belongs to $\mathscr{R}$. Then $\mathscr{R}_s$ contains every countable codimensional subspace of each of its members.*

Since $K^{(N)}$ is separable, $F \times K^{(N)}$ has density character at most α for any B-complete space F with this property, hence

Theorem 22′. *A countable codimensional subspace of a $G(\alpha)$-barreled space is itself $G(\alpha)$-barreled* [57].

As mentioned above the same result is true for $(c_0)_s$, see [57].

Let E be a space and let α be an infinite cardinal. E is said to be an α-weakly compact generated (α-WCG) space if there is a family of absolutely convex compact sets $(A_i: i \in I)$ in $(E, \sigma(E, E'))$ such that $\operatorname{card} I = \alpha$ and $\bigcup(A_i: i \in I)$ is dense in $(E, \sigma(E, E'))$. A space E is weakly compact generated (WCG) is there exists a weakly compact set total in E. It is easy to see that a Banach space is WCG iff it is $\aleph_0$-WCG. The following theorem was proved by Marquina [37], and it extends Kalton's result (Theorem 18).

Theorem 23. *Let E be a space such that $(E', \sigma(E', E))$ is sequentially complete and let F be an $\aleph_0$-WCG B_r-complete space. Then every linear mapping $f: E \to F$ with closed graph is continuous as a mapping from $(E, \mu(E, E'))$ into F.*

The following extension of Theorem 20 can be obtained by using Theorem 23.

Theorem 24. *Let E be a space such that every absolutely convex bounded metrizable subset of $(E', \sigma(E', E))$ is E-equicontinuous, and let F be an $\aleph_0$-WCG B_r-complete space. Then every linear mapping $f: E \to F$ with closed graph is continuous.*

Let α be an infinite cardinal number and E a space. E is α-barreled if every bounded set of cardinality less than or equal to α of $(E', \sigma(E', E))$ is E-equicontinuous. The following result [37], can be obtained in the same way as Theorem 23 by using, instead of Dieudonne-Schwartz's result, the weak compactness result proved in [81], Theorem 1:

Theorem 25. *Let E be a Mackey α-barreled space, F an α-WCG B_r-complete space and $f: E \to F$ a linear mapping with closed graph. Then f is continuous.*

If $\mathscr{R}^*(\alpha)$ is the class of all Banach spaces which are closed subspaces of α-WCG Banach spaces, Saiflu-Tweddle in [58] characterize the class $\mathscr{R}^*(\alpha)_s$ which is the maximal domain class for Marquina's closed graph theorem. First we need a definition.

Definition. *Let U be a barrel in a space E. U is an $M(\alpha)$-barrel if $E/N(U)^\wedge$ belongs to $\mathscr{R}^*(\alpha)$. E is $M(\alpha)$-barreled if every $M(\alpha)$-barrel is a 0-neighbourhood in E.*

Theorem 26 [58]. *E is $M(\alpha)$-barreled iff belongs to $\mathscr{R}^*(\alpha_s)$.*

By virtue of Theorem 25, every Mackey space which is α-barreled is $M(\alpha)$-barreled. In [58] examples of $M(\alpha)$-barreled spaces which are neither Mackey nor α-barreled as well as examples of α-barreled spaces which are not $M(\alpha)$-barreled are given. It is shown in [37] that, for a non countable cardinal α, $l^1(\alpha)$ does not belong to $\mathscr{R}^*(\aleph_0)_{sr}$. If I is an index set with $\operatorname{card} I > \alpha$, α being an infinite cardinal number, the space $l^1(I)$ is not in $\mathscr{R}^*(\alpha)$ as shown in [58]. Again, the class $\mathscr{R}^*(\alpha)_s$ is closed under the formation of inductive limits, products, completions and by countable codimensional subspaces [58]. Clearly, every $M(\alpha)$-barreled space is $G(\alpha)$-barreled and every $G(\alpha)$-barreled space can be endowed with a topology of the dual pair for which it is $M(\alpha)$-barreled [58].

The following result characterizes the $M(\aleph_0)$-barreled spaces and is due to Marquina (unpublished). Let K be a Hausdorff infinite compact set. K is an Eberlein-compact set if it is homeomorphic to a weakly compact subset of a Banach space. Amir-Lindenstrauss [3], prove the following characterization: "K is Eberlein-compact iff $C(K)$ is WCG." The continuous image of Eberlein-compact is also a set of this type [8] and [40]. Using these results and [93]; Theorem 2, p. 147 it is easy to show that if E is a closed subspace of a WCG Banach space, i.e. E belongs to $\mathscr{R}^*(\aleph_0)$, and if U is the closed unit ball of E, then U^0 endowed with the weak topology $\sigma(E', E)$ is Eberlein-compact. Thus, if E belongs to $\mathscr{R}^*(\aleph_0)$, there is an isometric embedding $J: E \to C(U_\sigma^0)$. According to the definition of the $M(\aleph_0)$-barreled space and Theorem 26, an argument similar to that employed in the proof of Theorem 17 gives

Theorem 27. *Let $\mathscr{R}$ be the class of all Banach spaces of the form $C(K)$, K being Eberlein-compact. Then $\mathscr{R}_s$ is the class of all $M(\aleph_0)$-barreled spaces.*

We shall now describe the Mackey spaces which belong to $(l^2)_s$ characterized through local completeness. The definitions and theorems that follow are all due to Valdivia [80].

A space E is dual locally complete if $(E', \sigma(E', E))$ is locally complete (for the notion of local completeness, see e.g. [28]). Suggested by Theorem 16, define a Λ_r-space E as a space for which every locally complete subspace G of $(E^*, \sigma(E^*, E))$ which intersects E' in a dense subspace of $(E', \sigma(E', E))$ contains E'. Then a closed graph theorem for dual locally complete spaces in the domain and Λ_r-spaces in the range is available and the Λ_r-spaces form a class which is maximal:

Theorem 28. *Let E be a dual locally complete space, F a Λ_r-space and $f: E \to F$ a linear mapping with closed graph. Then $f: E_\sigma \to F_\sigma$ is continuous.*

Theorem 29. *If F is not a Λ_r-space then there exists a dual locally complete space E and a linear mapping $f: E \to F$ with closed graph which is not weakly continuous.*

The maximality of the domain class is shown through a "Mahowald-type" theorem which is rather involved and exploits the existence of bounded Markushevich bases:

Theorem 30. *Let F be an infinite-dimensional Banach space. If E is not dual locally complete then there is a non-weakly-continuous linear mapping with closed graph $f: E \to F$.*

Reflexive Banach spaces are Λ_r-spaces: indeed, let E be a reflexive Banach space. Then $(E', \mu(E', E))$ is obviously locally complete. If H is a weakly dense subspace of E', its local completion is contained in $(E', \mu(E', E))$ and clearly coincides with E'. Other examples of Λ_r-spaces are the Fréchet-Schwartz spaces and the space $(K^N)^{(N)}$. Non-reflexive Fréchet spaces are not Λ_r-spaces (and therefore the class of Λ_r-spaces is strictly contained in the class of Γ_r-spaces): indeed, let E be a non-reflexive Fréchet space. Then the space E, endowed with the topology of the uniform convergence on the locally null sequences of E', is not complete and thus $(E', \mu(E', E))$ is not bornological. There is a locally bounded linear form on E' which is not continuous on $(E', \mu(E', E))$ and therefore $\ker f$ is a dense hyperplane of this space. If $\ker f$ is locally complete we reach our conclusion. Indeed, it is enough to show that $\ker f$ is locally closed (since $(E', \mu(E', E))$ is complete) which is obvious since f is sequentially continuous.

Corollary 30.1. *A space E is dual locally complete iff every linear mapping $f: E \to l^2$ with closed graph is weakly continuous.*

Thus, a Mackey space $E \in (l^2)_s$ iff E is dual locally complete.

2. AMEMIYA-KOMURA'S PROPERTY

Metrizable barreled spaces enjoy the following property as was shown by Amemiya-Komura [2].

Theorem 31. *If a metrizable barreled space E can be covered by the increasing union of a sequence (B_n) of closed, absolutely convex sets, then there is a natural p such that B_p is absorbing in E (and therefore a 0-neighbourhood).*

In order to extend this result the following concepts are useful: an increasing sequence $\mathcal{U} = (U_n)$ of absolutely convex sets in a space E is absorbing if $\bigcup(U_n: n = 1, 2, \ldots)$ is absorbing in E, and $\mathcal{U}$ is bornivorous if, for every bounded set B in E, there is a natural p such that U_p absorbs B.

Valdivia's extension of Theorem 31 [74], read as follows:

Theorem 32. *Let E be a barreled (quasi-barreled) space whose completion is Baire. If $\mathcal{U} = (U_n)$ is an absorbing (bornivorous) sequence of closed subsets of E, then there is a natural p such that U_p is a 0-neighbourhood in E.*

The proof of this result is accomplished by a deep understanding of the behaviour of absorbing sequences which can be particularized in the following two equivalent conditions (Valdivia and De Wilde-Houet): if L is a barreled (quasi-barreled) subspace of a space E and if $\mathcal{U} = (U_n)$ is an absorbing (bornivorous) sequence in L, then (i) the closure of $\bigcup(U_n: n = 1, 2, \ldots)$ in L coincides with the algebraic closure of $\bigcup(\overline{U}_n: n = 1, 2, \ldots)$; (ii) if $\mathcal{F}$ denotes a filter in $\bigcup(U_n: n = 1, 2, \ldots)$ and $\mathcal{G}$ the filter generated by the sets $M + U$, with M in $\mathcal{F}$ and U a 0-neighbourhood in L, then $\mathcal{G}$ induces a Cauchy filter in some $(1 + \varepsilon) U_p$ if $\mathcal{F}$ is a Cauchy filter in $\bigcup(U_n: n = 1, 2, \ldots)$.

