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Časopis pro pěstování matematiky, Vol. 111 (1986), No. 2, 156--176

Persistent URL: `http://dml.cz/dmlcz/118274`

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INVARIANT MANIFOLDS AND THE CONCEPT OF ASYMPTOTIC PHASE

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

(Received July 17, 1985)

1. INTRODUCTION

Invariant manifold theory is known to provide useful tools for the study of the flow of an ordinary differential equation $\dot{x} = f(x)$ near an equilibrium point x_0. In particular the notion of a center manifold plays a crucial role in this context. Roughly speaking the flow on a local center manifold through the equilibrium point x_0 bears valuable information about the flow in a full neighborhood of x_0. For simplicity but with only little loss of generality we suppose that the Jacobian of f at x_0 has no eigenvalues with positive real parts. Otherwise one simply has to restrict the considerations to the flow in a center-stable manifold through x_0. Historically the first description of an essential property of the center manifold – sometimes referred as "reduction principle" – appears in the work of Pliss [8] and Kelley [5]. This principle is related to the general concept of asymptotic phase (see [1]) and can be roughly phrased as follows. Each point x_1 in a neighbourhood of x_0 can be mapped into a point $A(x_1)$ on a center manifold, such that the difference between the solutions $x(t, x_1)$ and $x(t, A(x_1))$ decays to zero exponentially as $t \to \infty$. A somewhat different point of view is taken by Palmer [7, Theorem 4.1], where – roughly speaking – the inverse of the map A is studied.

The present paper is centered around ideas similar to the ones which underlie the above mentioned work. Our main result is Theorem 3.1 and the subsequent corollaries which are concerned with additional invariance properties of the manifolds whose existence follows from the theorem. The theorem itself demonstrates that one can find more invariant manifolds in the neighborhood of a stationary point than is commonly known. In fact we will show that an invariant submanifold $\overline{M}$ of the center manifold admits a continuation as a locally invariant manifold M which is transversal to the center manifold (see figure 1). This manifold M represents a new type of

*) Supported by the Volkswagenwerk foundation.

invariant manifold which in a particular case may be thought of as lying between the stable manifold and a center – stable manifold through x_0. In fact, if $\bar{M}$ contains the equilibrium point x_0 then M contains the stable manifold through x_0; on the other hand M is a submanifold of a center-stable manifold through x_0. That this result can be a useful tool in the discussion of concrete problems will be illustrated by an application given in Section 4. Here we consider an attractor which is a manifold of equilibria and establish some properties of its region of attraction.

Our method follows the approach to invariant manifold theory as developed in [6]. We eliminate certain shortcomings of the technique used in [6] which led to a loss of the degree of smoothness in the course of the construction. Furthermore we prove a reduction principle which is valid under rather weak assumptions.

The paper is organized as follows. The starting point of our considerations is [6, Satz 7.1] which we discuss in detail in Section 2. We also relate this theorem to well known concepts which can be found in Coddington-Levinson [2], Hartman [4] and Hale [3] (see Remark 2.4). We are then in a position to state and prove our main result.

As far notation is concerned there are only a couple of points worth mentioning. By $\Sigma(A)$ we denote the set of real parts of the eigenvalues of the matrix A. By a $C^{1,\mathrm{Lip}}$-function $f(t, x)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ we mean a continuous function which is of class C^1 as function of x and whose x-derivative has a global Lipschitz constant with respect to x on its entire domain of definition. A $C^{1,\mathrm{Lip}}$-manifold is the graph of a $C^{1,\mathrm{Lip}}$-function.

2. BASIC RESULTS

We first present the main theorem about integral manifolds from [6, Ch. V] since it will be used repeatedly in the sequel. In doing so we add two properties which we will prove below. Let us consider a pair of coupled differential equations

$$\dot{x} = g(t, x, y), \quad \dot{y} = h(t, x, y) \quad (x \in \mathbb{R}^k, \; y \in \mathbb{R}^m) \tag{2.1}$$

and assume that g and h are continuous and bounded together with their first partial derivatives with respect to x and y on the whole (t, x, y)-space. In particular we let

$$\sup \|g_x\| \leqq c, \quad \sup \|g_y\| \leqq \gamma_g, \quad \sup \|h_x\| \leqq \gamma_h, \quad \sup \|h_y\| \leqq c \tag{2.2}$$

where the supremum is taken over the (t, x, y)-space. Furthermore we assume that g_x, g_y, h_x and h_y satisfy a global Lipschitz condition with Lipschitz constant K with respect to x and y. Finally we ask for the following property:

(2.3) If $(x, y)(t)$ is any solution of (2.1) we ask of the transition matrices $\phi_1(t, \tau)$, $\phi_2(t, \tau)$ of

$$\dot{w} = g_x(t, x(t), y(t))\, w, \quad \dot{z} = h_y(t, x(t), y(t))\, z$$

respectively to satisfy the uniform estimates

$$\|\phi_1(t,\tau)\| \leqq \gamma_1 e^{\alpha(t-\tau)}, \quad t \geqq \tau,$$
$$\|\phi_2(t,\tau)\| \leqq \gamma_2 e^{\beta(t-\tau)}, \quad t \leqq \tau,$$

for $\alpha < \beta$ and some positive constants γ_1, γ_2.

Theorem 2.1. *Let $g(t, x, y)$ and $h(t, x, y) = By + q(t, x, y)$ satisfy all the above assumptions and let $\Sigma(B) > 0$. If there exists a $\varrho \geqq 0$ such that*

$$(2.4) \qquad \alpha < \nu\varrho < \beta, \quad \kappa_{\varrho\nu} := \gamma_1\gamma_2\gamma_g\gamma_h(\nu\varrho - \alpha)^{-1}(\beta - \nu\varrho)^{-1} < 1 \quad \textit{for} \quad \nu = 1, 2$$

then there exists a continuous and bounded function $y = s(t, x)$ that has continuous and bounded partial derivatives of first order with respect to t and x with the following properties:

(a) *$M := \{(t, x, y) : y = s(t, x), (t, x)$ arbitrary$\}$ is the maximal integral manifold for (2.1) containing all solutions $(x, y)(t)$ of (2.1) with bounded y-component for $t \to \infty$.*

(b) *If g and q are T-periodic in t, then s is.*
If g and q are ω-periodic in x_j, then s is.

(c) $\sup |s| \leqq \sup |q| \displaystyle\int_0^\infty \|e^{-Bt}\|\, dt,$

$\sup \|s_x\| \leqq (1 - \kappa_{\varrho 1})^{-1}(\beta - \varrho)^{-1}\gamma_2\gamma_h[1 + c\gamma_1(\varrho - \alpha)^{-1}] =: \bar{\kappa}.$

(d) *$s_x(t, x)$ satisfies a global Lipschitz condition with respect to x.*

(e) *If g, q and q_x, q_y vanish at $(t, 0, 0)$, $t \in \mathbb{R}$, then $s(t, 0) = 0$ and $s_x(t, 0) = 0$ for all $t \in \mathbb{R}$.*

Remark 2.2. (1) If the hypotheses of Theorem 1 are satisfied only on an interval of the form $[t_0, \infty)$ then the statements are valid on the same interval.

(2) By reversing the time t one can prove an analogous theorem for differential equations of the form

$$(2.5) \qquad \dot{x} = Ax + p(t, x, y), \quad \dot{y} = h(t, x, y)$$

with $\Sigma(A) < 0$. Thus by applying Theorem 2.1 twice, first to (2.1) and then to (2.5) one can prove that $C^{1,\mathrm{Lip}}$-systems possess a global $C^{1,\mathrm{Lip}}$ center manifold.

