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COMPARISON OF VARIOUS DISTANCES BETWEEN ISOMORPHISM CLASSES OF GRAPHS

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We shall compare three types of distances between isomorphism classes of graphs. (An isomorphism class of graphs is the class of all graphs which are isomorphic to a given graph.) These distances were introduced in [1], [2] and [3]. We consider finite undirected graphs without loops and multiple edges.

Let $\mathfrak{G}_2, \mathfrak{G}_2$ be two isomorphism classes of graphs with the same number n of vertices. The distance $\delta(\mathfrak{G}_1, \mathfrak{G}_2)$ introduced in [2] is equal to n minus the maximum number of vertices of a graph which is isomorphic to an induced subgraph of a graph from $\mathfrak{G}_1$ and simultaneously to an induced subgraph of a graph from $\mathfrak{G}_2$.

Let $\mathfrak{T}_1, \mathfrak{T}_2$ be two isomorphism classes of trees with the same number n of vertices. The distance $\delta_T(\mathfrak{T}_1, \mathfrak{T}_2)$ introduced in [3] is equal to n minus the maximum number of vertices of a tree which is isomorphic to a subtree of a tree from $\mathfrak{T}_1$ and simultaneously to a subtree of a tree from $\mathfrak{T}_2$.

Let again $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of graphs with the same number n of vertices and, moreover, with the same number m of edges. The edge rotation distance between $\mathfrak{G}_1$ and $\mathfrak{G}_2$ was introduced in [1] and will be denoted here by $\delta_e(\mathfrak{G}_1, \mathfrak{G}_2)$. Let v_0, v_1, v_2 be three distinct vertices of a graph G such that v_0 is adjacent to v_1 and not adjacent to v_2. If we delete the edge v_0v_1 from G and add the edge v_0v_2 to$G$, we say that we perform an edge rotation. The distance $\delta_e(\mathfrak{G}_1, \mathfrak{G}_2)$ is equal to the minimum number of edge rotations which are necessary in order to obtain a graph belonging to $\mathfrak{G}_2$ from a graph belonging to $\mathfrak{G}_1$.

In some cases we shall use symbols like $\delta(G_1, G_2)$, $\delta_T(G_1, G_2)$, $\delta_e(G_1, G_2)$, where G_1, G_2 are graphs; such a symbol denotes the corresponding distance between isomorphism classes to which the graphs G_1, G_2 belong.

We shall prove some theorems.

Theorem 1. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of graphs with the same number n of vertices and the same number m of edges. Then*

$$\delta(\mathfrak{G}_1, \mathfrak{G}_2) \leqq \delta_e(\mathfrak{G}_1, \mathfrak{G}_2)$$

and the equality may occur.

Proof. If $\delta_e(\mathfrak{G}_1, \mathfrak{G}_2) = 1$, then there exist graphs $G_1 \in \mathfrak{G}_1$ and $G_2 \in \mathfrak{G}_2$ such that G_2 is obtained from G_1 by an edge rotation. The graphs G_1 and G_2 have the same vertex set V and there exist vertices v_0, v_1, v_2 of V such that v_0 is adjacent to v_1 and non-adjacent to v_2 in G_1 and adjacent to v_2 and non-adjacent to v_1 in G_2, while any other pair of vertices is either adjacent in both G_1, G_2, or non-adjacent in both G_1, G_2. The set $V - \{v_0\}$ induces the same subgraph in both G_1 and G_2 and thus $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = \delta_e(\mathfrak{G}_1, \mathfrak{G}_2) = 1$. Now let k be an integer, $k \geqq 2$, and let $\delta_e(\mathfrak{G}_1, \mathfrak{G}_2) = k$. There exist graphs $H_0, H_1, \ldots, H_k$ such that $H_0 \in \mathfrak{G}_1$, $H_k \in \mathfrak{G}_2$ and the graph H_i is obtained from H_{i-1} by an edge rotation for $i = 1 \ldots, k$. We have $\delta_e(H_{i-1}, H_i) = 1$ and thus also $\delta(H_{i-1}, H_i) = 1$ for $i = 1, \ldots, k$. Inductively from the triangle inequality we obtain $\delta(H_0, H_k) = \delta(\mathfrak{G}_1, \mathfrak{G}_2) \leqq k = \delta_e(\mathfrak{G}_1, \mathfrak{G}_2)$. ■

Theorem 2. *Let N be a positive integer. Then there exist isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of graphs such that*

$$\delta_e(\mathfrak{G}_1, \mathfrak{G}_2) - \delta(\mathfrak{G}_1, \mathfrak{G}_2) = N .$$