Two interesting facts can be deduced: from Theorem 32 first, countable codimensional subspaces of barreled spaces are barreled (a result which was also obtained independently and with different techniques by Levin and Saxon [62]) and a version of Raikov's completeness theorem:

Theorem 33. *If a barreled (quasi-barreled) space E has an absorbing (bornivorous) sequence of complete sets, then E is complete.*

In 1962 Garling introduced the concept of the generalized inductive limit topology as follows: let E be a linear space covered by an increasing sequence of subspaces (E_n) and let, for every n, A_n be an absolutely convex subset of E_n absorbing in E_n. If t_n denotes a locally convex topology on E_n, suppose $2A_n$ contained in A_{n+1} and t_{n+1} induced on A_n coarser than t_n on A_n for every n. Then the generalized inductive limit topology t on E is defined as the finest locally convex topology on E such that its restriction to A_n is coarser than t_n. If t_{n+1} induces on A_n precisely t_n, t is called the generalized strict inductive limit topology. In this case t is a Hausdorff topology which coincides on A_n with t_n for every n. Moreover, if A_n is closed in A_{n+1} endowed with t_{n+1} for every n, then A_n is closed in (E, t) and (E, t) is complete if every (A_n, t_n) is complete. A similar idea, but for balanced sets, was studied by Turpin. For an absorbing (bornivorous) sequence $A = (A_n)$, provide each A_n with the topology induced by the original topology of the space and write t_A to denote the generalized inductive limit topology on E. Valdivia [74], shows

Theorem 34. *If (E, t) is a barreled (quasi-barreled) space, then for every absorbing (bornivorous) sequence $A = (A_n)$ in E, t coincides with t_A.*

Corollary 34.1. *If (E_n) is an increasing sequence of subspaces of a barreled space (E, t), then (E, t) is the inductive limit of the sequence (E_n, t).*

Those ideas were used by Ruess to introduce and study a natural generalization of (DF)-spaces, the (gDF)-spaces which have good permanence properties and contain large classes of spaces which are of interest in applications and which are not (DF)-spaces, e.g., spaces of type $C(X)$ endowed with the strict topology of Buck. This class appears also in Nouredinne under the name D_b-space and in Adasch and Ernst as σ-local topological locally convex paces. To be precise, a space (E, t) is a (gDF)-space if it admits a fundamental sequence A of bounded sets and if t coincides with t_A. Every (gDF)-space is quasi-normable and the class is stable under the formation of separated quotients, countable direct sums, countable inductive limits and completions. In this context, and interesting open-mapping theorem (due to Baernstein for (DF)-spaces) appears

Theorem 35. *Let E be a semi-Montel space, F a (gDF)-space and $f: E \to F$ a continuous linear mapping such that $f^{-1}(B)$ is bounded in E for every bounded set B of F. Then f is open and injective and, moreover, E is a (gDF)-space.*

For interesting applications of this result see (Bierstedt, Meise, Summers: "A projective description of weighted inductive limits". Trans. Amer. Math. Soc., 272 (1982), 107–160).

Another line of research starts with Saxon in 1972 whose purpose is two-fold: first, try to study unknown stability properties of Baire locally convex spaces such as finite products and finite-codimensional subspaces of Baire spaces through local convexity and second, classify the large amount of (normed) barreled spaces which are not Baire. The first line has not been successful (see Theorems 3, 4 and 5, whose proofs do not depend on the locally convex structure of the space), but the second has provided several new types of strong barreledness conditions, a classification of (LF)-spaces and several closed graph theorems which are interesting, if only from the structural point of view.

Saxon defines Baire-like (BL) spaces as those locally convex spaces satisfying Amemiya-Komura's property, i.e., E is BL if, when covered by an increasing sequence of closed, absolutely convex subsets (A_n), there is a natural p such that A_p is a 0-neighbourhood, or equivalently, if in every absorbing sequence of closed subsets (A_n) there is a member A_p which is a 0-neighbourhood. Theorem 31 is extended in the following way:

Theorem 36. *If E is barreled and does not contain $K^{(N)}$, E is* BL.

According to Theorem 32, every barreled space whose completion is BL is BL itself. It is unknown if there exists a complete BL space which is not Baire. Every non countable product of infinite-dimensional Banach spaces contains a copy of $K^{(N)}$ which shows that the reciprocal of Theorem 36 is not true. For BL spaces the following closed graph theorem holds.

Theorem 37. *Let E be a* BL *space, F an* (LB)*-space and $f: E \to F$ a linear mapping with closed graph. Then f is continuous.*

This result implies that every metrizable (LB)-space is necessarily complete. Smoljanov [69], proved that the (LF)-space $D(R)$ has a quotient which is a proper dense subspace of K^N. By application of Grothendieck's closed graph theorem, this quotient is again an (LF)-space and therefore metrizable and non-complete. Thus, (LB)-spaces cannot be replaced by (LF)-spaces in Theorem 37. Thus, to give a closed graph theorem which includes (LF)-spaces in the range class, a close look at Grothendieck's proof of his closed graph theorem shows that what you need in the domain class is that the following condition is satisfied: A space E is suprabarreled (SB) if, given an increasing sequence (E_n) of subspaces of E covering E there is a natural p such that E_p is barreled and dense in E, see [75]. By dropping the word "increasing" in the definition you get what Saxon and Todd called unordered-Baire-like (UBL) spaces. It is easy to show that De Wilde's theorem of the closed graph can be extended to UBL spaces in the domain. But a further extension in the domain is possible for totally barreled spaces (TB spaces) defined in [90]: a space E is totally barreled if, given a sequence of subspaces of E covering E, there is an integer p such that E_p is barreled and its closure in E is of finite codimension in E. TB spaces have the following property:

Let (E_n) be a sequence of subspaces of a TB space E covering E. There is a natural p such that E_p is TB and its closure in E is of finite codimension in E.

This property allows to extend De Wilde's result in the following sense:

Theorem 38. *Let E be* TB, *G a space admitting a web formed by absolutely convex sets, and $f: E \to G$ a linear mapping with closed graph, then f is continuous.*

The idea is to reduce the problem to a situation where De Wilde's method of proof can be applied.

Clearly we have the following chain of implications: Baire $\Rightarrow$UBL $\Rightarrow$ TB $\Rightarrow$ $\Rightarrow$ SB $\Rightarrow$ BL $\Rightarrow$ Quasi-Baire $\Rightarrow$ Barreled.

In what remains in this section we shall review their permanence properties, provide examples that show that the implications above are strict and classify (LF)-spaces in terms of those definitions.

The spaces we have introduced (BL, SB, TB, UBL) have remarkable hereditary properties. Although the proof of some of them is a simple routine, not all of them are of trivial nature. The stability of those classes by products is specially cumbersome, depending normally on several previous lemmas. We refer to the original papers for their proofs (see [61], [75], [90] and [70], respectively). A unified approach to this study can be found in [29].

We state the results.

Theorem 39. *Let F be a dense subspace of a space E. If F is* BL(SB, TB, UBL) *then E is* BL(SB, TB, UBL).

Theorem 40. *Let E be a space and let F be a closed subspace of E. If E is* BL(SB, TB, UBL) *then E/F is* BL(SB, TB, UBL) *as well.*

Theorem 41. *Let F be a countably codimensional subspace of a space E. If E is* BL(SB, TB, UBL), *then F is* BL(SB, TB, UBL) *as well.*

Lemma. *Let $(E_i: i \in I)$ be a family of spaces and set $E := \prod(E_i: i \in I)$.* If $\mathscr{B}$ is a family of a countable number of closed balanced subsets of E covering E then there is an element B of $\mathscr{B}$ and a finite part A of I such that B contains $\prod(E_i: i \in I \setminus A)$. Moreover, if the elements of $\mathscr{B}$ are also convex sets, the family $\mathscr{F} := ([B]: B \supset \supset \prod(E_i: i \in I \setminus J(B)$ with $J(B)$ finite and contained in $I)$ covers E.

Theorem 41′. *Let $(E_i: i \in I)$ be a family of* BL(SB, TB, UBL) *spaces. Then $E := \prod(E_i: i \in I)$ is* BL(SB, TB, UBL) *as well.*

Under a three -space problem we understand the following situation: Let E be a space such that there is a closed subspace F such that F and E/F satisfy a certain property P. Does E satisfy property P? The answer is affirmative if P is the property of being a barreled space, [55] and if P is the property of being BL, SB, TB and UBL [10].

Theorem 42. *The three-space-problem is true for P = barreled,* BL, SB, TB, UBL.

Natural extensions of all those concepts to the non-locally-convex setting can be easily given: we shall consider topological linear spaces E which are Hausdorff.

Let $\mathscr{U} = (U_n)$ be a family of subsets of E. $\mathscr{U}$ is a basic sequence if every U_n is absorbing, balanced and satisfies $U_{n+1} + U_{n+1} \subset U_n$. $\mathscr{U}$ is said to be closed if every U_n is closed in E. Let A be a subset of E: A is admissible if it is balanced and there is a basic sequence $\mathscr{A} := (A_n)$ in the linear hull of A with $A_1 = A$. If there is a sequence (U^m) of admissible sets in E such that $\bigcup U^m$ is absorbing in E and if $\mathscr{U}^m = (U_n^m)$ denote the associated basic sequences verifying $U_n^m \subset U_n^{m+1}$ for every m and n, we call $(U_n^m)_{m,n}$ a "nice double sequence" generated by (U^m). W. Robertson defined ultrabarreled spaces as those where every closed basic sequence is all constituted by 0-neighbourhoods. It is obvious that every closed basic sequence generates a nice double sequence (U_n^m) by setting $U_n^m := mU_n$.

Definition. (i) *A topological linear space E is* *-BL *if, whenever E is the increasing union of a sequence (U^m) of closed admissible subsets generating a nice double sequence, than there is a natural number p such that U^p is a* 0*-neighbourhood in E.* (ii) *E is* *-SB *if, whenever there is a sequence (U^m) of closed (in their linear hull) admissible subsets of E such that $\bigcup U^m$ is absorbing in E and such that $([U^m])$ is increasing, than there exists a natural number p such that the closure (in E) of U^p is a* 0*-neighbourhood in E.*

Similarly, *-TB and *-UBL spaces can be defined and it is possible to show that the above theorems hold also in the non-locally-convex case, see [46] for the study of the spaces from Definition 41. Ultrabarreled locally convex spaces form a strict subclass of barreled spaces as we shall see later.

Now we shall classify (LF)-spaces in terms of the Amemiya-Komura's property and related definitions. First, non-metrizable (LF)-spaces distinguish between quasi-Baire and BL spaces and between barreled and quasi-Baire spaces:

(i) Let E be the strong dual of a Fréchet-Montel space defined by continuous norms. Then $E = \operatorname{ind}(E_n, U_n)$ with (E_n, U_n) a Banach space, $n = 1, 2, \ldots$. Since E is not metrizable it is not BL. E is a quasi-Baire space; indeed, suppose E is not quasi-Baire. Then E can be covered by an increasing sequence of proper closed subspaces (H_n). E_1 can be covered by the sequence $(E_1 \cap H_n)$ of closed subspaces in (E_1, U_1). Since (E_1, U_1) is a Baire space there is an index p such that H_p contains E_1. But E_1 is dense in E and thus H_p coincides with E, a contradiction.

(ii) A strict (LF)-space is barreled but not quasi-Baire.