(3) Part (c) of Theorem 2.1 has the important consequence that – in loose terms – the function s depends continuously on the functions g and q. More precisely, let

$$(*)_\lambda \qquad \dot{x} = g(t, x, y) + \lambda\tilde{g}(t, x, y)$$
$$\dot{y} = By + q(t, x, y) + \lambda\tilde{q}(t, x, y)$$

satisfy all the assumptions of Theorem 2.1 for $\lambda = 0$ and $\lambda = 1$. Let $s_\lambda(t, x)$ be corresponding functions furnished by Theorem 2.1. Then the substitution $v = y - s_0(t, x)$ yields a system of the form

$$(**)_\lambda \qquad \begin{aligned} \dot{x} &= g(t, x, v + s_0(t, x)) + \lambda \bar{g}(t, x, v + s_0(t, x)) \\ \dot{v} &= [B + V(t, x, v)]\, v + \\ &\quad + \lambda[\tilde{q}(t, x, v + s_0(t, x)) + \tilde{g}(t, x, v + s_0(t, x))] \end{aligned}$$

where $\sup |V(t, x, v)| \leqq c + \bar{\kappa}\gamma_g$. Thus, for sufficiently small $c + \bar{\kappa}\gamma_g$, part (c) of Theorem 2.1 implies for the "integral manifold" $v = \sigma(t, x)$ of $(**)_1$

$$\sup |\sigma| \leqq \chi[\sup |\tilde{q}| + \bar{\kappa} \sup |\tilde{g}|]$$

for an approriate constant χ. As a consequence we have

$$\sup |s_1 - s_0| \leqq \chi[\sup |\tilde{q}| + \bar{\kappa} \sup |\tilde{g}|] .$$

Proof. The function $s(t, x)$ is defined with the help of solutions of boundary value problems of the following type

$$(2.6) \qquad \begin{aligned} \dot{x} &= g(t, x, y) , \quad x(t_0) = x_0 , \\ \dot{y} &= h(t, x, y) , \quad y(t_1) = y_1 , \quad t_0 < t_1 . \end{aligned}$$

Under the above hypotheses there exists a unique solution of (2.6) which we denote by $(x, y)(t; t_0, x_0, t_1, y_1)$. As shown in [6, pp. 239, 240] there exists a sequence (t_μ), $t_\mu \to \infty$ for $\mu \to \infty$, such that the expression

$$(2.7) \qquad s(t, x) = \lim_{\mu \to \infty} y(t; t, x, t_\mu, 0)$$

defines a function s satisfying all the statements of the theorem except for parts (d) and (e). In Satz 6.1 [6, pp. 230–235] it is shown that the solution of (2.6) is continuously differentiable with respect to all its variables t, t_0, x_0, t_1, y_1. From the proof of Satz 6.1 it follows that $y_{x_0}(t_0; t_0, x_0, t_1, y_1)$ admits the uniform bound given by $\bar{\kappa}$ from part (c). Thus it remains to establish the existence of a uniform Lipschitz constant for y_{x_0} not depending on t_0, t_1 and y_1. Then the relation (2.7) implies point (d).

We regard t_0, t_1, y_1 as fixed for the moment and write $(x, y)(t, x_0)$ instead of $(x, y)(t; t_0, x_0, t_1, y_1)$. We consider the partial derivative of this solution with respect to the i-th component x_0^i of x_0 and write

$$(2.8) \qquad w(t, x_0) = \frac{\partial x}{\partial x_0^i}(t, x_0) , \quad z(t, x_0) = \frac{\partial y}{\partial x_0^i}(t, x_0) .$$

Then $(w, z)(t, x_0)$ satisfies the variational equation of (2.6) which can be written as a linear boundary value problem of the form

$$(2.9) \qquad \begin{aligned} \dot{w} &= A(t)\, w + A_1(t)\, z , \quad w(t_0, x_0) = e_i , \\ \dot{z} &= B_1(t)\, w + B(t)\, z , \quad z(t_1, x_0) = 0 . \end{aligned}$$

(cf. [6, p. 230]). We can now apply the results of [6, pp. 220–224] in particular Hilfssatz 4.2 and 4.3, and obtain an estimate of the form

$$(w, z)(t, x_0) = O(e^{\varrho(t-t_0)}) \tag{2.10}$$

for $t_0 \leqq t \leqq t_1$. The symbol O from now on refers to estimates where the constants depend on $\alpha, \beta, \gamma_1, \gamma_2, c, \gamma_g, \gamma_h$ and K but not on t_0, t_1, y_1. Thus, from (2.8) and (2.10) we conclude

$$|(x, y)(t, x_0) - (x, y)(t, x_0')| = O(e^{\varrho(t-t_0)}|x_0 - x_0'|) \tag{2.11}$$

for $t_0 \leqq t \leqq t_1$. To complete the proof of part (d) we need to show

$$|z(t_0, x_0) - z(t_0, x_0')| = O(|x_0 - x_0'|). \tag{2.12}$$

Because of (2.9) the differences

$$\begin{aligned} \Delta_w(t, x_0, x_0') &:= w(t, x_0) - w(t, x_0'), \\ \Delta_z(t, x_0, x_0') &:= z(t, x_0) - z(t, x_0') \end{aligned} \tag{2.13}$$

satisfy

$$\begin{aligned} \Delta_w^{\bullet} &= A(t)\,\Delta_w + A_1(t)\,\Delta_z + a(t), \quad \Delta_w(t_0, x_0, x_0') = 0, \\ \Delta_z^{\bullet} &= B_1(t)\,\Delta_w + B(t)\,\Delta_z + b(t), \quad \Delta_z(t_1, x_0, x_0') = 0 \end{aligned}$$

where $A(t)$, $A_1(t)$, $B_1(t)$, $B(t)$ are given by g_x, g_y, h_x, h_y at $(t, (x, y)(t, x_0))$ respectively and where $a(t)$ is equal to

$$\begin{aligned} &[g_x(t, (x, y)(t, x_0)) - g_x(t, (x, y)(t, x_0'))]\, w(t, x_0') + \\ &+ [g_y(t, (x, y)(t, x_0)) - g_y(t, (x, y)(t, x_0'))]\, z(t, x_0'). \end{aligned}$$

A similar formula holds for $b(t)$. Because of (2.10), (2.11) and the assumed Lipschitz condition for g_x, g_y, h_x, h_y we have an estimate of the form $O(e^{2\varrho(t-t_0)}|x_0 - x_0'|)$ for $a(t)$ and $b(t)$ on $[t_0, t_1]$. By (2.4) and Hilfssatz 4.2 [6, p. 222] we have $\Delta_z(t, x_0, x_0') = O(e^{2\varrho(t-t_0)}|x_0 - x_0'|)$ on $[t_0, t_1]$ and thus (2.12). By taking (2.13) and (2.8) into account we finally arrive at

$$\frac{\partial y}{\partial x_0^i}(t_0, x_0) - \frac{\partial y}{\partial x_0^i}(t_0, x_0') = O(|x_0 - x_0'|)$$

where the symbol O is independent of t_0, t_1 and y_1. Part (d) now follows from (2.7).

Finally we turn to the proof of part (e). Because of $g(t, 0, 0) = 0$ and $q(t, 0, 0) = 0$ one has the trivial solution $(x, y) = (0, 0)$ for (2.1). By part (a) $s(t, 0)$ needs to be 0. In order to show $s_x(t, 0) = 0$ we consider $y_{x_0}(t; t, 0, t_1, 0)$. The z-equation in (2.9) now reduces to $\dot{z} = Bz$, $z(t_1, 0) = 0$, for all $z = (\partial y/\partial x_0^i)(t; t, 0, t_1, 0)$, $i = 1, \ldots, k$. Thus $y_{x_0}(t;, t, 0, t_1, 0) = 0$ holds uniformly in t_1, and relation (2.7) implies part (e). ∎

Next we consider systems of the form

$$(2.14)\qquad \begin{aligned} \dot{x} &= Ax + p(t, x, y, z), & x \in \mathbb{R}^k, \\ \dot{x} &= By + q(t, x, y, z), & x \in \mathbb{R}^m, \\ \dot{z} &= Z(t, z) + r(t, x, y, z), & z \in \mathbb{R}^n, \end{aligned}$$

that are defined for all (t, z) and for all (x, y) in a neighborhood of $(0, 0)$ in $\mathbb{R}^k \times \mathbb{R}^m$. We assume that the right-hand side belongs to $C^{1,\mathrm{Lip}}$ and that $\sup |Z(t, z)| < \infty$. Furthermore we pose the following hypotheses on (2.14):

There exists positive constants ϱ, γ', α' and ε with the following properties:

(2.15) $\Sigma(A) < -\varrho$, $\beta' := \min \Sigma(B) > -\varrho$,

If $z(t)$ is any solution of $\dot{z} = Z(t, z)$ the transition matrix $\Psi(t, \tau)$ of $\dot{z} = Z_z(t, z)\, z$ satisfies

$\|\Psi(t, \tau)\| \leqq \gamma' e^{\alpha'(t-\tau)}$ for $t \geqq \tau$.

Let w be any of the functions p, q or r.

(2.16) w tends to 0 uniformly in (t, z) as $(x, y) \to (0, 0)$.

(2.17) $\|w_x(t, x, y, z) - w_x(t, 0, 0, z)\|$, $\|w_y(t, x, y, z) - w_y(t, 0, 0, z)\|$ and $\|w_z(t, x, y, z)\|$ tend to 0 uniformly in (t, z) as $(x, y) \to (0, 0)$.

Moreover $\|p_x\|$, $\|p_y\|$, $\|q_x\|$, $\|q_y\|$ are less ε for all $(t, 0, 0, z)$.

(2.18) There exists a ϱ' with $\alpha' < \nu\varrho' < \beta' + \varrho$ for $\nu = 1, 2$.

