Časopis pro pěstování matematiky

Josef Janyška
Geometrical properties of prolongation functors

Časopis pro pěstování matematiky, Vol. 110 (1985), No. 1, 77--86

Persistent URL: http://dml.cz/dmlcz/118224

Terms of use:

© Institute of Mathematics AS CR, 1985

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Časopis pro pěstování matematiky, roč. 110 (1985), Praha

GEOMETRICAL PROPERTIES OF PROLONGATION FUNCTORS

JOSEF JANYŠKA, Brno

(Received November 4, 1983)

In the present paper we shall study geometrical properties of *prolongation functors*, Kolář [5]. Prolongation functors generalize *lifting functors* (= natural bundles) in the sense of Nijenhuis [9]. The properties of the natural bundles have been described by Epstein [1], Epstein, Thurston [2], Krupka [7, 8], Palais, Terng [10], Salvioli [11], Terng [12] and others. We shall describe some geometrical properties of the prolongation functors which generalize the known properties of the lifting functors.

All objects and maps are in the category C^∞.

1. Let $\mathcal{M}$ be the category of all manifolds and maps. Let $\mathcal{C}$ be a subcategory of $\mathcal{M}$ such that every inclusion $U \subset M$, where $M \in \mathrm{On}\, \mathcal{M}$ is any manifold and U is any open subset of M, is an element of $\mathrm{Hom}\, \mathcal{C}$. Let $\mathcal{FM}$ be the category of all fibre manifolds and all fibre manifold morphisms and let $B: \mathcal{FM} \to \mathcal{M}$ be the base functor.

Definition 1. *A prolongation functor on $\mathcal{C}$ is a covariant functor F from the category $\mathcal{C}$ into the category $\mathcal{FM}$ such that the following conditions are fulfilled:*

i) (*the prolongation condition*) $B \circ F = \mathrm{id}_{\mathcal{C}}$;

ii) (*the regularity condition*) *if $f: M \times P \to N$ is a smooth map such that $f(-, z) \in \mathrm{Hom}\, \mathcal{C}$ for all $z \in P$ and all $M, N \in \mathrm{Ob}\, \mathcal{C}$, then a map $\tilde{F}f: FM \times P \to FN$ defined by $\tilde{F}f(-, z) = F(f(-, z))$ is also smooth*;

iii) (*the localization condition*) *for any open subset $U \subset M$, $FU = p_M^{-1}(U)$ is fulfilled, $p_M: FM \to M$ is the projection, and the inclusion $i_{FU}: FU \to FM$ is the prolongation of the inclusion $i_U: U \to M$, i.e. $Fi_U = i_{FU}$.*

A prolongation functor on the whole category $\mathcal{M}$ will be called briefly a prolongation functor. If $\mathcal{C}$ is the category $\mathcal{M}_m$ of all m-dimensional manifolds and their embeddings, we shall call a prolongation functor on $\mathcal{M}_m$ a lifting functor (in dimension m). If F is a lifting functor then the quadruple (FM, p_M, F, M) is a natural bundle in the sense of Nijenhuis [9]. For lifting functors the prolongation property implies the regularity property, [2]. In the case when a prolongation functors has values in the category $\mathscr{V}\mathscr{B}$ of all vector bundles and all vector bundle morphisms, then the prolongation property implies the regularity property, [1].

Definition 2. *A prolongation functor F on $\mathscr{C}$ will be said to be of order r if for every $f, g \in \operatorname{Hom} \mathscr{C}$, $f, g: M \to N$, $j^r_x f = j^r_x g$ implies $Ff \mid F_x M = Fg \mid F_x M$, where $F_x M = p_M^{-1}(x)$ is the fibre over $x \in M$.*

We remark that the order of any lifting functor is finite, [10].

Examples. 1. The tangent functor T which associates the tangent bundle $TM \to M$ to a manifold M and the tangent map $Tf: TM \to TN$ to a map $f: M \to N$ is a prolongation functor of order one.

2. The functor T^r_k of k^r-velocities which associates the fibre bundle $T^r_k M = J^r_0(\mathbf{R}^k, M)$ to a manifold M and the map $T^r_k f: T^r_k M \to T^r_k N$, $T^r_k f(j^r_0 \varphi) = j^r_0(f \circ \varphi)$, to a map $f: M \to N$ is an r-th order prolongation functor.