Saxon-Narayanaswami (preprint) give the following classification of (LF)-spaces: A space (E, t) which is an (LF)-space is said to be of type (i) if it satisfies the condition (i) below; $i = 1, 2, 3$:

(1) (E, t) has a defining sequence none of whose members is dense in (E, t).

(2) (E, t) is non metrizable and has a defining sequence each of whose members is dense in (E, t).

(3) (E, t) is metrizable.

From what has been said above it is clear that $(LF)_1$-spaces are precisely those (LF)-spaces which distinguish between barreled and quasi-Baire spaces; $(LF)_2$-spaces are precisely those (LF)-spaces which distinguish between quasi-Baire and BL spaces; $(LF)_3$-spaces are precisely those (LF)-spaces which distinguish between BL and SB spaces: indeed, Amemiya-Komura's result implies that every metrizable (LF)-space is BL, but an obvious application of Pták's closed graph theorem shows that such a space is not SB (a Fréchet space cannot be barreled for a coarser topology). It is easy to see that an (LF)-space is metrizable iff it i sBL.

The following two results show that, in a sense, metrizable (LF)-spaces are the only possible examples distinguishing between BL and SB spaces (see [90] and [63]).

Theorem 43.1. *Let E be a Banach space and let F be a barreled dense subspace of E which is not suprabarreled. Then there is a proper subspace G of E containing F which is an* (LF)*-space for a topology t finer than the topology induced by E.*

Theorem 43.2. *Let E be a non complete metrizable* (LF)*-space and let F be a dense subspace of E which is barreled. Then F is not suprabarreled.*

Sketch of the proof of Theorem 43.1: Take an increasing sequence (F_n) of subspace of F covering F and such that F_n is not barreled, $n = 1, 2, \ldots$. There is a bounded barrel T_n in F_n which is not a 0-neighbourhood in F_n. Let U_n be its closure in E. Setting $V_n^p := U_p \cap U_{p+1} \cap \ldots \cap U_n$ with $p, n = 1, 2, \ldots$ and $n \geqq p$ and G_p for all those elements of E absorbed by V_n^p for all n, it follows that G_p is a linear subspace of E containing F_p with $G_p \subset G_{p+1}$. Set $G := \bigcup(G_p\colon p = 1, 2, \ldots)$. Every G_p can be endowed with a Fréchet topology t_p such that $(G_p, t_p) = \operatorname{proj}(EV_n^p\colon n = p, p + 1, \ldots)$. We define $(G, t) := \operatorname{ind}(G_p, t_p)$. If G coincidences with E, we apply Grothendieck's closed graph theorem to conclude that E can be continuously imbedded in G_q for a certain q and thus $V_q^q = U_q$ ia s a 0-neighbourhood in F_q, a contradiction. Q.E.D.

In $E = l^2$, there is a proper dense subspace $G = \operatorname{ind}(G_n, t_n)$ with (G_n, t_n) being Fréchet spaces with a basis of 0-neighboorhoods which are closed in E. There is a point x in E and not in G such that $H := [G \cup \{x\}]$ is not an (LF)-space (see [79]). But it is possible to construct a topology s on H such that (H, s) is an (LF)-space, s being finer than the topology of E. Indeed, since $G = \bigcup(G_n\colon n = 1, 2, \ldots)$ and since G_n is not barreled, there are bounded barrels T_n in G_n which are not 0-neighbourhoods in G_n but 0-neighbourhoods in (G_n, t_n). Therefore there are 0-neighbourhoods U_n in (G_n, t_n) which are closed in E and contained in T_n. Clearly U_n is not a 0-neighbourhood in G_n. Now we repeat the construction in the proof of Theorem 43.1 applied to $U_n + \{ax\colon |a| \leqq 1\}$ to obtain the required topology s.

Now we shall justify that there are plenty of metrizable (non-complete) (LF)-spaces. We mentioned that Smoljanov constructred such an example. Earlier, Grothendieck constructed also such an example (see Proposition 44). If E is a non-normable Fréchet space with no continuous norms, E contains K^N complemented and K^N contains Smoljanov's (LF)-space as a proper dense subspace. Now it is easy to construct an (LF)-space in the space E. Just to illustrate the techniques introduced by Valdivia in [74] we give the only full proof of this article (which is due to Valdivia and the author) which shows, in particular, that every non-normable Fréchet space can be considered as the completion of a metrizable non-complete (LF)-space. The same result can be obtained using the techniques of the three-space-problem situation (see [91]).

Let $E[\mathscr{U}]$ be an infinite-dimensional Fréchet space, F a closed subspace of $E[\mathscr{U}]$ and $T: E \to E/F$ the canonical surjection. $E[\mathscr{U}]/F$ denotes the quotient space E/F endowed with the quotient topology $\mathscr{U}^\wedge$. A Fréchet space $E[\mathscr{U}]$ has the property $(*)$ if:

There is a closed subspace F of $E[\mathscr{U}]$ such that there is a proper dense subspace M in $E[\mathscr{U}]/F$, which is an (LF)-space, i.e., $M[\mathscr{U}^\wedge] = \operatorname{ind} M_n[\mathscr{U}_n]$, satisfying i) for every n, there is a decreasing fundamental system of 0-neighbourhoods $\mathscr{B}_n = (V_p^n: p = 1, 2, \ldots)$ in $M_n[\mathscr{U}_n]$ which are closed in $M[\mathscr{U}^\wedge]$; (ii) for every n, M_n is dense in $M_{n+1}[U_{n+1}]$; (iii) for every n, if $V \in \mathscr{B}_n$, then $V^{-M_{n+p}[\mathscr{U}_{n+p}]}$ coincides with $V^{-M_{n+p}[\mathscr{U}^\wedge]}$, $p = 1, 2, \ldots$; (iv) for every n, if x is a vector not belonging to M_n, there is a 0-neighbourhood $V \in \mathscr{B}_n$ such that x does not belong to $[V^-]$, the closure being taken in $E[\mathscr{U}]/F$.

Proposition 44. *Every non-normable Fréchet space has the property* $(*)$.

Proof. We use a contruction method due to Grothendieck [24] (see also [31], p. 195, Problem D). The spaces ω and s (nuclear Fréchet space of the rapidly decreasing sequences) are now the "bricks" instead of the spaces l^p and l^q, $1 < q < p < \infty$, respectively. We consider the subspace of ω^N, M_n, of all double sequences $x = (x_{ij})$ with

$$p_{n,r,k}(x) = \sup\{|x_{ij}|: i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, r\} \vee$$
$$\vee \{\sum_{i=n+1}^{n+r} (\sum_{j=1}^{\infty} j^k |x_{ij}|)\}$$

finite for $r = 1, 2, \ldots$, $k = 0, 1, 2, \ldots$. The sequence (M_n) is increasing and $M = \bigcup(M_n: n = 1, 2, \ldots)$ is dense in ω^N. Let $\mathscr{U}_n$ be the topology on M_n defined by the family of seminorms $(p_{n,r,k}: r = 1, 2, \ldots,\ k = 0, 1, 2, \ldots)$. $M_n[\mathscr{U}_n]$ is isomorphic to $\omega \times \omega \times \ldots \times \omega \times s^N$ and therefore a Fréchet space. Moreover, $\mathscr{U}_{n+1}/M_n \leqq \mathscr{U}_n$. Then $M = \underrightarrow{\operatorname{ind}}\, M_n[\mathscr{U}_n]$ (see the reference above or [10]). Clearly (i), (ii), (iii) are satisfied. Given a natural number n, let x be a vector of ω^N which is not in M_n. Then there are natural numbers $r_0 > n$ and k_0 such that $\sum_{j=1}^{\infty} j^{k_0}|x_{r_0 j}| = \infty$. We set $V = \{y = (y_{ij}): p_{n,r_0,k_0}(y) \leqq 1\}$. Every element y of V as well as the elements y of the closure V^- of V in ω^N satisfy $\sum_{j=1}^{\infty} j^{k_0}|y_{r_0 j}| \leqq 1$ and thus x does not belong to $[V^-]$. Since ω^N is isomorphic to ω, we have a proper dense subspace of ω which is an (LF)-space with the desired properties. According to [21] (see also [33] p. 435) every non-Banach Fréchet space has a quotient isomorphic to ω. The conclusion follows. Q.E.D.

Lemma 45. *Let $E[\mathscr{U}]$ be an infinite-dimensional Fréchet space and let F be a closed subspace of $E[\mathscr{U}]$. Let G be a dense subspace of $E[\mathscr{U}]$ containing F such that there is a locally convex topology $\mathscr{T}$. finer than the topology induced by $\mathscr{U}$, for which $G[\mathscr{T}]$ is an* (LF)*-space. If $T: E \to E/F$ denotes the canonical surjection and if $T(G)$, as subspace of $E[\mathscr{U}]/F$, is an* (LF)*-space, then $G[\mathscr{U}]$ is barreled.*

Proof. We set $\mathscr{U}^\wedge$ to denote the topology of $E[\mathscr{U}]/F$. Clearly, $G[\mathscr{T}]/F$ is an (LF)-space. The identity $I: T(G)\,[\mathscr{U}^\wedge] \to G[\mathscr{T}]/F$ has closed graph and therefore, since $T(G)\,[\mathscr{U}^\wedge]$ is also an (LF)-space, we apply Grothendieck's closed graph theorem to conclude that both topologies coincide on $T(G)$. Let V be a barrel in G and let $\mathscr{S}$ be the topology on G obtained by adjoining the barrel V as a 0-neighbourhood to the topology on G induced by $\mathscr{U}$. $\mathscr{S}$ is a metrizable topology finer than the topology induced by $\mathscr{U}$ and coarser than $\mathscr{T}$ and therefore the quotient topology of $\mathscr{S}$ coincides on $T(G)$ with the topology induced by $\mathscr{U}^\wedge$. Applying [33], § 18.4.(4), $G[\mathscr{S}]^\wedge$ is a subspace of E. We shall see that $G[\mathscr{S}]^\wedge$ coincides with E algebraically and therefore coincides with $E[\mathscr{U}]$ topologically, by application of Banach's homomorphism theorem. Then V is a 0-neighbourhood in $G[\mathscr{U}]$. Indeed, let x be a vector in E which is not in $G[\mathscr{S}]^\wedge$. Then $T(x)$ is the limit in $E[\mathscr{U}]/F$ of a Cauchy sequence (z_n) in $T(G)\,[\mathscr{U}]$ and therefore there is a Cauchy sequence (x_n) in $G[\mathscr{S}]$ with $T(x_n) = z_n$, $n = 1, 2, \ldots$. The sequence (x_n) converges to a certain y in $G[\mathscr{S}]$ and, moreover, $T(y) = T(x)$. Thus $x - y$ belongs to G and we arrive at a contradiction. Q.E.D.