Theorem 2.3. *There exist positive constants ε_0, δ_0 and δ with $\delta_0 < \delta$ such that for $\varepsilon < \varepsilon_0$*

$$\{(t, x, y, z) : y = s(t, x, z),\ |x| < \delta\}$$

defines a local $C^{1,\mathrm{Lip}}$-integral manifold for (2.14) *having the following properties*:

(a) $s(t, 0, z) = 0$ *for all* $(t, 0, z)$.

(b) *If* $|x_0| < \delta_0$ *and* $y_0 = s(t_0, x_0, z_0)$ *then the solution* $(x, y, z)(t)$ *of* (1.14) *with initial value* (x_0, y_0, z_0) *at* $t = t_0$ *exists for all* $t \geqq t_0$ *and satisfies* $|x(t)| < \delta$, $|y(t)| < \delta$, $y(t) = s(t, x(t), z(t))$, $t \geqq t_0$. *Moreover,* $|x(t)|$ *and* $|y(t)|$ *tend exponentially to 0 as $t \to \infty$. More precisely, there exist positive constants Γ_0 and Γ that are independent of the chosen initial value with* $|(x, y)(t)| \leqq \Gamma_0 |x_0|\, e^{-\Gamma t}$, $t \geqq 0$.

(c) *If the right-hand side of* (2.14) *is periodic in t (in z), then s is periodic in t (in z).*

Remark 2.4. Theorem 2.3 generalizes two concepts which meanwhile may be considered classical. On the one hand invariant manifolds with description $y = s(t, x, z)$ are sought for differential systems of the form (2.14) where the respective spectra of A or B lie left or right to a vertical line in the left plane of $\mathbb{C}$. Coddington and Levinson [2, Ch. 13, Th. 4.4] and Hartman [4, X, Th. 8.3] consider the case where the z-component in (2.14) is absent. Moreover in [2] B is not allowed to have eigenvalues on the imaginary axis. The z-component as a parameter, i.e. the case when $Z(t, z) + r(t, x, y, z) \equiv 0$ in our notation, is treated in Knobloch and Kappel [6, Satz 9.2]. On the other hand systems of the form (2.14) with a general z-equation are considered by Hale [3], but there the matrices A and B are in a hyperbolic configuration, i.e. the vertical line separating the spectra of A and B may be chosen to be the imaginary axis. Obviously, both directions mentioned above are taken into account and generalized in Theorem 2.3.

If z represents an angular variable the global assumptions with respect to z are quite natural. Otherwise they can be satisfied after a modification of a system (2.14) that is originally only given in a neighborhood of $(x, y, z) = (0, 0, 0)$ (cf. [6, p. 248]). But then property (b) does in general not apply to solutions of the originally given local system.

Proof. Since we intend to use Theorem 2.1 we consider a system related to (2.14). To this end we shift the spectra of A and B to the right. That is done in the following way: We denote by $\xi(t) = \mu e^{-\varrho(t-\tau)}$ a nontrivial solution of $\dot{\xi} = -\varrho\xi$. Then the following is true near τ:

If $(x, y, z)(t)$ is a solution of (2.14) then $(\tilde{x}, \tilde{y}, \tilde{z})(t) := (x/\xi, y/\xi, z)(t)$ is a solution of

$$\begin{aligned} \dot{x} &= (A + \varrho I)\, x + \frac{1}{\xi(t)}\, p(t, \xi(t)\, x, \xi(t)\, y, z)\,, \\ \dot{y} &= (B + \varrho I)\, y + \frac{1}{\xi(t)}\, q(t, \xi(t)\, x, \xi(t)\, y, z)\,, \\ \dot{z} &= Z(t, z) + r(t, \xi(t)\, x, \xi(t)\, y, z) \end{aligned} \tag{2.19}$$

provided $|\tilde{x}(\tau)|$ and $|\tilde{y}(\tau)|$ are sufficiently small. Conversely, if $(x, y, z)(t)$ is a solution of (2.19) with $|\tilde{x}(t)|, |\tilde{y}(t)|$ sufficiently small, then $(x, y, z)(t) := (\xi\tilde{x}, \xi\tilde{y}, \tilde{z})(t)$ is a solution of (2.14).

We define for ξ in a neighborhood of $0 \in \mathbb{R}$

$$\begin{aligned} \tilde{p}(t, \xi, x, y, z) &= \frac{1}{\xi}\, p(t, \xi x, \xi y, z) \quad \text{for} \quad \xi \neq 0\,, \\ \tilde{p}(t, 0, x, y, z) &= p_x(t, 0, 0, z)\, x + p_y(t, 0, 0, z)\, y \\ \tilde{r}(t, \xi, x, y, z) &= r(t, \xi x, \xi y, z) \end{aligned}$$

and note the validity of

$$\tilde{p}(t, \xi, x, y, z) = \int_0^1 p_x(t, s\xi x, s\xi y, z)\, x \,\mathrm{d}s\,.$$

Analogous formulae hold for $\tilde{q}$. Thus, $\tilde{p}$ and $\tilde{q}$ belong to $C^{1,\mathrm{Lip}}$ with respect to x, y, z, but not necessarily with respect to ξ. We now modify the functions $\tilde{p}, \tilde{q}, \tilde{r}$ by using a C^∞ cut-off function for the variables ξ, x, y (cf. Hilfssatz 2.3 of [6, pp. 214/215]) such that the globally defined continuations p^*, q^*, r^* coincide with $\tilde{p}, \tilde{q}, \tilde{r}$ respectively for $|\xi| < \Delta_1$, $|x| < \Delta_1$, $|y| < \Delta_1$ and such that the following is true for any continuous function $\xi: \mathbb{R} \to \mathbb{R}$: The modified system

$$\begin{aligned} \dot{x} &= (A + \varrho I)\, x + p^*(t, \xi(t), x, y, z)\,, \\ \dot{y} &= (B + \varrho I)\, y + q^*(t, \xi(t), x, y, z)\,, \\ \dot{z} &= Z(t, z) + r^*(t, \xi(t), x, y, z) \end{aligned} \tag{2.20}$$

satisfies the global assumptions stated below system (2.1) with $(x, z) \to x$. Note that the Lipschitz constants of p^*, q^* and r^* as well as all the other constants there can be chosen independently from $\xi(t)$ since the cut-off function is applied to $\xi(t)$ too. As it is easily seen the conditions (2.2)–(2.4) can be satisfied independently from the chosen $\xi(t)$ provided ε is less than a certain ε_0 and δ_1 is sufficiently small. For the verification of (2.3) for the transition matrix of the (x, z)-variational equation we refer to [6, pp. 219/220]. Thus for such values of ε and δ_1 Theorem 2.1 can be applied to (2.20) yielding a function $y = s^*(t, \xi(t), x, z)$ with all the properties stated there. In particular part (a) implies

$$s^*(t, \xi(t), 0, z) = 0 \quad \text{for all} \quad (t, z)\,. \tag{2.21}$$

We are going to show now that s^* is in some restricted sense independent of the choice of $\xi(t)$ (see (2.24) below). Let $q > 1$ be the Lipschitz constant of s^* with respect to (x, z). We choose a $\delta_2 \in (0, \delta_1)$ and a $\delta_0 \in (0, \delta_2/2q)$ with the following property:

(2.22) With $\xi(t) = \mu e^{-\varrho(t-\tau)} \neq 0$ any solution $(\tilde{x}, \tilde{y}, \tilde{z})(t)$ of (2.20) with $|\mu| < \delta_0$, $|\tilde{x}(\tau)| < \delta_0$ and $\tilde{y}(\tau) = s^*(\tau, \mu, \tilde{x}(\tau), \tilde{z}(\tau))$ fulfills $|\xi(t)| < \delta_2/2q$, $|\tilde{x}(t)| < \delta_2/2q$, $|\tilde{y}(t)| < \delta_2/2$ for $t \geqq \tau$.