3. The r-th order frame bundle functor H^r which associates the fibre principal bundle $H^r M = \operatorname{inv} J^r_0(\mathbf{R}^m, M)$, $m = \dim M$, to a manifold M and the map $H^r f$: $H^r M \to H^r \overline{M}$, $H^r f(j^r_0 \varphi) = j^r_0(f \circ \varphi)$, to a diffeomorphism $f: M \to \overline{M}$ is a lifting functor of order r.

From Definition 2 we obtain that every r-jet $A \in J^r_x(M, N)_z$ defines a map FA: $F_x M \to F_z N$ by $FA(y) = (Ff)(y)$, $y \in F_x M$, $A = j^r_x f$. If $B \in J^r_z(N, P)_p$, then the composition of jets yields $B \circ A \in J^r_x(M, P)_p$ and we obtain

$$F(B \circ A) = (FB) \circ (FA)$$

where the right hand side is the composition of maps FA and FB.

It is known, [2], that $F\mathbf{R}^m$ is diffeomorphic with $\mathbf{R}^m \times S_m$, where $S_m = (F\mathbf{R}^m)_0 = F_0 \mathbf{R}^m$ is the fibre over the origin in $\mathbf{R}^m$. This diffeomorphism is given by $u = (Ft_x)(s)$, where $u \in F\mathbf{R}^m$, $p_{\mathbf{R}^m}(u) = x$ and $s \in S_m$. t_x denotes the translation $y \mapsto y + x$. Then we have

Lemma 1. *FM is a local trivial fibre manifold with the standard fibre S_m.*

Proof. If (U, φ) is a local chart on M Then we have a sequence of diffeomorphisms

$$U \times S_m \xrightarrow{\varphi \times \mathrm{id}_{S_m}} \mathbf{R}^m \times S_m \to F\mathbf{R}^m \xrightarrow{F\varphi^{-1}} FU$$

and the composed map of $U \times S_m$ into FU is also a diffeomorphism, QED.

If $U \subset M$ is a coordinate chart with coordinates (x^i) and $V \subset S_m$ is a coordinate chart with coordinates (y^p), then from the diffeomorphism $U \times S_m \approx FU$ we obtain local fibre coordinates (x^i, y^p) on $U \times V \subset FM$, which we shall call *adapted coordinates.*

Let $A \in J^r(M, N)$ and $y \in FM$ be such that $\alpha(A) = p_M(y)$, where α: $J^r(M, N) \to M$ is the source projection. Then $FA(y) \in FN$ and $p_N(FA(y)) = \beta(A)$, where β: $J^r(M, N) \to N$ is the target projection. We have the so-called *associated map*

$$(1) \qquad F_{M,N}: J^r(M, N) \oplus FM \to FN ,$$

where $\oplus$ is Whitney's sum over M with respect to the projections α and p_M.

Theorem 1. *The associated map* (1) *is smooth.*

Proof. In the first step we shall prove Theorem 1 in the special case $M = \mathbf{R}^m$, $N = \mathbf{R}^n$. We note that $L^r(m, n) = J_0^r(\mathbf{R}^m, \mathbf{R}^n)_0$. In the coordinates (x^i) on $\mathbf{R}^m$ and (z^α) on $\mathbf{R}^n$, any $A \in L^r(m, n)$ is given by the coefficients of Taylor's series

$$(2) \qquad z^\alpha = a_i^\alpha x^i + \frac{1}{2!} a_{ij}^\alpha x^i x^j + \ldots + \frac{1}{r!} a_{i_1 \ldots i_r}^\alpha x^{i_1} \ldots x^{i_r} .$$

For $S_m = F_0 \mathbf{R}^m$ and $x = 0$, $y = 0$ the associated map (1) is in the form

$$(3) \qquad L^r(m, n) \times S_m \to S_n , \quad w = \varphi(A, s) = FA(s) ,$$

$s \in S_m$, $w \in S_n$, $A = j_0^r f$, $f(0) = 0$. For any $A \in L^r(m, n)$ Taylor's series (2) is a smooth map $\psi: L^r(m, n) \times \mathbf{R}^m \to \mathbf{R}^n$. From the regularity condition for the prolongation functor F we obtain a smooth map $\tilde{F\psi}: L^r(m, n) \times F\mathbf{R}^m \to F\mathbf{R}^n$ and its restriction to S_m is (3), hence the map (3) is smooth.