Theorem 46. *Let $E[\mathscr{U}]$ be an infinite-dimensional Fréchet space satisfying condition* $(*)$*. Then there is a dense proper subspace G of $E[\mathscr{U}]$ which is an* (LF)*-space.*

Proof. There is a closed subspace F of $E[\mathscr{U}]$ such that there is a proper dense subspace M of $E[\mathscr{U}]/F$ which is an (LF)-space, i.e., $M[\mathscr{U}^\wedge] = \operatorname{ind} M_n[\mathscr{U}_n]$, with (i), (ii), (iii) and (iv). We write $G_n = T^{-1}(M_n)$, $n = 1, 2, \ldots$, and $G = \bigcup(G_n: n = 1, 2, \ldots)$. Clearly G is dense in $E[\mathscr{U}]$. Let $(R_p: p = 1, 2, \ldots)$ be a decreasing fundamental system of closed absolutely convex 0-neighbourhoods in $E[\mathscr{U}]$ and set $W_p^n = T^{-1}(V_p^n) \cap R_p$ for $n, p = 1, 2, \ldots$. For fixed n, $(W_p^n: p = 1, 2, \ldots)$ is a fundamental system of absolutely convex 0-neighbourhoods for a locally convex metrizable topology $\mathscr{T}_n$ on G_n, which are closed in $G_n[\mathscr{U}]$. Clearly $\mathscr{T}_n$ is finer than the topology induced by $\mathscr{U}$. $G_n[\mathscr{T}_n]$ is a Fréchet space. Indeed, let (x_s) be a Cauchy sequence in $G_n[\mathscr{T}_n]$ which obviously converges to a certain x in $E[\mathscr{U}]$. We fix p to obtain a natural number s_0 such that $x_m - x_h$ belongs to W_p^n if $m > h \geqq s_0$ and therefore $x - x_{s_0}$ belongs to W_p^{n-}, the closure being taken in $E[\mathscr{U}]$. Thus x belongs to $[W_p^{n-}]$. Then $T(x)$ is in $[V_p^{n-}]$, $p = 1, 2, \ldots$. We apply (iv) to obtain that $T(x)$ belongs to M_n and thus x belongs to G_n. Now apply [4], § 18.4.(4). Clealry $\mathscr{T}_{n+1}/G_n \leqq \mathscr{T}_n$. Let $\mathscr{T}$ be the topology on G, finer than the topology induced by $\mathscr{U}$, such that $G[\mathscr{T}] = \underrightarrow{\operatorname{ind}}\, G_n[\mathscr{T}_n]$. By Lemma 45, $G[\mathscr{U}]$ is barreled. $G^\wedge$ and H stand for the topological duals of $G[\mathscr{U}]$ and $G[\mathscr{T}]$ and $\circ$ and $*$ for the polars taken in G and H. We shall see that $G[\mathscr{U}] = \underrightarrow{\operatorname{ind}}\, G_n[\mathscr{T}_n]$. Let L be an absolutely convex 0-neighbourhood in $G[\mathscr{T}]$, i.e., there is a sequence of natural numbers $p_1 < p_2 < \ldots < p_n < \ldots$ such that $(1/2^2 L) \cap G_n \supset W^n_{p_n}$, $n = 1, 2, \ldots$. We set $S_n = T^{-1}(V^n_{p_n})$, $W_n = W^n_{p_n}$ and $Z_n = R_{p_n}$, $n = 1, 2, \ldots$. We fix n. First we show that $W^{-G_{n+1}[\mathscr{U}]}$ coincides with $W_n^{-G_{n+1}[\mathscr{T}_{n+1}]}$. Indeed, let T_n be the restriction of T to G_n. Since $T_n\colon G_n[\mathscr{T}_n] \to M_n[\mathscr{U}_n]$, $T_n\colon G_n[\mathscr{U}] \to M_n[\mathscr{U}^\wedge]$ are homomorphisms, we have

$$T^{-1}((V^n_{p_n})^{-M_{n+1}[\mathscr{U}_{n+1}]}) = (T^{-1}(V^n_{p_n}))^{-G_{n+1}[\mathscr{T}_{n+1}]} = (S_n)^{-G_{n+1}[\mathscr{T}_{n+1}]}$$

$$T^{-1}((V^n_{p_n})^{-M_{n+1}[\mathscr{U}^\wedge]}) = (T^{-1}(V^n_{p_n}))^{-G_{n+1}[\mathscr{U}]} = (S_n)^{-G_{n+1}[\mathscr{U}]}.$$

After applying (iii) the lefthand sides of the equalities coincide and therefore $(S_n)^{-G_{n+1}[\mathscr{T}_{n+1}]} = (S_n)^{-G_{n+1}[\mathscr{U}]}$. Now we show that $(Z_n \cap G_n)^{-G_{n+1}[\mathscr{T}_{n+1}]} = (Z_n \cap G_n)^{-G_{n+1}[\mathscr{U}]}$. Let x be a point of $[Z_n \cap G_n]^{-G_{n+1}[\mathscr{U}]}$. There is t_0 such that, if $t_1 \leqq t < 1$, the point tx is interior to the set $(Z_n \cap G_n)^{-G_{n+1}[\mathscr{U}]}$ for the topology $\mathscr{T}_{n+1}$ and, according to (ii) tx belongs to the closure of G_n in $G_{n+1}[\mathscr{T}_{n+1}]$; thus there is a sequence (x_s) in G_n converging to tx in $G_{n+1}[\mathscr{T}_{n+1}]$. There is a natural number s_0 such that x_s belongs to $Z_n \cap G_n$ for $s \geqq s_0$ and therefore tx belongs to $(Z_n \cap G_n)^{-G_{n+1}[\mathscr{T}_{n+1}]}$. Making t tend to 1, we have that x belongs to $(Z_n \cap G_n)^{-G_{n+1}[\mathscr{T}_{n+1}]}$. Since Z_n is a 0-neighbourhood in $E[\mathscr{U}]$ it follows that

$$(S_n \cap Z_n)^{-G_{n+1}[\mathscr{T}_{n+1}]} = (S_n)^{-G_{n+1}[\mathscr{T}_{n+1}]} \cap ((Z_n \cap G_n))^{-G_{n+1}[\mathscr{T}_{n+1}]} =$$
$$= (S_n)^{-G_{n+1}[\mathscr{U}]} \cap ((Z_n \cap G_n))^{-G_{n+1}[\mathscr{U}]} = ((S_n \cap Z_n))^{-G_{n+1}[\mathscr{U}]}.$$

Thus $(W_n)^{-G_{n+1}[\mathscr{T}_{n+1}]} = (W_n)^{-G_{n+1}[\mathscr{U}]}$. Having in mind (iii) and making repeated use of the argument above we prove that

(1) $$(W_n)^{-G_{n+p}[\mathscr{T}_{n+p}]} = (W_n)^{-G_{n+p}[\mathscr{U}]}, \quad p = 1, 2, \ldots.$$

Now we prove that

(2) $$(W_n)^- \subset \tfrac{1}{2}L.$$

Indeed, if $x \in (W_n)^-$ there is a natural number $p > n$ such that $x \in G_p$ and therefore $x \in (W_n)^- \cap G_p = (W_n)^{-G_p[\mathscr{U}]}$ which coincides with $(W_n)^{-G_p[\mathscr{T}_p]}$ because of (1). Since

$$(W_n)^{-G_p[\mathscr{T}_p]} \subset W_n + W_p \subset (1/2^2)\, L + (1/2^2)\, L = \tfrac{1}{2}L,$$

we have $x \in \frac{1}{2}L$.

The following equality is holds:

(3) $$(\Gamma(W_1^- \cup \ldots \cup W_n^-))^- = (\Gamma(W_1^- \cup \ldots \cup W_n^-))^{-G[\mathscr{T}]}, \quad n = 1, 2, \ldots.$$

Indeed, fix n and let $1 \leqq p \leqq n$. Clearly $W_p^- = (S_p \cap Z_p)^- = S_p^- \cap ((Z_p \cap G))^- = S_p^- \cap Z_p$ since Z_p is a closed 0-neighbourhood of $E[\mathscr{U}]$. The mapping $T\colon G[\mathscr{U}] \to M[\mathscr{U}^\wedge]$ is a homomorphism. Let $T'\colon M' \to G'$ be its transposed mapping. Clearly T' is injective and $T'(M')$ is $\sigma(G', G)$-closed and $T'(M')$ coincides with $F^\perp$ (orthogonal in G'). Since $G[\mathscr{T}]/F$ and $G/F[\mathscr{U}^\wedge]$ are (LF)-spaces, we apply Grothendieck's closed graph theorem to conclude that they coincide topologically. Therefore $T\colon G[\mathscr{T}] \to$

$\to M[\mathscr{U}^\wedge]$ is also a homomorphism and thus $T'(M') = F^\perp$ is $\sigma(H, G)$-closed. Since $(S_p^-)^*$ is the $\sigma(H, G)$closure of $(S_p^-)^\circ$ and since $(S_p^-)^\circ$ is contained in $F^\perp$ we have that $(S_p^-)^* = (S_p^-)^\circ \subset G'$. Clearly $Z_p^* = Z_p^0$, since Z_p is a 0-neighbourhood in $E[\mathscr{U}]$. Clearly

$$\{\bigcap_{p=1}^{n}(\Gamma((S_p^-)^\circ \cup Z_p^0))^{-\sigma(G',G)}\}^\# = (\Gamma(W_1^- \cup \ldots \cup W_n^-))^- ,$$

$$\{\bigcap_{p=1}^{n}(\Gamma((S_p^-)^* \cup Z_p^*))^{-\sigma(H,G)}\}^\# = (\Gamma(W_1^- \cup \ldots \cup W_n^-))^{-G[\mathscr{T}]}$$

$\#$ being the polar taken in $G[\mathscr{U}]$.

Since the lefthand sides of both equalities coincide, (3) follows.

By (2), $\Gamma(W_1^- \cup \ldots \cup W_n^-) \subset \frac{1}{2}L$, $n = 1, 2, \ldots$. Then $(\Gamma(W_1^- \cup \ldots \cup W_n^-))^{-G[\mathscr{T}]} \subset L$, $n = 1, 2, \ldots$. Then $(\Gamma(W_1^- \cup \ldots \cup W_n^-))^- \subset L$, because of (3). Clearly $G = \bigcup_{n=1}^{\infty} n(\Gamma(W_1^- \cup \ldots \cup W_n^-))^-$. $G[\mathscr{U}]$ is barreled and metrizable and by Amemiya-Kōmoura's theorem there is an index m such that $(\Gamma(W_1^- \cup \ldots \cup W_m^-))^-$ is a 0-neighbourhood in $G[\mathscr{U}]$ and thus L is a 0-neighbourhood in $G[\mathscr{U}]$. The proof is complete. Q. E. D.