Next, we take any $\eta \in \mathbb{R}$ with $0 < |\eta| < 2$, $|\mu/\eta| < \delta_2/2q$, $0 < \mu < \delta_0$. Then

$$(\hat{\xi}, \hat{x}, \hat{y}, \hat{z})(t) := (\xi/\eta, \eta\tilde{x}, \eta\tilde{y}, \tilde{z})(t) \tag{2.23}$$

satisfies

$$|\hat{\xi}(t)| < \frac{\delta_2}{2q}\,, \; |\hat{x}(t)| < \frac{\delta_2}{q}\,, \quad |\hat{y}(t)| < \delta_2 \quad \text{for} \quad t \geqq \tau\,.$$

Thus $(\hat{x}, \hat{y}, \hat{z})(t)$ is a solution of (2.20) for $\hat{\xi}(t) = (\mu/\eta)\, e^{-\varrho(t-\tau)}$. Since its $\hat{y}$-component is bounded for $t \to \infty$ we have

$$\hat{y}(t) = s^*(t, \hat{\xi}(t), \hat{x}(t), \hat{z}(t)) = \eta s^*(t, \xi(t), \tilde{x}(t), \tilde{z}(t)) = \eta\, \tilde{y}(t)$$

for $t \geqq \tau$. We define $x(t) := \xi(t)\,\hat{x}(t)$ and $\hat{\mu} := \mu/\eta$ and thus can rewrite the last equation in a more symmetric form

$$(2.24)\quad \hat{\xi}(t)\, s^*\left(t, \hat{\xi}(t), \frac{x(t)}{\hat{\xi}(t)}, z(t)\right) = \xi(t)\, s^*\left(t, \xi(t), \frac{x(t)}{\xi(t)}, z(t)\right) \quad \text{for} \quad t \geqq \tau$$

under the conditions

$$0 < \mu < \delta_0\,, \quad 0 < |\mu/\hat{\mu}| < 2\,, \quad |\hat{\mu}| < \frac{\delta_2}{2q}\,, \quad |x(\tau)/\mu| < \delta_0\,.$$

We fix such a μ and define

$$(2.25a)\qquad s(\tau, x, z) = \mu s^*(\tau, \mu, x/\mu, z)\,.$$

Because of (2.24) we have

$$(2.25b)\qquad s(\tau, x, z) = \xi(t)\, s^*(\tau, \xi(t), x/\xi(t), z)$$

$$\text{for}\quad 0 < |\mu/\xi(t)| < 2\,, \quad 0 < |\xi(t)| < \frac{\delta_2}{2q}\,, \quad |x/\mu| < \delta_0\,.$$

Thus, part (a) (cf. (2.21)) and part (c) of Theorem 2.3 follow. The smoothness of s is the same as the one of s^*.

Next, we are going to show how a local integral manifold for (2.14) can be defined with the help of this function s. With the fixed μ in (2.25a)and with $\xi(t) = \mu e^{-\varrho(t-\tau)}$ we have the following because of (2.25b):

If $(x, y, z)(t)$ is a solution of (2.14) with $|x(\tau)| < \delta_0\mu$ and $y(\tau) = s(\tau, x(\tau), z(\tau))$ then there exists an interval I, $\tau \in I^\circ$, with

$$(2.26)\qquad s(t, x(t), z(t)) = \xi(t)\, s^*(t, \xi(t), x(t)/\xi(t), z(t)) \ \text{on}\ I\,.$$

On the other hand $(\tilde{x}, \tilde{y}, \tilde{z})(t) := (x/\xi, y/\xi, z)(t)$ is a solution of (2.20) on some interval I^*, $\tau \in (I^*)^\circ$, satisfying $\tilde{x}(\tau) = x(\tau)/\mu$, $|\tilde{x}(\tau)| < \delta_0$, $\tilde{z}(\tau) = z(\tau)$, $\tilde{y}(\tau) = s^*(\tau, \mu, \tilde{x}(\tau), \tilde{z}(\tau))$. Thus (2.22) implies $\tilde{y}(t) = s^*(t, \xi(t), \tilde{x}(t), \tilde{z}(t))$ on I^*. By (2.26) we have

$$(2.27)\qquad y(t) = s(t, x(t), z(t)) \quad \text{on} \quad I \cap I^*$$

so that $y = s(t, x, z)$ defines a local integral manifold for (2.14).

It remains to establish part (b) of Theorem 2.3. For this purpose we consider the reduced equation

$$(2.28)\qquad \begin{aligned} \dot{x} &= Ax + p(t, x, s(t, x, z), z)\,, \\ \dot{z} &= Z(t, z) + r(t, x, s(t, x, z), z) \end{aligned}$$

for $|x| < \delta_0\mu$. Because of (2.15)–(2.17) and $s(t, 0, z) = 0$ there exists a uniform $\delta_0' > 0$ such that $|x(t_0)| < \delta_0'$ implies $|x(t)| < \delta_0\mu$ for $t \geqq t_0$ and the exponential decay of $|x(t)|$ towards 0 as $t \to \infty$. The constants in the exponential estimate will not depend on t_0. Thus the x-restrictions in (2.25b) are satisfied. With $\xi(t) = \xi_0(t) = \mu e^{-\varrho(t-t_0)}$ (2.27) holds for $t \in [t_0^*, t_1]$ where t_0^* is an appropriate time less t_0 and $t_1 = t_0 + (1/2\varrho) \ln 2$. By using $\xi_1(t) = \mu e^{-\varrho(t-t_1)}$ one can extend (2.27) to $[t_0^*, t_2]$ with $t_2 = t_1 + (1/2\varrho) \ln 2$. Continuing in this manner one shows that (2.27) holds for all $t \geqq t_0^*$. By taking the Lipschitz condition of s and $s(t, 0, z) = 0$ into account all the statements of part (b) follow. ∎

Remark 2.5. A careful analysis of the above proof shows that the constants $\varepsilon_0, \delta, \delta_0, \Gamma_0$ and Γ just depend on the constants in the estimates for the transition matrices Φ_1, Φ_2, Ψ in (2.3) and (2.15).

3. A NEW TYPE OF INVARIANT MANIFOLD

Since we are only interested in the behavior on the center stable manifold of our underlying system we restrict ourselves to $C^{1,\mathrm{Lip}}$-systems of the form

$$\begin{aligned} \dot{x} &= Ax + P(x, z), \quad x \in \mathbb{R}^k, \\ \dot{z} &= Cz + Q(x, z), \quad z \in \mathbb{R}^n, \end{aligned} \tag{3.1}$$

defined for (x, z) in a neighborhood $\mathcal{N}$ of $(0, 0)$. We assume

$$\Sigma(A) < -\varrho < 0, \quad \Sigma(C) = 0, \tag{3.2}$$

$$P \text{ and } Q \text{ are of order } \quad O(|x|^2 + |z|^2), \quad (x, z) \to (0, 0).$$

By Theorem 2.1 (see Remark 2.2) there exists a $C^{1,\mathrm{Lip}}$ center manifold

$$CM = \{(x, z) : x = c(z) = O(|z|^2), \; |z| < \delta_0\} \tag{3.3}$$

for (3.1) in $\mathcal{N}$. The reduced system on CM is then

$$\dot{z} = Cz + Q(c(z), z) = Cz + O(|z|^2), \quad z \to 0. \tag{3.4}$$

Now we suppose that for a $\delta_1 \in (0, \delta_0)$

$$\overline{M} = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} : u = \bar{s}(v) = \lambda + Sv + O(|v|^2), \quad v \in \mathbb{R}^d, \; |v| < \delta_1 \right\} \tag{3.5}$$

defines a locally invariant $C^{1,\mathrm{Lip}}$-manifold for (3.4) inside $\{z \colon |z| < \delta_0\}$. We denote points of $\overline{M}$ by $\zeta(v) = (\bar{s}(v), v)$ and define

$$\chi(v) = (c(\zeta(v)), \; \zeta(v)) \tag{3.6}$$

so that $\chi(\overline{M})$ is the lift of $\overline{M}$ to CM. If we write

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

and $Q = (U, V)^T$ in accordance to $z = (u, v)^T$ the flow on $\overline{M}$ is generated by

$$\dot{v} = Z(v) := C_{21}\, \bar{s}(v) + C_{22} v + V(\chi(v))\,. \tag{3.7}$$

We would like to point out that $\overline{M}$ only passes through $z = 0$ if λ is zero. Thus $Z(0)$ does not vanish unless λ is equal to 0.

Theorem 3.1. *Under the above assumptions there exist a* $\lambda_0 > 0$ *and a* $\delta_2 \in (0, \delta_1)$ *such that system* (3.1) *has a locally invariant* $C^{1,\mathrm{Lip}}$*-manifold*

$$M = \{(x, u, v) : u = f(x, v),\ |x| < \delta_2,\ |v| < \delta_2\} \tag{3.8}$$

inside $\mathcal{N}$ *provided* $|\lambda| < \lambda_0$. *Moreover the inclusion* $M \cap CM \subset \chi(\overline{M})$ *holds.*