Now, let $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$ be arbitrary points. We can identify $J^r(\mathbf{R}^m, \mathbf{R}^n)$ with $\mathbf{R}^n \times L^r(m, n) \times \mathbf{R}^m$ by the rule $B = j_x^r f \approx (y, A, x)$, where $B \in J_x^r(\mathbf{R}^m, \mathbf{R}^n)_y$, $A = j_0^r(t_y^{-1} \circ f \circ t_x) \in L^r(m, n)$ and t_x is the translation $u \mapsto u + x$. We have the identification $\mathbf{R}^m \times S_m \approx F\mathbf{R}^m$, $(x, s) \approx (Ft_x)(s)$. Then every element $B \in J_x^r(\mathbf{R}^m, \mathbf{R}^n)_y$ can be identified with $j_x^r(t_y \circ g \circ t_x^{-1})$, $g(0) = 0$, and every element $u \in F\mathbf{R}^m$ can be identified with $(Ft_x)(s)$, $p_{\mathbf{R}^m}(u) = x$, for some $s \in S_m$. Then

$$FB(u) = F(t_y \circ g \circ t_x^{-1})((Ft_x)(s)) = (Ft_y)((FA)(s)) = (y, \varphi(A, s)) ,$$

where $A = j_0^r g \in L^r(m, n)$. Hence $F_{\mathbf{R}^m, \mathbf{R}^n}$ is in the form

$$(4) \qquad (B, u) \mapsto (y, \varphi(A, (Ft_{p_{\mathbf{R}^m}(u)}^{-1})(u))) = (Ft_y)(FA((Ft_{p_{\mathbf{R}^m}(u)}^{-1})(u))) ,$$

which is smooth because Ft_y, FA nad $Ft_{p_{\mathbf{R}^m}(u)}^{-1}$ are smooth.

Suppose that local coordinate charts (U, φ) on M and (V, ψ) on N are such that $F_{M,N}(J^r(U, V) \oplus FU) \subset FV$. With any r-jet $A \in J^r(M, N)$, $\alpha(A) \in U$, $\beta(A) \in V$, we associate a jet from $J^r(\mathbf{R}^m, \mathbf{R}^n)$,

$$A = j_x^r f \mapsto j_{\varphi(x)}^r(\psi \circ f \circ \varphi^{-1}) .$$

This map is a diffeomorphism $\theta: J^r(U, V) \to J^r(\mathbf{R}^m, \mathbf{R}^n)$. We have a sequence of maps

$$J^r(\mathbf{R}^m, \mathbf{R}^n) \oplus F\mathbf{R}^m \xrightarrow{\theta^{-1} \oplus F\varphi^{-1}} J^r(U, V) \oplus FU \xrightarrow{F_{U,V}} FV \xrightarrow{F\psi} F\mathbf{R}^n .$$

$\theta^{-1} \oplus F\varphi^{-1}$ and $F\psi$ are smooth and $F_{\mathbf{R}^m, \mathbf{R}^n}$ is a composed map which is also smooth. Hence $F_{U,V} = F_{M,N} \mid J^r(U, V) \oplus FU$ is smooth, QED.

Let (y^p) be local coordinates on S_m and (w^λ) local coordinates on S_n. Then (3) has the coordinate form

$$w^\lambda = \varphi^\lambda(a_i^\alpha, \ldots, a_{i_1 \ldots i_r}^\alpha, y^p) .$$

If (x^i, y^p) are local fibre coordinates on FM and (z^α, w^λ) are local fibre coordinates on FN. Then $F_{M,N}$ has the coordinate form

$$(5) \qquad w^\lambda = \Phi^\lambda(x^i, z^\alpha, a_i^\alpha, \ldots, a_{i_1 \ldots i_r}^\alpha, y^p)$$

and Ff: $z^\alpha = f^\alpha(x^i)$

$$w^\lambda = \Phi^\lambda\left(x^i, f^\alpha(x^i), \frac{\partial f^\alpha(x)}{\partial x^i}, \ldots, \frac{\partial^r f^\alpha(x)}{\partial x^{i_1} \ldots \partial x^{i_r}}, y^p\right).$$

Equations (5) are called the equations of the prolongation functor F. If (x^i, y^p) and (z^α, w^λ) are adapted coordinates, then (4) implies that $\Phi^\lambda = \varphi^\lambda$ and the equations of a prolongation functor in the adapted coordinates do not depend on $x \in M$ and $z \in N$.