Corollary 46.1. *Every non-Banach Fréchet space contains a proper dense subspace which is an* (LF)-*space.*

Theorem 47. *Let $E[\mathscr{U}]$ be a non-Banach reflexive (Montel, Schwartz, nuclear) Fréchet space. Then there is a proper dense subspace G which is an* (LF)-*space i.e. $G[\mathscr{U}] = \underrightarrow{\operatorname{ind}}\, G_n[\mathscr{T}_n]$, such that $G_n[\mathscr{T}_n]$ is a reflexive (Montel, Schwartz, nuclear) Fréchet space, $n = 1, 2, \ldots$.*

Proof. In any case $E[\mathscr{U}]$ satisfies condition $(*)$ and therefore there is a proper dense subspace G of $E[\mathscr{U}]$ which is an (LF)-space, i.e. $G[\mathscr{U}] = \underrightarrow{\operatorname{ind}}\, G_n[\mathscr{T}_n]$. We mantain the notations introduced in the proof of Theorem 46.

In what follows we fix a natural number n. The topological dual G_n' of $G_n[\mathscr{T}_n]$ can be identified with the linear hull of $E' \cup T'(M_n')$, M_n' being the topological dual of $M_n[\mathscr{U}_n]$. We shall show that if $E[\mathscr{U}]$ is a non-Banach reflexive (Montel, Schwartz, nuclear) Fréchet space, then $G_n[\mathscr{T}_n]$ is a reflexive (Montel, Schwartz, nuclear) Fréchet space.

(1) $E[\mathscr{U}]$ is a non-Banach reflexive space. Let A be a bounded closed absolutely convex subset of $G_n[\mathscr{T}_n]$ and let (x_α) be a net in A. Since $E[\mathscr{U}]$ is reflexive, there is a subnet (y_α) of (x_α) converging to a certain point z in $E[\sigma(E, E')]$. The net $(T(y_\alpha))$ converges to $T(z)$ in ω and is contained in the bounded set $T(A)$. Since $M_n[\mathscr{U}_n]$ is reflexive, there is a subnet (z_α) of (y_α) such that $(T(z_\alpha))$ converges to a certain point v in $M_n[\sigma(M_n, M_n')]$. Clearly $T(z) = v$ and thus z belongs to G_n. For every u in M_n' we have that $\langle u, T(z_\alpha)\rangle = \langle T'(u), z_\alpha\rangle$ and, since $G_n' = [E' \cup T'(M_n')]$, the net (z_α) converges to z in $G_n[\sigma(G_n, G_n')]$. Since the absolutely convex set A is closed in $G_n[\mathscr{T}_n]$, z belongs to A.

(2) $E[\mathscr{U}]$ is a Fréchet Montel space. Let A be a bounded closed subset of $G_n[\mathscr{T}_n]$. Since $E[\mathscr{U}]$ is Montel, A is relatively compact in $E[\mathscr{U}]$. Given a sequence (x_p) in A there is a subsequence (y_p) of (x_p) such that (y_p) converges to a certain z in $E[\mathscr{U}]$. The sequence $(T(y_p))$ converges to $T(z)$ in ω. Since $M_n[\mathscr{U}_n]$ is Montel there is a subsequence (z_p) of (y_p) such that $(T(z_p))$ converges to a certain v in $M_n[\mathscr{U}_n]$ which coincides with $T(z)$. Thus z belongs to G_n. Fixed a natural number r. Then there is a natural number $p_0(r)$ such that $z_p - z \in Z_r$ if $p \geqq p_0(r)$. There is also a natural number $p_1(r)$ such that $T(z_p) - T(z) \in V_r^n$ if $p \geqq p_1(r)$. Thus $z_p - z \in W_r^n$ if $p \geqq$ $\geqq p_1(r)$ and therefore (z_p) converges to z in $G_n[\mathscr{T}_n]$. Since A is closed, the conclusion follows.

(3) $E[\mathscr{U}]$ is a Fréchet Schwartz space. We write H to denote the topological dual of $G_n[\mathscr{T}_n]$. Let W_p^n be a 0-neighbourhood in $G_n[\mathscr{T}_n]$ with $W_p^n = S_p^n \cap Z_p$, $S_p^n =$ $= T^{-1}(V_p^n)$. Since $E[\mathscr{U}]$ is Schwartz there is a natural number $h > p$ such that Z_p^o is compact in $E'_{Z^o_p} = H_{Z^o_r}$ if $r \geqq h$. Since $M_n[\mathscr{U}_n]$ is Schwartz, there is a natural number $m > h$ such that V_p^{n0} is compact in $(M_n')_{V_m{}^{n0}}$ which is isomorphic to $H_{S_m{}^{n0}}$. Thus $\Gamma(Z_0^p \cup S_p^{n0})$ is compact in $H_{W_m{}^{n0}}$ and thus W_p^{n0} is compact in $H_{W_m{}^{n0}}$.

Similar ideas can be used to prove (4).

Roelcke showed the existence of dense subspaces if l^p, $1 \leqq p < \infty$, which are (LF)-spaces. The ideas used above can be applied to construct such subspaces in every Banach space having l^p as a quotient (e.g., l^∞).

Saxon-Narayanaswami (preprint) gave the following result: "Let (F, t) be a Banach space with a sequence (P_n) of orthogonal projections such that each of the subspaces $P_n(F)$ has an infinite-dimensional separable Hausdorff quotient. Then F contains a proper dense subspace which is an (LF)-space." Then all Banach spaces with an unconditional basis have a proper dense subspace which is an (LF)-space. The same conclusion follows for $C([0, 1])$, $L^p([0, 1])$ and l^∞.

It is very easy to show that, for a Banach space, the existence of a proper dense subspace which is not barreled is equivalent to the existence of a proper dense subspace which is not SB. According to Theorem 7, should every Banach space have a proper dense subspace which is an (LF)-space, then this would imply the existence of an infinite-dimensional separable Hausdorff quotient, which is a wellknown open problem in Banach space theory.

Whether this problem is equivalent to the existence of proper dense subspaces which are (LF)-spaces, is unknown to us.

Metrizable (LF)-spaces was never sequentially retractive [22], since it is always possible to construct a sequence converging to the origin and such that it is not localized in any of the spaces of the defining sequence: suppose $E = \text{ind } E_n$ and select points y_n in E_{n+1} but not in E_n, $n = 1, 2, \ldots$, and scalars (a_n) such that $(a_n y_n)$ converge to the origin. This sequence is bounded but not localized.

A classification of (LF)-spaces is outlined (see [63] for a full discussion) which is useful to distinguish between barreled, quasi-Baire and BL spaces. In the proof of

Proposition 44 it is showed that K^N contains plenty of proper dense subspaces which are (LF)-spaces and therefore BL spaces which are not SB (see also [10]). The following result can be found in [47].

Theorem 48. *Every countable product of infinite-dimensional Fréchet spaces contains a proper dense subspace which is* SB *but not* TB.

Set (e_n) for the canonical unit vectors in K^N and consider the subspace E of K^N defined as the linear hull of $K^{(N)}$ union with $\{(x_n): x_n = 0$ save for a sequence of indices (n_k) with $\lim n_k/k = +\infty\}$. Clearly E is the union of a countable family of closed hyperplanes and thus E is not UBL. To show that E is TB it is enough to prove that if E is covered by a sequence (E_n) of subspaces, then there is an index p such that E_p is barreled. Indeed, suppose the assertion is true and suppose E is covered by a sequence (H_n) of subspaces of E. Considering their closures (H_n^-) in E we claim the existence of an index p such that H_p^- is of finite codimension in E: if this is not true, for every index p select an infinity of linearly independent vectors $(y_1^p, y_2^p, \dots)$ which are not in H_p^- and denote by G_p the linear hull of $H_p^- \cup (y_1^p, y_2^p, \dots)$. Since E is covered by (G_p) there is an index p_0 such that G_{p_0} is barreled by assumption. According to Cor. 34.1, G_{p_0} carries the inductive topology of the sequence $([H_{p_0}^- \cup (y_1^{p_0}, \dots, y_m^{p_0})])_m$ and thus G_{p_0} contains a copy of $K^{(N)}$ which is in contradiction with G_{p_0} being metrizable. If $\mathscr{F}_1$ is the family of subspaces of (H_n) such that their closures in E are of finite codimensions in E and if $\mathscr{F}_2$ denotes $(H_n) \setminus \mathscr{F}_1$, then $\mathscr{F}_1$ has to cover E since $\mathscr{F}_2$ does not. We apply again our assumption and we are through.

If the assumption is not true, let T_n be barrels in E_n which are not 0-neighbourhoods and denote by V_n their closures in E, $n = 1, 2, \dots$. Denote by R_i the linear hull of e_i, $i = 1, 2, \dots$. Suppose that no V_n contains all R_i except a finite number of them. We select a sequence of naturals $(n, 1) < (n, 2) < \dots < (n, s) < \dots$ such that $R_{(n,s)}$ is not contained in V_n, $n, s = 1, 2, \dots$. Let $1 < q_1 < q_2 < \dots$ be a sequence of null density, select $(1, p_1) \geqq 1$ and set $p_1' = p_1$. Suppose we have constructed $(1, p_1^1)$, $(2, p_1^2), (1, p_2^2) \dots (n, p_1^n), (n-1, p_2^n) \dots (1, p_n^n)$ such that a finite subsequence of (q_n), $1 < r_1 < r_2 < \dots < r_\beta$ has been selected satisfying $1 \leqq (1, p_1^1) \leqq r_1 \leqq (2, p_1^2) \leqq \leqq r_2 \leqq (1, p_2^2) \leqq \dots \leqq r_\alpha \leqq (n, p_1^n) \leqq r_{\alpha+1} \leqq \dots \leqq (1, p_n^n) \leqq r_\beta$. Take an element $(n+1, p_1^{n+1})$ in the sequence $(n+1, 1), (n+1, 2), \dots$ such that $(n+1, p_1^{n+1}) \geqq r_\beta$ and select in (q_n) the first element r_γ such that $r_\gamma \geqq (n+1, p_1^{n+1})$. In the sequence $(n, 1)$, $(n, 2), \dots$ select an element $(n, p_2^{n+1}) \geqq r_\gamma$ and continue in this way until $(1, p_{n+1}^{n+1})$ is selected. Let F be the Fréchet space of all those sequences of E that are null except for the positions by the extracted sequence. F is clearly a Fréchet space and $F = \bigcup(mV_n \cap F: m, n = 1, 2, \dots)$. Then there is a natural n_0 such that $V_{n_0} \cap F$ is a 0-neighbourhood in F. Therefore there is a 0-neighbourhood W in E such that $W \cap F \subset V_{n_0}$ and there is an index p such that R_i is contained in $W \cap F$ for $i \geqq p$. Thus V_{n_0} contains all $R_{(n_0, p_1^{n_0})}, \dots, R_{(n_0, p_{k+1}^{n_0+k})}, \dots$ except a finite number of them and this is a contradiction. Thus there is an index q such that V_q contains $(R_i: i \geqq p)$

and therefore $V_q \supset (\oplus(R_i\colon i \geqq p))^-$, the closure taken in E, which is a closed subspace of E of finite codimension in E. Then T_q is a barrel in E_q which contains a closed subspace of finite codimension and thus a 0-neighbourhood.