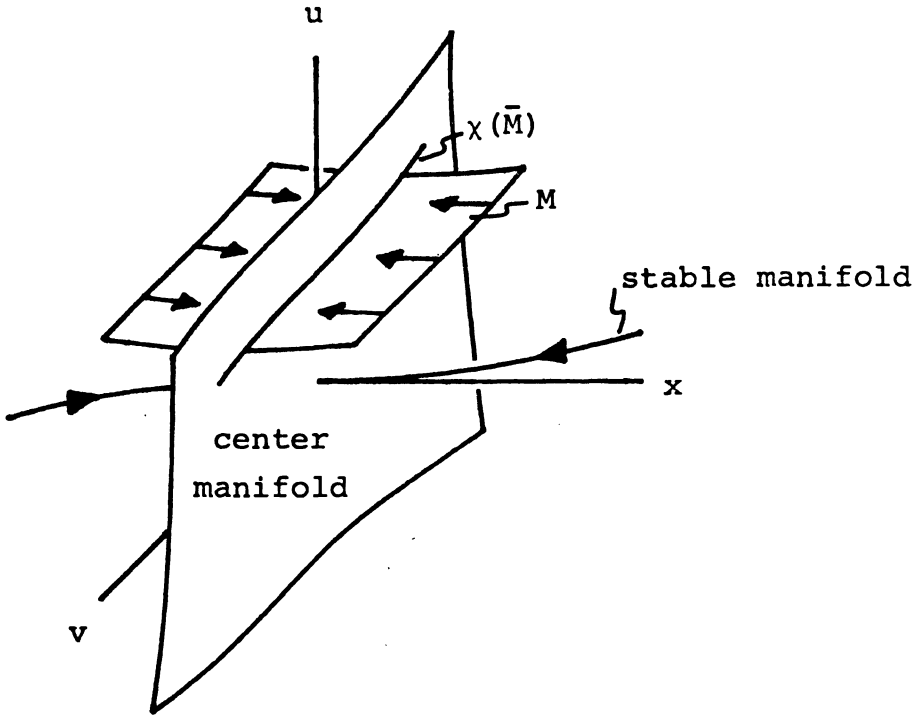

Fig. 1

Remark 3.2. (1) Loosely speaking Theorem 3.1 tells that local invariant manifolds of dimension d within the n-dimensonal center manifold of (3.1) can be continued to local invariant manifolds of dimension $(k + d)$, where k is the dimension of the stable manifold of (3.1). See Figure 1. Below we present sufficient conditions for these $(k + d)$-dimensional manifolds to posses an additional invariance property like part (b) of Theorem 2.3.

(2) A similar result can be proved in the case that $\bar{M}$ is an invariant closed curve. If a moving orthonormal coordinate system $(y, \theta) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is introduced for z (see [3, Ch. VI]) the resulting system will be of the form (2.14). Under appropriate conditions on the closed curve $\bar{M}$ which hold if $\bar{M}$ is generated by a periodic solution Theorem 2.3 is applicable. In this case one arrives at an invariant cylinder of the form $\{(x, y, \theta) : y = f(x, \theta), |x| < \delta_2\}$ over $\bar{M}$. Here, we also like to refer to the work of S. Graff on the continuation of invariant tori (cf. [9]).

Proof. The substitutions

$$x \to x + c(z), \quad u = y + \bar{s}(v) \tag{3.9}$$

transform (3.1) into a system of the form

$$\begin{aligned} \dot{x} &= Ax + p(x, y, v) \\ \dot{y} &= By + q(x, y, v) \\ \dot{v} &= Z(v) + r(x, y, v) \end{aligned} \tag{3.10}$$

with

$$\begin{gathered} B = C_{11} - SC_{21}, \quad \Sigma(B) = 0 \\ Z(v) = Dv + Z_1(v) \quad \text{with} \quad D = C_{21}S + C_{22}, \quad \Sigma(D) = 0 . \end{gathered} \tag{3.11}$$

The functions p, q and r have the following properties:

$$\begin{aligned} p(0, y, v) &= 0, \quad q(0, 0, v) = 0, \quad r(0, 0, v) = 0 \\ p_x(0, 0, v) &= P_x(\chi(v)) - c_z(\zeta(v))\, Q_x(\chi(v)), \\ p_x(0, 0, v) &= U_x(\chi(v)) - \bar{s}_v(v)\, V_x(\chi(v)), \\ q_y(0, 0, v) &= U_x(\chi(v))\, c_u(\zeta(v)) + U_u(\chi(v)) - \\ &\quad - \bar{s}_v(v)\, [V_x(\chi(v))\, c_u(\chi(v)) + V_u(\chi(v))] . \end{aligned}$$

We now use a C^∞ cut-off function for v so that (3.10) and the modified equation

$$\begin{aligned} \dot{x} &= Ax + \tilde{p}(x, y, v) \\ \dot{y} &= By + \tilde{q}(x, y, v) \\ \dot{v} &= \tilde{Z}(v) + \tilde{r}(x, y, v) \end{aligned} \tag{3.12}$$

coincide for $|v| < \tilde{\Delta}$ (see [6, p. 248]). Because of the spectral properties in (3.2) and (3.11) and the representation of CM and $\bar{M}$ in (3.3) and (3.5) all the hypotheses of Theorem 2.3 can be satisfied provided $|\lambda|$ and $\tilde{\Delta}$ are sufficiently small. Thus we arrive at an invariant $C^{1,\mathrm{Lip}}$-manifold

$$\{(x, y, v) : y = s(x, v),\ |x| < \delta_3\} \tag{3.13}$$

for (3.12) where $\delta_3 > 0$ is appropriately chosen. Because of the modification in v the manifold (3.13) just defines a locally invariant manifold for (3.10), i.e.

(3.14) there exist a $\Delta > 0$ and a $\Delta_0 \in (0, \Delta)$ such that the manifold $\{(x, y, v) : y = s(x, v),\ |x| < \Delta,\ |v| < \Delta\}$ has the following property: If $|x_0| < \Delta_0$, $|v_0| < \Delta_0$, $y_0 = s(x_0, v_0)$ then there exists an interval I, $t_0 \in I^\circ$, such that the solution $(x, y, v)(t)$ of (3.10) with initial value (x_0, y_0, v_0) at $t = t_0$ satisfies

$$|x(t)| < \Delta\,, \quad |v(t)| < \Delta\,, \quad y(t) = s(x(t), v(t)) \quad \text{on} \quad I\,.$$

Reversing the transformations (3.9) yields an implicit representation of (3.13) in the form

$$F(x, y, v) := u - \bar{s}(v) - s(x - c(u, v), v) = 0$$

satisfying $F(\chi(v)) = 0$ (cf. (3.6)). Since $F_u(\chi(0)) = I + s_x(0, 0)\, c_u(\lambda, 0)$ is nonzero (provided $|\lambda|$ is sufficiently small) an application of the implicit function theorem leads to the locally invariant manifold M in (3.8) for system (3.1) with $M \cap CM \subset \chi(\overline{M})$. ∎

Remark 3.3. Once again we note that the constants λ_0, Δ_0 and Δ just depend on the estimates for the transition matrices (cf. Remark 2.5).

In order to obtain the additional invariance property of M mentioned in Remark 3.2 (1) one has to ensure that in (3.14) the x- and v-component of a solution of (3.1) satisfy $|x(t)| < \Delta$, $|v(t)| < \Delta$ for all $t \geqq 0$. For this purpose we will impose conditions on the flow on $\overline{M}$ generated by (3.7). We will use the notations of (3.1)–(3.8).

Corollary 3.4. *We assume that the flow on $\overline{M}$ generated by (3.7) satisfies the following condition: There exists a solution $v^*(t)$ of (3.7) with $|v^*(t)| < \Delta_1$ for all $t \geqq 0$. Then there exist positive constants Δ_1^*, λ^* with the following property: If $\Delta_1 < \Delta_1^*$ and if λ in (3.5) is in norm less λ^* then there exist positive constants δ_3 and δ_4 and a $C^{1,\mathrm{Lip}}$ function $f^*(t, w)$ on $t \geqq 0$, $|w| < \delta_3$ such that the manifold M has the additional invariance property:*

If $(x_0, u_0, v_0) \in M$ and $v_0 = v^*(0) + f^*(0, x_0 - c(u_0, v_0))$, $|x_0 - c(u_0, v_0))| < \delta_4$, then the solution $(x, u, v)(t)$ of (3.1) with initial value (x_0, u_0, v_0) at $t = 0$ exists for all $t \geqq 0$ and satisfies for all $t \geqq 0$ $u(t) = f(x(t), v(t))$, $v(t) = v^*(t) + f^*(t, x(t) - cu(t), v(t)))$, $|(x, u, v)(t) - \chi(v^*(t))| \leqq \Gamma_0 |x_0 - c(u_0, v_0)|\, \mathrm{e}^{-\Gamma t}$.

The positive constants Γ_0, Γ do not depend on the chosen initial value.

Corollary 3.5. *We assume that the flow on $\overline{M}$ generated by (3.7) satisfies the following condition: There exist positive constants Δ_0 and Δ_1 such that any solution $v(t, v_0)$ of (3.7) with initial value $v(0, v_0) = v_0$, $|v_0| < \Delta_0$, satisfies $|v(t, v_0)| < \Delta_1$ for all $t \geqq 0$. Then there exist positive constants Δ_1^*, $\lambda^*$with the following property: If $\Delta_1 < \Delta^*$ and if λ in (3.5) is in norm less λ^* then there exists a $\delta_0^* > 0$ such that the manifold M in (3.8) has the additional invariance property:*

If $(x_0, u_0, v_0) \in M$, $|x_0| < \delta_0^*$, $|v_0| < \delta_0^*$, *then the solution* $(x, u, v)(t)$ *of* (3.1) *with initial value* (x_0, u_0, v_0) *at* $t = 0$ *exists for all* $t \geqq 0$ *and satisfies for all* $t \geqq 0$ $(x, u, v)(t) \in M$.