Proof. We shall construct graphs $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$ with a common vertex set $V = \{u_1, \ldots, u_{N+1}, v_1, \ldots, v_{N+1}, w\}$. In G_1 the set $\{u_1, \ldots, u_{N+1}, w\}$ induces a clique and the vertices $v_1, \ldots, v_{N+1}$ are isolated. In G_2 two vertices are adjacent if and only if either they both belong to the set $\{u_1, \ldots, u_{N+1}\}$, or one of them is w and the other belongs to the set $\{v_1, \ldots, v_{N+1}\}$. Evidently each of the graphs G_1, G_2 has $\frac{1}{2}(N + 1) . (N + 2)$ edges. The set $V - \{w\}$ induces the same subgraph in both G_1 and G_2, hence $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = 1$. The graph G_2 can be obtained from G_1 by $N + 1$ edge rotations; at each of them we delete the edge $u_i w$ and add the edge $v_i w$ for some $i \in \{1, \ldots, N + 1\}$. If we perform less than $N + 1$ edge rotations, starting at G_1, then at least one of the vertices $v_1, \ldots, v_{N+1}$ remains isolated and no graph isomorphic with G_2 is obtained. Hence $\delta_e(\mathfrak{G}_1, \mathfrak{G}_2) = N + 1$ and the assertion holds. ■

Before proving the next theorem, we shall prove a lemma.

Lemma. *Let T be a finite tree with the edge set E, let T_0 be its proper subtree with the edge set E_0. Then there exists a bijection f of the set $E - E_0$ onto the number set $\{1, \ldots, |E - E_0|\}$ with the property that the set $E_i = E_0 \cup \{e \in E - E_0 \mid f(e) \leqq i\}$ is the edge set of a subtree of T for each $i = 1, \ldots, |E - E_0|$.*

Proof. We shall carry out the proof by induction according to the cardinality of $E - E_0$. If this cardinality is equal to one, then $E_1 = E$ and the assertion holds trivially. Now let $k \geqq 2$ and suppose that the assertion is true for $|E - E_0| \leqq k - 1$. There exists at least one edge $e_1 \in E - E_0$ which has one end vertex in T_0. Evidently $E'_0 = E_0 \cup \{e_1\}$ is the edge set of a subtree T'_0 of T. We have $|E - E'_0| = k - 1$ and by the induction hypothesis there exists a bijection f' of $E - E'_0$ onto $\{1, \ldots, k - 1\}$ such that the set $E'_i = E_0 \cup \{e \in E - E'_0 \mid f'(e) \leqq' i\}$ is the edge set of a subtree of T for each $i = 1, \ldots, k - 1$. We define a bijection f of $E - E_0$

onto $\{1, \ldots, k\}$ in such a way that $f(e_1) = 1$ and $f(e) = f'(e) + 1$ for each $e \in E - E'_0$. Then evidently $E_i = E'_{i-1}$ for $i = 1, \ldots, k$ and the assertion holds. ∎

Theorem 3. *Let $\mathfrak{T}_1, \mathfrak{T}_2$ be two isomorphism classes of trees with the same number n of vertices. Then*

$$\delta_e(\mathfrak{T}_1, \mathfrak{T}_2) \leqq \delta_T(\mathfrak{T}_1, \mathfrak{T}_2).$$

Proof. Let $\delta_T(\mathfrak{T}_1, \mathfrak{T}_2) = k$. This means that the maximum number of vertices of a tree which is isomorphic to a subtree of a tree from $\mathfrak{T}_1$ and simultaneously to a subtree of a tree from $\mathfrak{T}_2$ is equal to $n - k$. We may consider the trees $T_1 \in \mathfrak{T}_1$ and $T_2 \in \mathfrak{T}_2$ whose intersection is a tree T_0 with $n - k$ vertices. Let f_1 (or f_2) be a mapping corresponding to the mapping f from Lemma provided we consider T_1 (or T_2, respectively) instead of T. Both f_1 and f_2 are bijections onto the set $\{1, \ldots, k\}$. For each $i = 1, \ldots, k$ let $e_1(i)$ (or $e_2(i)$) be the edge which is mapped by f_1 (or f_2, respectively) onto the number i. The end vertices of $e_1(i)$ (or $e_2(i)$) will be denoted by $v_1(i)$ and $w_1(i)$ (or $v_2(i)$ and $w_2(i)$) in such a way that the distance of $w_1(i)$ (or $w_2(i)$) from a vertex of T_0 is greater than the distance of $v_1(i)$ (or $v_2(i)$, respectively) from the same vertex. Now we identify $w_2(i)$ with $w_1(k + 1 - i)$ for $i = 1, \ldots, k$. After this identification the trees T_1, T_2 have the same vertex set. For $i = 1, \ldots, k$ let $\mathscr{R}_i$ be the edge rotation which deletes the edge $e_1(i) = v_1(i)\, w_1(i)$ and adds the edge $e_2(k + 1 - i) = v_2(k + 1 - i)\, w_2(k + 1 - i) = v_2(k + 1 - i)\, w_1(i)$. If we start from the tree T_1 and subsequently perform the edge rotations $\mathscr{R}_1, \ldots, \mathscr{R}_k$, we obtain the tree T_2 and thus $\delta_e(\mathfrak{T}_1, \mathfrak{T}_2) \leqq k = \delta_T(\mathfrak{T}_1, \mathfrak{T}_2)$. ∎

