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NOTE ON GENERALIZED MULTIPLE PERRON INTEGRAL

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We show that every real-valued function which is GP-integrable in the sense of [2] must be Lebesgue measurable. Using this result we obtain a dominated convergence theorem for the GP-integral which answers the question posed in [2], Remark 3 (cf. also Remark 11 in [1]).

By an interval (in $\mathbb{R}^m$) we mean a Cartesian product of m closed one-dimensional intervals of positive length. Given such an interval I we choose a cube $K \supset I$ of minimal volume and put

$$r(I) = \operatorname{vol} I / \operatorname{vol} K ;$$

if a point $x \in I$ has been specified in I, then the interval is termed a pointed interval and will be denoted by (x, I). By a P-patrition of an interval J we mean any finite system

$$(x^1, I^1), \ldots, (x^p, I^p) \tag{1}$$

of mutually non-overlapping pointed intervals whose union equals J. If $x = [x_1, \ldots, x_m] \in \mathbb{R}^m$ and $\varrho > 0$, then we adopt the notation

$$B[x, \varrho] = \mathop{\mathsf{X}}_{j=1}^{m} \langle x_j - \varrho, x_j + \varrho \rangle$$

for the cube of side-length 2ϱ centered at x. A positive function on J is called a gauge. If δ is a gauge on J, then the P-partition (1) is termed δ-fine provided

$$I^j \subset B[x^j, \delta(x^j)] , \quad j = 1, \ldots, p . \tag{2}$$

Let now J be a fixed interval and consider a real-valued function

$$F : I \mapsto F(I)$$

of an interval $I \subset J$. Given $x \in J$, $\alpha \in (0, 1\rangle$ and $\varrho > 0$, we put

$$_*F^{\varrho}_{\alpha}(x) = \inf_I F(I)/\operatorname{vol} I ,$$

where $I \subset J$ runs over all intervals satisfying

$$x \in I \subset B[x, \varrho] , \quad r(I) \geqq \alpha ;$$

further we define

$$_*F_\alpha(x) = \sup_{\varrho>0} {_*F_\alpha^\varrho}(x)\,, \quad _*F(x) = \inf_{0<\alpha\leqq 1} {_*F_\alpha}(x)\,,$$

$$^*F(x) = -_*(-F)(x)\,.$$

If $^*F(x) = {_*F}(x) \in \mathbb{R}$, then F is said to be derivable at x and the common value of $^*F(x)$ and $_*F(x)$ is denoted by $F'(x)$ and termed the derivative of F at x.

Let us recall, for the case of real-valued functions, the definition of the GP-integral from [2].

Definition. We say that a real-valued function f on J is GP-*integrable over* J if there exists a real number k satisfying the following condition:

For any $\varepsilon > 0$ and $\alpha \in (0, 1)$ there exists a gauge δ on J such that

$$\left|k - \sum_{j=1}^{p} f(x)^j \operatorname{vol} I^j\right| < \varepsilon$$

holds for each δ-fine P-partition (1) of J fulfilling

$$r(I^j) \geqq \alpha\,, \quad j = 1, \ldots, p\,. \tag{3}$$

The corresponding k is called the GP-*integral of* f *over* J and denoted by

$$\mathrm{GP}\int_J f\,. \tag{4}$$

Remark 1. Let us recall some basic facts established in [2].

The existence of the integral (4) guarantees that GP $\int_I f$ exists for each interval $I \subset J$ and

$$I \mapsto \mathrm{GP}\int_I f \tag{5}$$

is an additive function of an interval $I \subset J$.

If f is a function on J with a convergent Lebesgue integral

$$\mathrm{L}\int_J f\,, \tag{6}$$

then the integral (4) exists as well and coincides with (6).

For later use let us rephrase Proposition 9 from [2] in the following form.

Saks-Henstock lemma. *Let f be a real-valued function which is* GP-*integrable over J and suppose that $\varepsilon > 0$, $\alpha \in (0, 1)$. If δ is a gauge on J corresponding to ε and α as in the above definition, then*

$$\left|\sum_{j=1}^{p}\left[\mathrm{GP}\int_{I^j} f - f(x^j)\operatorname{vol} I^j\right]\right| < \varepsilon$$

holds for each finite system of mutually non-overlapping pointed intervals (1) *in J fulfilling the conditions* (2), (3).

Proof. If (1) is any system of non-overlapping pointed intervals in J satisfying (2), (3), then we can complete $I^1, \ldots, I^p$ by adding some intervals $I^{p+1}, \ldots, I^{p+q}$ so as to get a partition $I^1, \ldots, I^{p+q}$ of J formed by mutually non-overlapping intervals. If $r(I^n) < \alpha$ for some n, then I^n can be further subdivided into non-overlapping intervals I_t^n with $r(I_t^n) \geqq \alpha$.