Moreover, on M there exists an asymptotic phase, i.e. for each solution $(x, u, v)(t)$ *of* (3.1) *as above there exists a solution* $\chi(v^*(t))$ *on* $\chi(\overline{M})$ *such that*

$$|(x, u, v)(t) - \chi(v^*(t))| \leqq \Gamma_0 |x_0 - c(u_0, v_0)| \, e^{-\Gamma t}, \quad t \geqq 0 .$$

The positive constants Γ_0, Γ *do not depend on the chosen initial value.*

Remark 3.6. If $\overline{M}$ passes through 0 then (3.7) has the trivial solution $v = 0$. Thus the assumption of Corollary 3.5 is certainly fulfilled if $v = 0$ is a stable solution of (3.7). In this case M is a "weakly" stable manifold which is not necessarily maximal and in which the decay is not necessarily exponential.

To prove the invariance properties stated in the previous corollaries we just have to investigate the reduced equation on the locally invariant manifold M, which reads in terms of (3.12), (3.13) and (3.14):

$$\begin{aligned} \dot{x} &= Ax + P(x, s(x, v), v), \\ \dot{v} &= Z(v) + r(x, s(x, v), v) \end{aligned} \tag{3.15}$$

on $|x| < \Delta$, $|v| < \Delta$. We need to show the existence of a $\Delta_0' > 0$ such that the interval I in (3.14) becomes $[0, \infty)$ if we restrict the initial values to $|x_0| < \Delta_0'$, $|v_0| < \Delta_0'$ and $y_0 = s(x_0, v_0)$. We now investigate a general class of system containing (3.15) as a special case. The corollaries will then follow from the subsequent lemmata. So, we now move on to two results about the existence of an asymptotic phase for systems like (3.15).

We consider systems of the form

$$\begin{aligned} \dot{x} &= Ax + P(x, v), \quad x \in \mathbb{R}^k, \\ \dot{v} &= Bv + Q(x, v), \quad v \in \mathbb{R}^d, \end{aligned} \tag{3.16}$$

belonging to $C^{1,\mathrm{Lip}}$ which are defined on a neighborhood $\mathcal{N} = \{(x, v) : |x| < a, |v| < a\}$ of $(0, 0)$ and which satisfy

$$\begin{aligned} &\Sigma(A) < -\varrho < 0, \quad \Sigma(B) = 0 \\ &P(0, v) = 0, \quad |P_x(0, 0)| < \varepsilon, \\ &|Q(0, 0)| < \mu, \quad |Q_x(0, 0)| < \varepsilon, \quad |Q_v(0, 0)| < \varepsilon \end{aligned} \tag{3.17}$$

for appropriate positive ε and μ. We consider the reduced system of (3.16) on the center manifold $\{(x, v) : x = 0\}$ in $\mathcal{N}$

$$\dot{v} = Bv + Q(0, v) \tag{3.18}$$

and denote the solution $\eta(t)$ of (3.18) with initial value $\eta(0) = z$ by $\eta(t, z)$. Then we ask for the existence of positive constants Δ_0 and Δ_1 in $(0, a)$ with the following property:

(3.19) "Any solution $\eta(t, z)$ of (3.18) with $|z| < \Delta_0$ satisfies

$$|\eta(t, z)| < \Delta_1 \quad \text{for all} \quad t \geqq 0."$$

We also investigate (3.16), (3.18) under the weaker condition

(3.20) "There exists a solution $\eta^*(t) = \eta(t, z^*)$ of (3.18) satisfying

$$|\eta^*(t)| < \Delta_1 \quad \text{for all} \quad t \geqq 0."$$

where (3.19) is only required to hold for a particular initial value z^*.

In both cases the substitution $y = v - \eta(t, z)$, $t \geqq 0$, $|z| < \Delta_0$, leads to the system

$$\begin{aligned} \dot{x} &= Ax + p(t, x, y, z), \quad x \in \mathbb{R}^k, \\ \dot{y} &= By + q(t, x, y, z), \quad y \in \mathbb{R}^d, \\ \dot{z} &= 0 \end{aligned} \tag{3.21}$$

defined for $t \geqq 0$, (x, y) in a neighborhood of $(0, 0)$ and z with $|z| < \Delta_0$. The right-hand side is again in $C^{1,\mathrm{Lip}}$ with respect to x and y but not necessarily with respect to z. The functions p and q have the following properties:

$$\begin{aligned} &p(t, x, y, z) := P(x, y + \eta(t, z)), \\ &q(t, x, y, z) := Q(x, y + \eta(t, z)) - Q(t, \eta(t, z)), \\ &p(t, 0, y, z) = 0, \quad p_x(t, 0, 0, z) = P_x(0, \eta(t, z)), \\ &q(t, x, 0, z) = 0, \quad q_y(t, 0, 0, z) = Q_v(0, \eta(t, z)). \end{aligned} \tag{3.22}$$

Thus, in general, Theorem 2.3 applies to (3.21) only if we consider (3.16) under the condition (3.20) where $z = z^*$ is fixed. In this case there exist positive constants ε_1 and Δ_1^* such that for $\varepsilon < \varepsilon_1$ in (3.17) and $\Delta_1 < \Delta_1^*$ in (3.20) Theorem 2.3 implies the existence of a local $C^{1,\mathrm{Lip}}$ integral manifold

$$M^*(z^*) = \{(t, x, y) : y = y^*(t, x, z^*),\ t \geqq 0,\ |x| < \delta'\}$$

for (3.21) where $\delta' > 0$ is appropriately chosen. Now, part (b) of Theorem 2.3 implies that solutions of (3.21) with sufficiently small initial data on $M^*(z^*)$ tend exponentially to the trivial solution of (3.21). More precisely, we have arrived at the following result.

Lemma 3.7. *Under the notations and assumptions of* (3.16)–(3.18) *and* (3.20) *there exist positive constants* ε_1 *and* Δ_1 *such that the following is true*: *If* $\varepsilon < \varepsilon_1$ *in* (3.17) *and if* $\Delta_1 < \Delta_1^*$ *in* (3.20) *then there exist positive constants* δ_0, δ *and a local* $C^{1,\mathrm{Lip}}$ *integral manifold*

$$M^*(z^*) = \{(t, x, v) : v = \eta(t, z^*) + y^*(t, x, z^*),\ t \geqq 0,\ |x| < \delta\}$$

for (3.16) *with the following property*:

If (x_0, v_0) *satisfies* $|x_0| < \delta_0$ *and* $v_0 = z^* + y^*(0, x_0, z^*)$ *then the solution* $(x, v)(t)$ *of* (3.16) *with initial value* (x_0, v_0) *at* $t = 0$ *satisfies* $(t, x(t), v(t)) \in M^*(z^*)$ *and*

$$|(x(t), v(t) - \eta(t, z^*))| \leqq \Gamma_0 |x_0| \, \mathrm{e}^{-\Gamma t}$$

for all $t \geqq 0$.

The positive constants Γ_0, Γ *do not depend on the chosen initial value. In particular one has* $(x, v)(t) \in \mathcal{N}$ *for all* $t \geqq 0$ *provided* δ_0 *is sufficiently small.*

In the next lemma we present the existence of an asymptotic phase for system (3.16) under the condition (3.19).

Lemma 3.8. *Under the notations and assumptions of* (3.16)–(3.19) *there exist positive constants* ε_1, μ_1 *and* Δ_1^* *such that the following is true:*

If $\varepsilon < \varepsilon_1$ *and* $\mu < \mu_1$ *in* (3.17) *and if* $\Delta_1 < \Delta_1^*$ *in* (3.19) *then there exists a* $\delta_0^* > 0$ *with the following property:*

If $|x_0| < \delta_0^*$, $|v_0| < \delta_0^*$ *then there exists a* $z_0 = z_0(x_0, v_0)$ *such that the solution* $(x, v)(t)$ *of* (3.16) *through* (x_0, v_0) *at* $t = 0$ *and the solution* $\eta(t, z_0)$ *of* (3.18) *satisfy*

$$|(x(t), v(t) - \eta(t, z_0))| \leqq \Gamma_0 |x_0| \, \mathrm{e}^{-\Gamma t}, \quad t \geqq 0 .$$

The positive constants Γ_0, Γ *do not depend on the chosen initial value. In particular one has* $(x, v)(t) \in \mathcal{N}$ *for all* $t \geqq 0$ *provided* δ_0^* *is sufficiently small.*

Proof. The proof is rather involved because the system (3.21) above need not be in $C^{1,\mathrm{Lip}}$ with respect to z.