Theorem 4. *Let N be a positive integer. Then there exist isomorphism classes $\mathfrak{T}_1, \mathfrak{T}_2$ of trees such that*

$$\delta_T(\mathfrak{T}_1, \mathfrak{T}_2) - \delta_e(\mathfrak{T}_1, \mathfrak{T}_2) = N.$$

Proof. We shall construct trees $T_1 \in \mathfrak{T}_1$, $T_2 \in \mathfrak{T}_2$ with a common vertex set $V = \{u_1, u_2, u_3, u_4, u_5, u_6, v_1, \ldots, v_{2N+4}, w_1, \ldots, w_{N+2}\}$. Both T_1 and T_2 contain the edges $u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6$ and u_2v_i for $i = 1, \ldots, 2N + 4$. Further, T_1 contains the edges u_4w_i and T_2 contains the edges u_5w_i for $i = 1, \ldots, N + 2$. No other edges than those described are contained in T_1 and T_2. The subtree T_0 induced in both T_1 and T_2 by the set $\{u_1, u_2, u_3, u_4, u_5, u_6, v_1, \ldots, v_{2N+4}\}$ has $2N + 10$ vertices; evidently no tree with more vertices can be isomorphic simultaneously to a subtree of T_1 and to a subtree of T_2. The set V contains $3N + 12$ vertices, hence $\delta_T(\mathfrak{T}_1, \mathfrak{T}_2) = N + 2$.

Now take the tree T_2. We perform the edge rotation which substitutes the edge u_2u_3 by u_2u_4. In the graph thus obtained we perform another edge rotation which substitutes the edge u_3u_4 by u_3u_6. The graph obtained by these two edge rotations will be denoted by T'_1. Define the bijection $f\colon V \to V$ in such a way that $f(u_3) = u_6$, $f(u_4) = u_3, f(u_5) = u_4, f(u_6) = u_5$ and $f(x) = x$ for each $x \in V - \{u_3, u_4, u_5, u_6\}$.

The mapping f is an isomorphism of T_1' onto T_1. Hence $T_1' \cong T_1$ and T_1' was obtained from T_2 by two edge rotations. Evidentla one edge rotation does not suffice and therefore $\delta_e(\mathfrak{T}_1, \mathfrak{T}_2) = 2$. ■

At the end of the paper we add some results concerning the edge rotation distance between isomorphism classes of trees.

Theorem 5. *Let S be a star with n vertices, let T be an arbitrary tree with n vertices. Let the maximum degree of a vertex of T be Δ. Then*

$$\delta_e(S, T) = n - 1 - \Delta .$$

Proof. Let u be a vertex of degree Δ in T. Evidently the subtree S_0 of T whose edge set is set of all edges incident with u is a star with $\Delta + 1$ vertices and is isomorphic to a subtree of S. As any subtree of S with at least 3 vertices is a star, the tree T cannot contain a subtree with more than $\Delta + 1$ vertices isomorphic to a subtree of S and $\delta_T(S, T) = n - 1 - \Delta$. According to Theorem 3 we have $\delta_e(S, T) \leqq \leqq n - 1 - \Delta$. As the maximum degree of a vertex of S is $n - 1$ and that of T is Δ, it is necessary to perform at least $n - 1 - \Delta$ edge rotations in order to obtain from T a graph isomorphic to S. (If they are exactly $n - 1 - \Delta$, then at each of them one edge is added to a vertex of the maximum degree.) We have $\delta_e(S, T) = = n - 1 - \Delta$. ■

Theorem 6. *Let P be a path with n vertices, let T be an arbitrary tree with n vertices. Let the diameter of T be d. Then*

$$\delta_e(P, T) = n - 1 - d .$$

Proof. Let P_0 be a diametral path in T. This is a subtree of T which is isomorphic to a subtree of P and has $d + 1$ vertices. As any subtree of P is a path, the tree T cannot contain a subtree with more than $d + 1$ vertices isomorphic to a subtree of P and $\delta_T(P, T) = n - 1 - d$. According to Theorem 3 we have $\delta_e(P, T) \leqq n - 1 - d$. As the diameter of P is $n - 1$ and that of T is d, it is necessary to perform at least $n - 1 - d$ edge rotations in order to obtain P from T. (If they are exactly $n - 1 - d$, then at each of them one edge is added to a vertex of degree 1.) We have $\delta_e(P, T) = = n - 1 - d$. ■

Corollary. *Let S be a star with n vertices, let P be a path with n vertices. Then*

$$\delta_e(P, S) = n - 3 .$$

In [3] it was proved that $\delta(P, S) = [n/2]$ at $n \geqq 7$. Hence $\delta_e(\mathfrak{T}_1, \mathfrak{T}_2)$ need not be equal to $\delta(\mathfrak{T}_1, \mathfrak{T}_2)$.

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