Therefore K^N contains proper dense subspaces which are SB but not TB and TB but not UBL. K^N contains also proper dense subspaces which are UBL but not Baire: indeed, $K^N \otimes_\pi K^N$ is a proper dense subspace of K^N which is UBL (see Theorem 66) but not Baire (see Theorem 67). According to a result of Eidelheit [21], we can prove

Theorem 49. *Every non-normable Fréchet space contains proper dense subspaces which are* (i) BL *but not* SB (ii) SB *but not* TB (iii) TB *but not* UBL (iv) UBL *but not Baire.*

Proof. There is a closed subspace F of E, E being the non-normable Fréchet space, such that E/F is isomorphic to K^N. According to our previous comments, proper dense subspaces H, L, G, R of K^N can be found such that H is BL but not SB, L is SB but not TB, G is TB but not UBL and R is UBL but not Baire. If $q\colon E \to E/F$ denotes the canonical surjection, we set $H_1 := q^{-1}(H)$, $L_1 := q^{-1}(L)$, $G_1 := q^{-1}(G)$ and $R_1 := q^{-1}(R)$. The triples (H_1, F, H), (L_1, F, L), (G_1, F, G) and (R_1, F, R) constitute a three-space problem. According to Theorem 42, H_1 is BL, L_1 is SB, G_1 is TB and R_1 is UBL. Since SB, TB, UBL and Baire are stable under the formation of quotients, H_1 is not SB, L_1 is not TB, G_1 is not UBL and R_1 is not Baire. Q.E.D.

The space l^p, $0 < p < 1$, endowed with the topology induced by l^1 is an UBL space which is not Baire (see [18]). Since every separable infinite dimensional Banach space is a quotient of l^1 it is natural to ask whether such a space contains always a proper dense subspace which is UBL but not Baire.

Our next example deserves a separate treatment since it is of impotance in the theory of vector measures. Let $\mathscr{A}$ be a σ-algebra of subsets of a set X and let $e(S)$ be the characteristic function of $S \in \mathscr{A}$. If H denotes the family of all characteristic functions over the elements of $\mathscr{A}$ we write $l_0^\infty(X, \mathscr{A})$ to denote the linear hull of H endowed with the norm $\|f\| := \sup\{|f(x)|\colon x \in X\}$. Its strong dual is the space $M(X, \mathscr{A})$ of all bounded finite additive measures on $\mathscr{A}$, i.e. finite additive set functions m on $\mathscr{A}$ such that there exists a constant $M > 0$ such that $|m(S)| \leqq M$ for every S in $\mathscr{A}$ endowed with the norm $\|m\| := |m|(X) = \sup\{\sum_{j=1}^{n} |m(S_j)|\colon (S_1, \ldots, S_n)$ a finite partition of $X\}$. If $\mathscr{A}$ is the σ-algebra of all subsets of the natural numbers N we write l_0^∞ to denote $l_0^\infty(N, \mathscr{P}(N))$.

Grothendieck showed that $l_0^\infty(X, \mathscr{A})$ is of the first category: consider $B_n := \{f \in l_0^\infty(X, A)\colon \operatorname{card}(f(X)) \leqq n\}$ which is closed and balanced and satisfies $B_n + B_n \subset 2B_{n^2}$. Moreover, (B_n) covers $l_0^\infty(X, \mathscr{A})$. Thus there is one of them which is a neighbourhood and therefore there is a natural number k such that $l_0(X, \mathscr{A}) = \bigcup(mB_k\colon m = 1, 2, \ldots)$ which is included in B_k, a contradiction. On the other hand, Dieudonne showed that a subset T of $M(X, \mathscr{A})$ is bounded iff $\sup\{|m(S)|\colon m \in T\} < +\infty$ for every S of $\mathscr{A}$ (see, e.g. [73], p. 131) which means that $l_0^\infty(X, \mathscr{A})$ is barreled.

Iyahen extended some results of [74] and proved that if an ultrabarreled space E is the union of a sequence (B_n) of closed balanced subsets satisfying $B_n + B_n \subset B_{n+1}$ and if its completion is Baire, then some B_n is a neighbourhood. Thus the same reasoning used above shows that $l_0^\infty(X, \mathscr{A})$ is not ultrabarreled. Valdivia [72], refined Dieudonne's result by showing that $l_0^\infty(X, \mathscr{A})$ is even SB, and Arias [6], has shown that it is not TB.

$l_0^\infty(X, \mathscr{A})$ contains $K^{(X)}$ as a closed subspace since $K^{(X)} = l_0^\infty(X, \mathscr{A}) \cap c_0(X)$, which is not barreled [59], and therefore even closed subspaces of normed barreled spaces need not be barreled. Pełczyński showed that $l_0^\infty(X, \mathscr{A})$ contains no infinite dimensional separable barreled subspace for $X = N$ but the result is also true for arbitrary X (see [18]).

Suitable modifications of the proof of the classical result of Schur (see [33], p. 283) show that the topology $\sigma(l^1(X), l_0^\infty(X, \mathscr{A}))$ and the norm topology on $l^1(X)$ have the same convergent sequences. In [7] it is proved that every sequence which is subseries-summable in $l_0^\infty(X, \mathscr{A})$ generates a finitedimensional space which yields that every Banach disc in $(l_0^\infty(X, \mathscr{A}), \sigma(l_0^\infty(X, \mathscr{A}), K^{(X)}))$ is finite-dimensional which in turn implies that the ultrabornological topology associated with $\sigma(l_0^\infty(X, \mathscr{A}), K^{(X)})$ is the finest locally convex topology. Thus $l_0^\infty(X, \mathscr{A})$ is not ultrabornological but clearly barreled and bornological.

The space l_0^∞ can be covered by a sequence of closed hyperplanes [59]: it is enough to show that every B_n can be covered by a sequence of proper closed subspaces (H_k^n) of l_0^∞ which are of finite codimensions in l_0^∞. If (e_n) denote the unit vectors of l_0^∞ define $P_n: l_0^\infty \to l_0^\infty$ by means of $P_n(x) := \sum_{j=1}^{n} x_j e_j$ which is a linear and continuous projection. Fixing n, E_k^n denotes the linear hull of A_k^n, where $A_1^n, A_2^n, \ldots$ denotes a finite enumeration of all subsets of $P_n(H)$ having less than n elements. Setting $H_k^n := P_n^{-1}(E_k^n)$ we are through. According to Theorem 66 and Theorem 68, the space $l_0^\infty \otimes_\pi l^2$ is SB but not TB.

The incidence of the barreledness of the space $l_0^\infty(X, \mathscr{A})$ in the theory of vector measures can be illustrated as follows: the classical theorem of Orlicz-Pettis assures that a set-function $m: \mathscr{A} \to E$, $\mathscr{A}$ being a σ-algebra of parts of a set X and E a Banach space, which is finitely additive and such that $f \circ m$ is a scalar measure for every f in E', is a measure (i.e., countably additive). Relaxing the hypothesis by asking that $f \circ m$ is a scalar measure for f belonging to some dense subspace of $(E', \sigma(E', E))$, Diestel-Faires prove that m is a vector measure under the additional condition that E does not contain l^∞. A functional-analytic proof of this fact can be provided as follows: we assume $\mathscr{A} = \mathscr{P}(N)$ for simplicity. Let g be the mapping which associates with every A in $\mathscr{A}$ its characteristic function $e(A)$. Then m factorizes through l_0^∞ by a linear mapping $T: l_0^\infty \to E$ which by the hypothesis has closed graph. The barreledness of l_0^∞ assures the continuity of T which extends continuously in $h: l^\infty \to E$. Since E does not contain l^∞, a result due to Rosenthal [56] assures that h is weakly compact which in turn assures that $f \circ m$ is a scalar measure for every f

in E'. The Orlicz-Pettis result implies that m is a measure. Since l_0^∞ is SB and, according to Th. 37 the above result can be seen to be true for E being a countable inductive limit of B_r-complete spaces not containing l^∞, since Orlicz-Pettis's theorem and Rosenthal's result hold in the category of locally convex spaces, as was shown by McArthur and Drewnowski [17], respectively. Unfortunately l_0^∞ is not TB and Theorem 38 cannot be used here, but l_0^∞ may belong to some intermediate class of spaces between the TB and SB spaces for which De Wilde's closed graph theorem holds.

For certain Boolean rings $\mathscr{A}$, Molto has shown that $l_0^\infty(X, \mathscr{A})$ is also SB which yields certain uniform boundedness properties for G-valued exhausting additive set functions, G being a commutative topological group, on such Boolean rings [41].