First step: To overcome the problem that, in general, $\eta(t, z)$ and thus the right-hand side of (3.21) do not possess a Lipschitz constant with respect to z which holds uniformly in $t \geqq 0$ we need some modifications of the right-hand side of (3.16). We first replace the matrix B by

$$B(t, \lambda) = B - \frac{\varrho}{3} \psi(t - \lambda) I , \quad t \in \mathbb{R} , \quad \lambda \geqq 0 , \tag{3.23}$$

where ψ is an increasing scalar C^∞-function with

$$\psi(t) \equiv 0 \quad \text{for} \quad t \leqq 0 , \qquad \psi(t) \equiv 1 \quad \text{for} \quad t \geqq 1 .$$

We note $B(t, \lambda) = B(t + \lambda', \lambda + \lambda')$ for all $t \geqq 0$ and all $\lambda, \lambda' \geqq 0$. The transition matrices $\tilde{\Phi}(t, \tau)$ of $\dot{x} = Ax$ and $\tilde{\Psi}(t, \tau, \lambda)$ of $\dot{v} = B(t, \lambda) v$ then satisfy uniform estimates of the form

$$\begin{aligned} \|\tilde{\Phi}(t, \tau)\| &\leqq \tilde{\gamma}_1 \mathrm{e}^{-\varrho_1 (t - \tau)} , \quad t \geqq \tau , \\ \|\tilde{\Psi}(t, \tau, \lambda)\| &\leqq \tilde{\gamma}_2 \mathrm{e}^{-\varrho_2 (t - \tau)/2} , \quad t \leqq \tau . \end{aligned} \tag{3.24a}$$

Next we choose numbers ϱ_1, r, ϱ_2 with $\varrho/2 < \varrho_1 < r < \varrho_2 < \varrho$ and $\varepsilon_0 \in (0, 1)$ such that for any continuous matrices $A_1(t), B_1(t)$ with $\|A_1(t)\| \leqq \varepsilon$, $\|B_1(t)\| \leqq \varepsilon$ the transition matrices $\Phi(t, \tau)$, $\Psi(t, \tau, \lambda)$ of

$$\dot{x} = (A + A_1(t))\,x\,, \quad \dot{v} = [B(t, \lambda) + B_1(t)]\,v \quad \text{resp.}$$

satisfy the uniform estimates

$$\|\Phi(t, \tau)\| \leqq \gamma_1 e^{-\varrho_2(t-\tau)}\,, \quad t \geqq \tau\,, \tag{3.24b}$$
$$\|\Psi(t, \tau, \lambda)\| \leqq \gamma_2 e^{-\varrho_1(t-\tau)}\,, \quad t \leqq \tau\,.$$

We now turn to the modification of the functions P and Q. We replace them in the usual way by $\tilde{P}$, $\tilde{Q}$ (cf. [6, pp. 214/5]) so that $P = \tilde{P}$, $Q = \tilde{Q}$ for $|x| < a_1$, $|v| < 2a_1$ and so that the suprema of $|\tilde{P}_x|$, $|\tilde{P}_v|$, $|\tilde{Q}_x|$, $|\tilde{Q}_v|$ are less $\varepsilon_1 < \varepsilon_0/2$ for all (x, v) with $|x| < a_1$, $|v| < 2a_1$. ε_1 is to be so small that the system (2.20) associated to

$$\dot{x} = Ax + \tilde{P}(x, v) \tag{3.25}$$
$$\dot{v} = B(t, \lambda)\,v + \tilde{Q}(x, v) - \tilde{Q}(x, 0)$$

satisfies the global assumptions there without further modifications in (x, v). Thereby ϱ in (2.20) is to be replaced by r now .Note that (3.24) may replace the eigenvalue condition in (2.15). All this can be achieved if ε in (3.17) is less then a certain $\tilde{\varepsilon} > 0$. By the Theorems 2.3 and 2.1 the function $v = \tilde{\sigma}(t, x, \lambda)$ representing the integral manifold for (3.25) satisfies estimates of the form

$$\sup|\tilde{\sigma}| \leqq K' \sup|\tilde{Q}|\,, \quad K' > 1\,,$$
$$\sup\|\tilde{\sigma}_x\| \leqq K \sup\|\tilde{Q}_x\|\,, \quad K > 1\,,$$

where K holds uniformly in $\lambda \geqq 0$. By choosing appropriate $\varepsilon^* \in (0, \tilde{\varepsilon}]$ and $a_2 \in (0, a_1]$ we can achieve the following:

$$P = \tilde{P}\,, \quad Q = \tilde{Q} \quad \text{for} \quad |x| < a_2\,, \quad |v| < 2a_2\,,$$
$$|\tilde{P}_x|,\ |\tilde{P}_v|,\ |\tilde{Q}_x|,\ |\tilde{Q}_v| \text{ are less } \varepsilon_1/4K \text{ for all } (x, v)\,.$$

So much for the modifications.

Now let $\eta(t, z, \lambda)$ denote the solution of

$$\dot{v} = B(t, \lambda)\,v + Q(0, v)\,, \quad v(0) = z\,. \tag{3.26}$$

The above modifications imply that the solution matrix $W(t, z, \lambda)$ of

$$\dot{W} = [B(t, \lambda) + \tilde{Q}_v(0, \eta(t, z, \lambda))]\,W,\ W(0) = I\,,$$

is bounded by some λ-dependent bound which holds uniformly in $t \geqq 0$, $|z| < \Delta_0$. To conclude that the Lipschitz condition

$$|\eta(t, z_1, \lambda) - \eta(t, z_2, \lambda)| \leqq L(\lambda)\,|z_1 - z_2|$$

holds uniformly in $t \geqq 0$ and $|z| \leqq \Delta_0' < \Delta_0$ we only need to show that any solution of (3.26) with $|z| \leqq \Delta_0'$ satisfies $|\eta(t, z, \lambda)| < a_2$ for all $t \geqq 0$, $\lambda \geqq 0$. But the existence of such a $\Delta_0' > 0$ is guaranteed if μ in (3.17) is less than a certain $\mu' > 0$. This follows from condition (3.19) and the properties derived for $B(t, \lambda)$.

Second step: We put

$$(3.27) \qquad y = v - \eta(t, z, \lambda), \quad t \geqq 0, \quad |z| \leqq \Delta_0', \quad \lambda \geqq 0,$$

and construct an integral manifold for the system corresponding to (3.21), i.e. for

$$(3.28) \qquad \begin{aligned} \dot{x} &= Ax + \tilde{P}(x, y + \eta(t, z, \lambda)) \\ \dot{y} &= B(t, \lambda)\, y + q(t, x, y, z, \lambda) \end{aligned}$$

for $t \geqq 0$ where q is given by

$$q(t, x, y. z, \lambda) = \tilde{Q}(x, y + \eta(t, z, \lambda)) - \tilde{Q}(x, \eta(t, z, \lambda)).$$

An application of the proof of Theorem 2.3 yields the following result for $|z| \leqq \Delta_0'$, $\lambda \geqq 0$:

There exists a function $y = \sigma(t, x, z, \lambda)$ belonging to $C^{1,\mathrm{Lip}}$ with respect to x such that

$$M(z, \lambda) = \{(t, x, y) : y = \sigma(t, x, z, \lambda),\ t \geqq 0\}$$

defines an integral manifold for (3.28). Moreover σ has the following properties:

$$\sigma(t, 0, z, \lambda) = 0, \quad \sup \|\sigma_x\| \leqq \varepsilon_1/2.$$

If we restrict the (x, y)-region to $|x| < a_2$, $|y| < a_2$ we obtain a local integral manifold for $|x|$ less than a suitable $\delta \in (0, a_2]$ with the property:

There exists a $\delta_0 > 0$ such that $|x_0| < \delta_0$ and $y_0 = \sigma(0, x_0, z, \lambda)$ imply that the solution $(x, y)(t)$ of (3.28) through (x_0, y_0) at $t = 0$ stays on $M(z, \lambda)$ for all $t \geqq 0$ and satisfies $|(x, y)(t)| \leqq \Gamma_0 |x_0| \, \mathrm{e}^{-\Gamma t}$ for all $t \geqq 0$.

We note that the above relations hold uniformly in (z, λ) with $|z| \leqq \Delta_0'$, $\lambda \geqq 0$.