Finally, each interval I of the new partition which did not occur in the original system $\{I^1, \ldots, I^p\}$ can be replaced by its δ-fine P-partition which is formed by intervals $\tilde{I}$ similar to I, so that $r(\tilde{I}) = r(I) \geqq \alpha$. In such a way we arrive at a δ-fine P-partition of J including the given system (1) and to this P-partition Proposition 9 from [2] applies.

Theorem 1. *Let f be* GP-*integrable over J. Then* (5) *is a function of an interval $I \subset J$ which is derivable at almost every $x \in J$ and its derivative coincides with f a.e. in J; in particular, f is Lebesgue measurable. If, moreover, the Lebesgue integral of f over J exists, then it necessarily converges and* (6) *coincides with* (4).

Corollary. *For any non-negative real-valued function f on J, the existence of the* GP-*integral* (4) *implies the convergence of the Lebesgue integral* (6) *and the equality of both.*

Proof. Let us denote by F the function of an interval $I \subset J$ defined by (5). Fix an arbitrary $\alpha \in (0, 1)$ and $\varrho > 0$ and consider the set

$$M_\varrho = \{x \in J;\ {}_*F_\alpha(x) \leqq f(x) - 2\varrho\}\ .$$

Admitting that the outer Lebesgue measure of M_ϱ equals $2\sigma > 0$ we choose $\varepsilon > 0$ small enough to guarantee that

$$\varepsilon < \varrho\sigma\ . \tag{7}$$

Now let δ be a gauge on J corresponding to ε and α as in the above definition. Associating with each $x \in M_\varrho$ the system of all intervals $I \subset J$ satisfying the conditions

$$x \in I \subset B[x, \delta(x)]\ , \quad r(I) \geqq \alpha\ , \quad F(I)/\mathrm{vol}\, I \leqq f(x) - \varrho\ ,$$

we obtain, as x runs over M_ϱ, a system of intervals which covers M_ϱ in the sense of Vitali. By Vitali's covering theorem, there exists a finite disjoint subsystem of pointed intervals (1) satisfying (2), (3) such that

$$\sum_{j=1}^{p} \mathrm{vol}\, I^j \geqq \sigma\ .$$

Employing the Saks-Henstock lemma we arrive at

$$\varepsilon > \sum_{j=1}^{p} [f(x^j)\, \mathrm{vol}\, I^j - F(I^j)] \geqq \varrho \sum_{j=1}^{p} \mathrm{vol}\, I^j \geqq \varrho\sigma$$

which contradicts (7). Thus each M_ϱ has vanishing Lebesgue measure and, in particular, the same is true for

$$M_\infty = \{x \in J;\ {}_*F_\alpha = -\infty\}\ .$$

By Ward's theorem (cf. [3], p. 139), F is derivable at almost all points in $J \setminus M_\infty$, i.e. almost everywhere in J. We have seen that the derivative satisfies the inequality

$$F' \geqq f \quad \text{a.e.} \quad \text{in } J .$$

Since f may be replaced by $-f$, we have $F' = f$ a.e. in J which means that f is Lebesgue measurable (cf. Theorem (4.2) in [3], p. 112).

If $f \geqq 0$, then the corresponding F is a non-negative additive function of an interval whose derivative $F'(= f \text{ a.e.})$ is known to be Lebesgue summable (cf. Theorem (7.4) in [3], p. 119); consequently, (6) is convergent and coincides with (4).

If f is of variable sign and its Lebesgue integral exists, then at least one of the functions $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$ must have a convergent Lebesgue integral; let it be f^+. The equality $f^- = f^+ - f$ implies that f^- is GP-integrable and, being non-negative, must also have a convergent Lebesgue integral.

As a consequence of Theorem 1 we get the following dominated convergence theorem for the GP-integral.

Theorem 2. *Let* $\{f_n\}$ *be a pointwise convergent sequence of* GP-*integrable functions over* J. *If there exist* GP-*integrable functions* g, h *such that*

$$g \leqq f_n \leqq h$$

on J *for all* n, *then* $f = \lim_n f_n$ *is also* GP-*integrable over* J *and*

$$\mathrm{GP}\int_J f = \lim_n \mathrm{GP}\int_J f_n .$$

Proof. We know that all the functions $f_n - g \geqq 0$ are Lebesgue summable and are dominated by $h - g$ which is Lebesgue summable as well. As $n \to \infty$, $f_n - g \to f - g$ pointwise on J, whence it follows by the Lebesgue dominated convergence theorem that

$$\mathrm{GP}\int_J f_n - \mathrm{GP}\int_J g = \mathrm{L}\int_J (f_n - g) \to \mathrm{L}\int_J (f - g) = \mathrm{GP}\int_J f - \mathrm{GP}\int_J g .$$

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