A Boolean characterization of those rings $\mathscr{A}$ for which $l_0(X, \mathscr{A})$ is barreled seems to be unknown. For a ring $\mathscr{A}$, let $S(\mathscr{A}, E)$ be the space of E-valued $\mathscr{A}$-simple functions defined on X endowed with the topology of the uniform convergence, E being a locally convex space. Freniche (preprint) has shown that *$S(\mathscr{A}, E)$ is barreled iff both $l_0(X, \mathscr{A})$ and E are barreled and E is nuclear*, as a consequence of the following result on barreledness of injective tensor products:

Theorem 50. *Let F be a normed barreled space containing a subspace topologically isomorphic to $(K^{(N)}, \|\cdot\|_\infty)$ and let E be a space. $E \otimes_\varepsilon F$ is barreled iff E is barreled and nuclear.*

We shall study the barreledness of the space $c_0(E)$, E being a space. In the Oberwolfach meeting in 1975, K. D. Bierstedt posed the following problem: Is the space $C(K, E)$ of all E-valued continuous functions defined on a Hausdorff compact space, endowed with the compact-open topology, barreled for E barreled? We consider first a particular case. Let $N^\#$ be the one-point compactification of the natural numbers N and consider the linear mapping $T: C(N^\#, E) \to c_0(E)$ defined by means of $T(f)(n) := f(\infty)$ if $n = 1$ and $T(f)(n) := f(n-1) - f(\infty)$ if $n > 1$, $c_0(E)$ being the space of all sequences converging to the origin in E, its topology defined by the system of seminorms $p(f^-) := \sup(p(f^-)(n): f^- \in c_0(E))$ with p varying over the family of seminorms defining the topology of E. T is clearly a topological isomorphism. Following Pietsch, we denote by $l_\pi^1\{E\}$ the linear space of all absolutely summable sequences in E provided with the topology defined by the system of seminorms

$$p_U^-(f^-) := \sum_{n=1}^{\infty} p_U(f^-(n))$$

for f in $l^1\{E\}$ and U being a closed absolutely convex 0-neighbourhood in E. The following result gives information on the dual of $c_0(E)$ and is due to Marquina-Sanz Serna ("Barrelledness conditions on $c_0(E)$", Arch. Math. *31* (1978), 589–596).

Theorem 51. *The following statements are true:*

(1) *$c_0(E)'$ is sequentially dense in $l_\pi^1\{E'_\beta\}$;*

(2) $l^1_\pi\{E'_\beta\}$ *induces on* $c_0(E)'$ *the strong topology.*

To insure the coincidence of $c_0(E)'$ and $l'\{E'_\beta\}$ we recall a property defined by Pietsch: E has property (B) if, for every bounded set M in $l^1_\pi\{E\}$, there is a closed bounded set B in E such that all sequences in M are uniformly absolutely summable with respect to the Minkowski functional p_B. The following result is due to Marquina-Sanz Serna (loc. cit.) and to Mendoza ("A barrelledness criterion for $c_0(E)$", Arch. Math. *40* (1983), 156–158).

Theorem 52. *The following two conditions are equivalent:*
(1) $c_0(E)$ *is barreled (quasi-barreled);*
(2) E *is barreled (quasi-barreled) and* E'_β *has property* (B).

Property (B) is enjoyed by metrizable and dual-metric spaces; not every space has this property as R^R shows. Thus the problem mentioned earlier has a negative solution when taking the space $R^{(R)}$ as E. On the other hand, there are plenty of spaces E for which $c_0(E)$ is barreled: Fréchet spaces, barreled DF spaces, barreled nuclear spaces and also arbitrary products and countable inductive limits of such spaces.

A complete solution to the problem stated earlier was given by Mendoza (("Necessary and sufficient conditions for $C(X, E)$ to be barreled or infrabarreled", S. Stevin *57* (1983)):

Theorem 53. *The following conditions are equivalent:*
(1) $c_0(E)$ *is barreled;*
(2) *there is an infinite Hausdorff compact space* K_0 *such that* $C(K_0, E)$ *is barreled;*
(3) *for every Hausdorff compact topological space* K, $C(K, E)$ *is barreled.*

It would be interesting to study under which conditions on E, $c_0(E)$ is BL, SB, TB, UBL.

By using an intriguing device ("desintegration theorem") which permits to transmit information known for the π-topology, to a coarser tensornorm topology r (due to Chevet-Saphar) which, under the conditions of the next theorem, coincides with the injective topology, Defant-Govaerts ([98] preprint) give the following result.

Theorem 54. *Let* F *be a complete space. The following assertions are equivalent:*
(1) *there is an infinite-dimensional* $\mathscr{L}^\infty$*-space* H *such that* $H \hat{\otimes}_\varepsilon F$ *is barreled;*
(2) $F \hat{\otimes}_\varepsilon F$ *is barreled for every* $\mathscr{L}^\infty$*-space* E;
(3) F *is barreled and* F'_β *has property* (B).

3. BARRELEDNESS AND TOPOLOGICAL TENSOR PRODUCTS

Given two spaces E and F, the inductive topology i is defined as the finest locally convex topology on $E \otimes F$ such that the canonical linear mappings $\{x\} \times F \rightarrow$

$\to E \otimes F, E \times \{y\} \to E \otimes F$ are continuous for every x in E and y in F. A topology t on $E \otimes F$ is compatible if it is finer than the ε-topology and coarser than the i-topology. A tensor topology associates with every pair of spaces (E, F) a c ompatible topology $t(E, F)$ such that the mapping property is satisfied, i.e., for every $D \subset \subset L(E_1, F_1)$ and every $D' \subset L(E_2, F_2)$ which are equicontinuous it follows that $\{T \otimes T': T \in D, T' \in D'\}$ is equicontinuous in $L(E_1 \otimes_t E_2, F_1 \otimes_t F_2)$. The topologies ε, π, i are tensor topologies. Another tensor topology of importance on a tensor product is the bi-hypocontinuous topology β which is the finest locally convex topology on $E \otimes F$ such that the bilinear canonical mapping $E \times F \to E \otimes F$ is hypocontinuous. One has $\varepsilon < \pi < \beta < i$ where $<$ means "coarser than". For every tensor topology t, the spaces E and F are isomorphic to complemented subspaces to $E \otimes_t F$. If $\mathscr{F}$ denotes a class of locally convex spaces which is stable by complemented subspaces and if $E \otimes_t F$ belongs to $\mathscr{F}$ for a pair of spaces E and F, then E and F also belong to the class $\mathscr{F}$. On the other hand, if $\mathscr{A}$ is a class of locally convex spaces stable under the formation of inductive limits (e.g., barreled, quasi-barreled, bornological, ultrabornological spaces) and if E and F are in $\mathscr{A}$, then $E \otimes_i F$ is in $\mathscr{A}$.

We start with studying conditions on E and F which imply coincidence of tensor topologies.

Theorem 55. a) *If E is normed, then $E \otimes_\beta F$ coincides with $E \otimes_\pi F$. Moreover, if F is barreled, then $E \otimes_i F$ coincides with $E \otimes_\pi F$.*

b) If E and F are barreled, then $E \otimes_\beta F$ coincides with $E \otimes_i F$.

c) If E and F are (gDF) spaces, then $E \otimes_\pi F$ coincides with $E \otimes_\beta F$.

d) If E is a metrizable barreled space and F is BL, then $E \otimes_\pi F$ coincides with $E \otimes_i F$.

Result c) is due to Hollstein and Ruess (and to Grothendieck for the (DF)-spaces). Result d) is due to Valdivia [87]. The following result is due to Defant and Govaerts [98], for barreled, quasi-barreled and bornological spaces.

Theorem 56. *Let E and F be spaces. Then $E \otimes_\beta F$ is barreled (quasi-barreled, bornological, $\aleph_0$-barreled, $\aleph_0$-quasi-barreled, quasi-Baire) if and only if E and F satisfy the same property.*

Corollary 56.1. a) *Let E be a normed barreled space and F any space. Then $E \otimes_\pi F$ is barreled ($\aleph_0$-barreled, quasi-Baire) if and only if F is barreled ($\aleph_0$-barreled, quasi-Baire).*

b) *If E is a normed space and F any space, then $E \otimes_\pi F$ is bornological (quasi-barreled, $\aleph_0$-quasi-barreled) if and only if F is bornological (quasi-barreled, $\aleph_0$-quasi-barreled).*

For (gDF)- spaces, one has the following result due to Groethendieck and Ruess (see [28]).

Theorem 57. a) *E and F are* ((DF)-*spaces* (gDF)-*spaces*) *if and only if* $E \otimes_\pi F$ *is a* (DF)-*space* ((gDF)-*space*).

b) *If E and F are* (DF)-*spaces*, $E \otimes_\pi F$ *is bornological* (*quasi-barreled, barreled,* $\aleph_0$-*barreled*) *if and only if E and F are also bornological* (*quasi-barreled, barreled,* $\aleph_0$-*barreled*).

For l^∞-barreledness one has

Theorem 58. *If E is a separable normed barreled space and if F is* l^∞-*barreled, then* $E \otimes_\pi F$ *is* l^∞-*barreled.*

In [11] one can find an example of a Banach space E and an l^∞-barreled space F such that $E \otimes_\pi F$ is not l^∞-barreled.

Directly from Theorem 55 we have

Theorem 59. a) *Let E be a normed ultrabornological space and F an ultrabornological space. Then* $E \otimes_\pi F$ *is ultrabornological.*

b) *If E and F are ultrabornological* (DF)-*spaces, then* $E \otimes_\pi F$ *is ultrabonological.*

c) *If E is an ultrabornological metrizable space and F an ultrabornological* BL *space, then* $E \otimes_\pi F$ *is ultrabornological.*

Corollary 59.1. *If E and F are Fréchet spaces, then* $E \otimes_\pi F$ *is ultrabornological.*

Our next three results relate the stability of certain locally convex properties with respect to projective tensor products to the coincidence of the projective topology with other tensor topologies (see [98], [96], [96]).

Theorem 60. *Let E and F be spaces such that E has the bounded approximation property* (b.a.p.). *Then* $E \otimes_\pi F$ *is barreled if and only if E and F are barreled and* $E \otimes_\pi F$ *coincides with* $E \otimes_\beta F$.

Observe that if E is a non-normable space with b.a.p., then $E \otimes_\pi E'_\beta$ is not barreled. For $E = K^N$ one gets Hòllstein's result [26]. (On the other hand, observe that the fact that $K^N \otimes_\pi K^{(N)}$ is not barreled can be deduced easily by application of Pták's closed graph theorem.) Its completion is isomorphic to $(K^{(N)})^N$ which is obviously barreled.

Theorem 61. *If E has* b.a.p. *and* $E \otimes_\varepsilon F$ *is barreled, then E and F are barreled and* $E \otimes_\varepsilon F$ *coincides with* $E \otimes_\pi F$.

The third result concerns ultrabornological spaces and requires two lemmas due to Grothendieck and Valdivia [87] respectively.

Lemma 62. *If* $R_n(E, F)$ *denotes the set* $\{z \in E \otimes F$: *rank* $(z) \leqq n\}$, *then* $R_n(E, F)$ *is closed in* $E \otimes_\varepsilon F$.