Example. 4. For local coordinates (x^i) on M and (y^p) on N, $(x^i, dx^i = \xi^i)$ are the adapted coordinates on TM and $(y^p, dy^p = \eta^p)$ are the adapted coordinates on TN. Then the associated map $T_{M,N}$: $J^1(M, N) \oplus TM \to TN$ has the form $T_{M,N}(A, \xi) = \eta \in TN$, $\eta \equiv \eta^p = a_i^p \xi^i$ for $A = (a_i^p) \in J^1(M, N)$, $\xi = (\xi^i) \in TM$ and Tf: $TM \to TN$ is given by $y^p = f^p(x^i)$,

$$\eta^p = \frac{\partial f^p}{\partial x^i} \xi^i , \quad \xi \in T_x M .$$

2. Let $L_m^r \subset L^r(m, m)$ be the r-th order differential group in a dimension m. Let F be a lifting functor of order r. Then (3) defines the map

$$(6) \qquad \varphi_m: L_m^r \times S_m \to S_m , \quad (A, s) \mapsto FA(s) ,$$

$A \in L_m^r$, $s \in S_m$. For any two elements $A, B \in L_m^r$ we obtain $\varphi_m(B, \varphi_m(A, s)) = FB(FA(s)) = F(B \circ A)(s) = \varphi_m(B \circ A, s)$ and $\varphi_m(j_0^r \, \mathrm{id}_{R^m}, s) = F \, \mathrm{id}_{R^m}(s) = s$. Hence (6) is a left action of the group L_m^r on S_m.

If M is an m-dimensional manifold, then $H^r M = \mathrm{inv}\, J_0^r(R_m, M)$ is the principal fibre bundle with the structure group L_m^r and the base M, which is called the holonomic r-th order frame bundle. With any element $u \in H^r M$, $u = j_0^r f$, we associate a diffeomorphism Fu: $S_m \to F_x M$ which is the restriction of Ff: $FR^m \to FM$ to $S_m = F_0 R^m$, $x = f(0)$. For any $A \in L_m^r$ and $s \in S_m$ we have $Fu(s) = Fu((FA \circ FA^{-1})(s)) = F(u \circ A)((FA^{-1})(s))$ and hence we have the equivalence

$$(u, s) \sim (u \circ A, (FA^{-1})(s)) .$$

Hence FM is the fibre manifold associated with $H^r M$ with the standard fibre S_m and the action (6) of L_m^r on S_m.

On the other hand, it is known, [8], [10], that if S is a left L^r_m-space, then the rule which with any m-dimensional manifold M associates the associated manifold $FM := (H^rM, S)$ and with any diffeomorphism $f \in \operatorname{Hom} \mathcal{M}_m$, $f: M \to \overline{M}$, associates the morphism of fibre manifolds $Ff: FM \to F\overline{M}$ given by $Ff := (H^rf, \mathrm{id}_S): (H^rM, S) \to$ $\to (H^r\overline{M}, S)$, is a lifting functor of order r.

Consider the category $\mathcal{PB}_m(G)$ (of all principal fibre bundles with the structure group G and m-dimensional bases and morphisms of such principal fibre bundles over diffeomorphisms). Then for any left G-space S we can define a covariant functor $\tilde{S}: \mathcal{PB}_m(G) \to \mathcal{FM}_m(S, G)$ (into the category of all associated fibre manifolds with m-dimensional bases, the standard fibre S and the structure group G and morphisms of such fibre manifolds over diffeomorphisms) given by $\tilde{S}(P) = (P, S)$ and $\tilde{S}(\varphi) =$ $= (\varphi, \mathrm{id}_S)$, $P \in \operatorname{Ob} \mathcal{PB}_m(G)$, $\varphi \in \operatorname{Hom} \mathcal{PB}_m(G)$. Then we can summarize:

Proposition 1. *Any lifting functor $F: \mathcal{M}_m \to \mathcal{FM}$ of order r has values in the subcategory $\mathcal{FM}_m(S_m, L^r_m)$, $S_m = F_0R^m$, and $F = \tilde{S}_m \circ H^r$. On the other hand, for any left L^r_m-space S the composed functor $\tilde{S} \circ H^r$ is a lifting functor of order r.*

Now, we shall prove an analogous description for an r-th order prolongation functor. Let us define the category L^r. $\operatorname{Ob} L^r$ is the set of natural numbers $1, 2, \ldots$. $\operatorname{Hom} L^r(m, n) = L^r(m, n) = J^r_0(R^m, R^n)_0$ and composition in L^r is given by the composition of jets.