Third step: For each fixed $\lambda \geqq 0$ we now determine the asymptotic phase of the solutions $(x, v)(t, \lambda)$ of

$$(3.29) \qquad \begin{aligned} \dot{x} &= Ax + P(x, v) \\ \dot{v} &= B(t, \lambda)\, v + Q(x, v) \end{aligned}$$

with sufficiently small initial data. To this end we want to associate to each (x_0, v_0) an initial value $z_0 = z^*(x_0, v_0, \lambda)$ for (3.26) such that the solution $(x, v)(t, \lambda)$ of (3.29) with initial value (x_0, v_0) tends exponentially to $(0, \eta(t, z_0, \lambda))$ as $t \to \infty$. For this purpose we need to solve

$$(3.30) \qquad v_0 - z = \sigma(0, x_0, z, \lambda)$$

for z as a function of (x_0, v_0, λ). In order to apply Rouché's theorem from degree theory we first need to prove that σ is continuous in z. This now follows from Remark 2.2 (3) since the expression $c + \bar{\kappa}\gamma_g$ there can be estimated here by $\varepsilon_1/2K + \varepsilon_1/2 \,.\, \varepsilon_1/2K < \varepsilon_1 < \varepsilon_0/2$. Thus we have

$$|\sigma(t, x, z_1, \lambda) - \sigma(t, x, z_2, \lambda)| \leqq C(\lambda)\, |z_1 - z_2|$$

where we have incorporated the constants K', K and the Lipschitz constants for $\tilde{P}$, $\tilde{Q}$ and η into $C(\lambda)$. Thus Rouché's theorem is applicable to (3.30) if we restrict v to $|v| \leqq \Delta_0'/3$ and x to $|x| < \delta_1$ where $\delta_1 > 0$ is chosen in such a way that

$$|\sigma(0, x, z, \lambda)| \leqq \Delta_0'/3 \quad \text{for all} \quad |x| < \delta_1\,, \quad |z| \leqq \Delta_0'\,, \quad \lambda \geqq 0\,.$$

That δ_1 can be chosen independently of λ follows from the properties of σ. Therefore to each (x_0, v_0) with $|x_0| < \delta_1$ and $|v_0| < \Delta_0'/3$ we can find a solution $z_0 = z^*(x_0, v_0, \lambda)$ of (3.30) in $|z| < \Delta_0'$. Because of our results in step 2 the solution $(x, v)(t, \lambda)$ of (3.29) with such an initial value approaches the solution $(0, \eta(t, z_0(\lambda))$ as $t \to \infty$:

$$(3.31) \qquad |(x(t, \lambda), v(t, \lambda) - \eta(t, z_0, \lambda))| \leqq \Gamma_0 |x_0| \, e^{-\Gamma t}\,, \quad t \geqq 0\,.$$

Final step: We choose a sequence (λ_n), $\lambda_n \to \infty$ as $n \to \infty$, and assume without loss of generality that a corresponding sequence $(z_{0n}) = (z^*(x_0, v_0, \lambda_n))$ converges to z_0^*, $|z_0^*| \leqq \Delta_0'$, for $n \to \infty$. We fix an interval $[0, T]$. Then for n greater than a certain n_0, $(x, v)(t, \lambda_n)$ and $\eta(t, z_{0n}, \lambda_n)$ are not only solutions of (3.28) and (3.26), but also of the original equations (3.16) and (3.18) respectively on the interval $[0, T]$. Moreover they satisfy the estimate (3.21) where Γ_0 and Γ neither depend on n nor on T. For $n \to \infty$ these solution converge on $[0, T]$ to solutions $(x, v)(t)$ and $\eta(t, z_0^*)$ of (3.16) and (3.18) respectively with initial values $(x, v)(0) = (x_0, v_0)$ and $\eta(0, z_0^*) = z_0^*$. Therefore we have arrived at

$$|(x(t), v(t) - \eta(t, z_0^*))| \leqq \Gamma_0 |x_0| \, e^{-\Gamma t} \quad \text{for all} \quad t \geqq 0$$

which implies for $|x_0| < \min(\delta_1, (a - a_2)/\Gamma_0)$ that $(x, v)(t)$ belongs for all $t \geqq 0$ to the neighborhood $\mathcal{N}$. ∎

4. AN APPLICATION

We keep the notations of Section 3 and consider a system (3.1) satisfying the hypotheses of Theorem 3.1. Moreover we assume that the manifold $\bar{M}$ in (3.4) is a singular $C^{1,\mathrm{Lip}}$-curve consisting of stationary points. Then the reduced equation on the manifold M of (3.8) will be of the form

$$(4.1) \qquad \begin{aligned} \dot{x} &= Ax + X(x, v)\,, \\ \dot{v} &= \qquad V(x, v)\,. \end{aligned}$$

The functions X and V will be defined in a neighborhood $G = \{(x, v) \in \mathbb{R}^k \times \mathbb{R}^1 : |x| < \varepsilon, |v| < \varepsilon\}$ and will vanish for $(x, v) = (0, v)$. For simplicity we assume that X and V are of order $O(|x|^2 + |v|^2)$ for $(x, v) \to (0, 0)$.

Proposition 4.1. *Under the above assumptions let* $L = \{(x, v) : x = 0, |v| < \varepsilon_1\}$ *correspond to a piece of the singular curve* $\bar{M}$ *and let* $\varepsilon_1 > 0$ *be sufficiently small.*

Then L is an attractor for (4.1) and the region of attraction is open in G. In particular, each solution $(x, v)(t)$ of (4.1) having an ω- limit point $(0, \hat{v})$ on L tends to $(0, \hat{v})$ for $t \to \infty$.

Proof. We fix any v^* with $|v^*| < \varepsilon_1$ and perform the transformation $w = v - v^* - z$, $|v^* + z| < \varepsilon_1$. This leads to a system of the form

$$\begin{aligned}\dot{x} &= Ax + p(x, w, z) \\ \dot{w} &= \qquad\quad q(x, w, z) \\ \dot{z} &= 0\end{aligned} \tag{4.2}$$

where p and q vanish for $(0, w, z)$ and where $p_x(0, 0, 0)$, $q_x(0, 0, 0)$ are given by $X_x(0, v^*)$ and $V_x(0, v^*)$ respectively. Thus for sufficiently small ε_1 Theorem 3.1 yields a local $C^{1,\mathrm{Lip}}$ invariant manifold

$$\{(x, w, z) : w = W(x, z),\ |x| < \Delta,\ |z| < \Delta\} \tag{4.3}$$

for an appropriate $\Delta > 0$. In the (x, v)-coordinates of (4.1) the manifold (4.3) reads as

$$M^* = \{(x, v) : v = \sigma(x, z) := v^* + z + W(x, z),\ |x| < \Delta,\ |z| < \Delta\}.$$

We now show that M^* furnishes a foliation of a neighborhood $U(L)$ of the form

$$U(L) = \{(x, v) : |x| < \eta(v),\ |v| < \varepsilon_1\}, \quad \eta(v) > 0$$

into stable manifolds. To this end we consider a pair (x_0, v_0) with sufficiently small $|x_0|$, $|v_0 - v^*|$ and prove the existence of a z_0 depending on (x_0, v_0) such that the solution $(x, v)(t)$ of (4.1) with initial value $(x, v)(0) = (x_0, v_0)$ belongs to the sheet

$$M^*_{z_0} = \{(x, v) : v = \sigma(x, z_0),\ |x| < \Delta\}$$

for $t \geqq 0$. In order to see this it suffices to note that $F^*(x, v, z) := v - \sigma(x, z)$ belongs to $C^{1,\mathrm{Lip}}$ and satisfies $F^*(0, v^*, 0) = 0$, $F^*_z(0, v, 0) = -1$. Then the implicit function theorem implies the existence of a $\delta > 0$ such that for $|x| < \delta$, $|v - v^*| < \delta$, $|z| < \delta$ there exists a unique solution $z = z^*(x, v)$ of $F^*(x, v, z) = 0$. Thus $z_0 = z^*(x_0, v_0)$ determines the sheet $M^*_{z_0}$ in which the solution $(x, v)(t)$ through (x_0, v_0) stays for all $t \geqq 0$ with $\lim_{t\to\infty} (x, v)(t) = (0, v^* + z_0)$. Since v^* has been an arbitrary choice we have shown the existence of a positive function $\eta(v)$, $|v| < \varepsilon_1$, such that $U(L)$ is filled with sheets M^*_z of M^*.

Now let $(x, v)(t)$ be a solution of (4.1) with an ω-limit point in L. Thus there exists a finite time $T = T(x(0), v(0))$ such that $(x, v)(T)$ belongs to $U(L)$. Any neighborhood $\mathcal{N}$ of $(x, v)(t)$ belonging to $U(L)$ then yields a full neighborhood $\mathcal{N}_0$ of (x_0, v_0) belonging to the region of attraction of L. $\mathcal{N}_0$ can be obtained by mapping $\mathcal{N}$ by the flow of (4.1) backwards by time T. Thereby proposition 4.1 is proved. ∎

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