Lemma 63. *If E and F are spaces and if B is a Banach disc in $E \otimes_\varepsilon F$ (or $E \otimes_\pi F$) then either there is a finite-dimensional subspace E_0 of E such that $(E \otimes_\pi F)_B$ is contained in $E_0 \otimes F$, or there is a finite-dimensional subspace F_0 of F such that $(E \otimes_\pi F)_B$ is contained in $E \otimes F_0$.*

Theorem 64. a) *If $E \otimes_\pi F$ $(E \otimes_\varepsilon F)$ is ultrabornological, then E and F are ultrabornological and $E \otimes_\pi F(E \otimes_\varepsilon F)$ coincides with $E \otimes_i F$.*

b) *For a space E, $E \otimes_\pi E'_b$ is ultrabornological if and only if E is a normed ultrabornological spaces.*

The aforementioned result of Hollstein shows that the barreledness of the projective tensor product of a metrizable space E and a space F is not as simple as the case when E is a normed space. Theorems 55d) and 56 provide a positive result: if E is a metrizable barreled space and F is BL then $E \otimes_\pi F$ is barreled. Our next four positive results are due to Valdivia [87].

Theorem 65. *If E is a metrizable barreled space and if F is* BL, *then $E \otimes_\pi F$ is* BL.

Theorem 66. *If E is a metrizable space and F a space, then* a) *$E \otimes_\pi F$ is* SB *if and only if E and F are* SB, b) *$E \otimes_\pi F$ is* UBL *if and only if E and F are* UBL.

For Baire spaces the situation is rather different. One has

Theorem 67. *$E \otimes_\pi F$ is a Baire space if and only if one of the following conditions is satisfied:* a) *E is finite-dimensional and $F^{\dim E}$ is a Baire space;* b) *F is finite-dimensional and $E^{\dim F}$ is a Baire space.*

For TB spaces, we have [90],

Theorem 68. *If E is a* TB *space which is not* UBL *and if F is a space, then $E \otimes_\pi F$ is* TB *if and only if F is finite-dimensional.*

Theorem 66 and 67 imply that, for infinite-dimensional Fréchet spaces E and F, $E \otimes_\pi F$ is UBL but not Baire.

Our next aim is to characterize those barreled spaces F for which $K^N \otimes_\pi F$ is barreled. One has [9],

Theorem 69. *Let F be a barreled space. $K^N \otimes_\pi F$ is barreled if and only if F is quasi-Baire.*

Theorem 70. *If E is a metrizable barreled space, then $E \otimes_\pi K^{(N)}$ is barreled if and only if E is normable.*

The last result sugests that the problem to study now is the following: characterize those Fréchet spaces E for which $E \hat{\otimes}_\pi K^{(N)}$ is barreled.

Grothendieck showed that if a Fréchet space E admits a non-normable quotient with a continuous norm, then $E \hat{\otimes}_\pi K^{(N)}$ is not barreled. In particular, if E stands for the space $\mathscr{E}(\Omega)$ of the K-valued infinitely differentiable functions defined on a non-void open set of the euclidean space, one has that E is isomorphic to s^N, see [73], Ch. III. 12, and hence E'_b is isomorphic to $(s')^{(N)}$. Thus $E \hat{\otimes}_\pi E'_b$ has $E \hat{\otimes}_\pi K^{(N)}$ as a complemented subspace. Since this last space is not barreled, $E \hat{\otimes}_\pi E'_b$ is not barreled.

Following [94], a Fréchet space is a quojection if every quotient with a continuous norm is a Banach space. Every countable product of Banach spaces is a quojection but there are quojections which are not isomorphic to such a product [99]. The following characterization can be found in [9].

Theorem 71. *Let E be Fréchet space. Then $E \hat{\otimes}_\pi K^{(N)}$ is barreled if and only if E is a quojection.*

The following two results are presented in [98].

Theorem 72. *Let E and F be bornological spaces and suppose E has b.a.p.. If t is a tensor topology, $E \otimes_t F$ is quasi-barreled if and only if $E \otimes_t F$ is bornological.*

Theorem 73. *Let E be a complete space, F a space with b.a.p. and t a tensor topology. Then every bounded set of $E \hat{\otimes}_t F$ is contained in the closure in $E \hat{\otimes}_t F$ of a bounded set of $E \otimes_t F$. In particular, $E \otimes_t F$ is quasi-barreled if and only if $E \hat{\otimes}_t F$ is barreled.*

Since $E \otimes K^{(N)}$ is locally dense in $E \hat{\otimes}_\pi K^{(N)}$, one has

Theorem 74. *Let E be a Fréchet space. The following conditions are equivalent:* (a) *E is a quojection;* (b) *$E \hat{\otimes}_\pi K^{(N)}$ is barreled;* (c) *$E \hat{\otimes}_\pi K^{(N)}$ is ultrabornological;* (4) *$E \otimes_\pi K^{(N)}$ is bornological;* (5) *$E \otimes_\pi K^{(N)}$ is quasi-barreled.*

These results have a certain incidence in the study of the barreledness for spaces of vector-valued infinitely differentiable functions. If E is a locally complete space and if X is a non-void open set of the euclidean space, Bonet [95] has shown that $D(X, E)$ and $\mathscr{E}(X, E)$ are isomorphic to $s(E)^{(N)}$ and $s(E)^N$, respectively. The barreledness of those spaces is then equivalent to the barreledness of $s(E)$. In particular, Theorem 65 says that if E is a locally complete BL space, then $D(X, E)$ and $\mathscr{E}(X, E)$ are barreled.

On the other hand, the following observation (which follows from Theorem 71) is useful:

Observation 75. If E is a Fréchet space which is not a quojection and if F is a barreled space which is not quasi-Baire, then $E \hat{\otimes}_\pi F$ is not barreled.

Thus if E is a locally complete barreled space which is not quasi-Baire (e.g., every strict (LF)-space), then $D(X, E)$ and $\mathscr{E}(X, E)$ are not barreled. In particular, $D(X_1, D(X_2))$ is not barreled for open sets X_1 and X_2. Thus the mapping h: $D(X_1 \times X_2) \to D(X_1, D(X_2))$ defined by $(h(f)(x))(y) := f(x, y)$ for x in X_1 and y in X_2 is not open (clearly it is linear, bijective and continuous).

Now we turn our attention to the following problem: under which circumstances is $s(E)$ barreled for a quasi-Baire space E which is not BL? Grothendieck [23], proved that $s(s') = s \hat{\otimes}_\pi s'$ is barreled. Thus if X denotes a non-void open set of R^n, one has that $D(X) \hat{\otimes}_\pi D(X)' = [(s \hat{\otimes}_\pi s')^{(N)}]^N$ is barreled. On the other hand, $D(X) \otimes_\pi D(X)'$ is not barreled since it contains a non-barreled complemented subspace $K^N \otimes_\pi K^{(N)}$. Also in [23], Grothendieck shows that if E is the strong dual of the space of all holomorphic functions on the unit disc, then $s(E)$ is not barreled.

If we take E to be the strong dual of a nuclear Fréchet space, the problem of the barreledness of $s(E)$ was first treated by Grothendieck for Köthe sequence spaces. In [100], Vogt gives the following

Theorem 76. *If E and F are nuclear Fréchet spaces and if one of them is a power series space $\Lambda_r(\alpha)$ with α satisfying the stability condition* $\lim \alpha_{n+1}/\alpha_n = 1$ *then the following conditions are equivalent*:

(a) $E'_b \hat{\otimes}_\pi F$ *is bornological;*

(b) $E'_b \hat{\otimes}_\pi F$ *is barreled;*

(c) (c.1) *if* $F := \Lambda_r(\alpha)$ *then E satisfies condition* (DN);

(c.2) *if* $E := \Lambda_\infty(\alpha)$ *then F satisfies condition* (Ω);

(c.3) *if* $E := \Lambda_1(\alpha)$ *then F satisfies condition* $(\bar{\Omega})$.

For information on conditions (DN), (Ω) and $(\bar{\Omega})$ see [102].

Theorems 69 and 71 suggest that quojections may form the adequate class of Fréchet spaces to characterize the barreledness of projective tensor products and their completions. The following results can be found in [97].

Theorem 77. a) *If E is a quojection and if F is barreled, then* $E \hat{\otimes}_\pi F$ *is barreled.*

b) *If E is a quojection and F is quasi-Baire, then* $E \otimes_\pi F$ *is barreled.*

c) *If E is a reflexive Fréchet space such that* $E \otimes_\pi F$ *is barreled for every quasi-Baire space F, then E is a quojection.*

Theorem 71 can also be applied to the study of the barreledness of certain spaces of continuous linear mappings. If E is a Fréchet space, then $L_b(K^N, E) = K^{(N)} \hat{\otimes}_\pi (E)$ is barreled if and only if E is a quojection. On the other hand, if E is a (df)-space (see [28]) with the Mackey topology, then $L_b(E, K^{(N)}) = E'_b \hat{\otimes}_\pi K^{(N)}$ is barreled if and only if E'_b is a quojection. In [101], Vogt characterizes the pairs of Fréchet spaces (E, F) such that every continuous linear mapping between them sends a certain 0-neighbourhood of E in a bounded set of $F(L(E, F) = \mathrm{LB}(E, F))$. In particular, if E is relexive and if $L(E, F) = \mathrm{LB}(E, F)$ then $E'_b \otimes_\pi F$ is barreled. Finally, observe that the three equivalent conditions of Theorem 76 are also equivalent to the barreledness of $L_b(E, F)$.

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Added in proof: Meanwhile the content of this article has been expanded in book form by the author and J. Bonet ("Barrelled Locally Convex Spaces", North-Holland Math. Studies, Amsterdam 1987)

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Souhrn

NĚKTERÉ OTÁZKY TEORIE SUDOVITÝCH PROSTORŮ

PEDRO PEREZ CARRERAS

Jedná se o přehledný článek, ve kterém jsou diskutovány některé otázky teorie sudovitých (angl. Barreled) prostorů. Je uveden přehled nedávných výsledků týkajících se spojitosti zobrazení s uzavřeným grafem.

Dalším tématem je analýza výsledků souvisejících s následující vlastností metrizovatelných sudovitých prostorů: jestliže je takový prostor sjednocením neklesající posloupnosti uzavřených absolutně konvexních množin, pak alespoň jedna z nich je okolím nuly.

Část výkladu je věnována zachování „sudovitosti" prostoru v projektivných tensorových součinech.

Резюме

НЕКОТОРЫЕ ВОПРОСЫ ТЕОРИИ БОЧЕЧНЫХ ПРОСТРАНСТВ

PEDRO PEREZ CARRERAS

Обзорная статья, в которой дискутируются некоторые вопросы теории бочечных пространств. Кроме обзора недавних результатов о непрерывности отображения с замкнутым графиком в ней содержится анализ результатов связанных со следующим свойством метризуемых бочечных пространств: если такое пространство является объединением неубывающей последовательности замкнутых абсолютно выпуклых множеств, то по крайней мере одно из множеств есть окрестность нуля. Часть изложения посвящена также вопросу сохранения бочечности пространства в проективных тензорных произведениях.

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