If $F: \mathcal{M} \to \mathcal{FM}$ is an r-th order prolongation functor, we shall denote $\mathcal{S} =$ $= \{S_1, S_2, \ldots\}$, $S_i = F_0R^i$. The action λ of the category L^r on $\mathcal{S}$ is defined as a system of maps

$$\lambda_{m,n}: L^r(m, n) \times S_m \to S_n$$

which satisfy

$$\lambda_{m,p}(B \circ A, s) = \lambda_{n,p}(B, \lambda_{m,n}(A, s)) \tag{7}$$

for all $A \in L^r(m, n)$, $B \in L^r(n, p)$, $s \in S_m$. The map (3) defines such an action with $\lambda_{m,n}(A, s) = FA(s)$. Hence we associate an action of the category L^r on $\mathcal{S}$ with an r-th order prolongation functor F.

Now, consider a sequence of manifolds $\mathcal{S} = \{S_1, S_2, \ldots\}$ and an action λ of the category L^r on $\mathcal{S}$. With any m-dimensional manifold M we can associate a fiber manifold $FM = (H^rM, S_m)$. The equivalence on (H^rM, S_m) is given by

$$(u, s) \sim (u \circ A, \lambda_{m,m}(A^{-1}, s)), \tag{8}$$

where $u \in H^rM$, $A \in L^r_m$, $s \in S_m$. If $f: M \to N$ and $f(x) = y$, we define $Ff: FM \to FN$ by

$$Ff(u, s) = (v, \lambda_{m,n}(v^{-1} \circ A \circ u, s)), \tag{9}$$

where $u \in H^r_xM$, $v \in H^r_yN$, $A \in J^r_x(M, N)_y$. It is easy to see that Ff is correct. From (8) and (9) we can deduce $F(g \circ f) = (Fg) \circ (Ff)$ for all $f: M \to N$, $g: N \to P$, and $F(\mathrm{id}_M) = \mathrm{id}_{FM}$. Then F is an r-th order prolongation functor and we have proved

Theorem 2. *There is a bijective correspondence between the set of r-th order prolongation functors and the set of actions of the category L^r.*

3. Let F, G be two r-th order lifting functors and $S = F_0\mathbf{R}^m$, $R = G_0\mathbf{R}^m$. We have the left action λ of the group L^r_m on S and the left action μ of L^r_m on R given by (6). Consider a natural transformation Φ of F into G. Then for any diffeomorphism $f: M \to \overline{M}$ we have $Gf \circ \Phi_M = \Phi_{\overline{M}} \circ Ff$ and rewriting it for $M = \overline{M} = \mathbf{R}^m, f(0) = 0$, using restriction to the fibres over the origin and notation $\Phi_{\mathbf{R}^m} \mid S = \varphi_m$, $(Ff \mid S)(s) = FA(s) = \lambda_m(A, s)$, $(Gf \mid R)(r) = GA(r) = \mu_m(A, r)$, $A = j^r_0 f$, $s \in S$, $r \in R$, we obtain

$$\varphi_m(\lambda_m(A, s)) = \mu_m(A, \varphi_m(s)) .$$

Hence $\varphi_m = \Phi_{\mathbf{R}^m} \mid S$ is an L^r_m-equivariant map of the L^r_m-space S into the L^r_m-space R.

On the other hand, consider a left L^r_m-space S (or R) and an L^r_m-equivariant map $\varphi: S \to R$. According to Proposition 1 we have an r-th order lifting functor F (or G), $FM = (H^rM, S)$ (or $GM = (H^rM, R)$), $Ff = (H^rf, \mathrm{id}_S)$ (or $Gf = (H^rf, \mathrm{id}_R)$) for all $M \in \mathrm{Ob}\, \mathcal{M}_m$, $f \in \mathrm{Hom}\, \mathcal{M}_m$. Then for any manifold $M \in \mathrm{Ob}\, \mathcal{M}_m$ we define $\Phi_M: FM \to GM$ by

$$\Phi_M := (\mathrm{id}_{H^rM}, \varphi) .$$

It is easy to prove that such Φ_M define a natural transformation of F into G.

Thus we have proved a result known from [7], [12]:

Proposition 2. *There is a bijective correspondence between the set of natural transformations of two r-th order lifting functors and the set of L^r_m-equivariant maps of the standard fibres determined by these lifting functors.*

Now, we describe analogous properties of natural transformations of r-th order prolongation functors. Consider two r-th order prolongation functors F and G. A natural transformation Φ of F into G is called projectable if $\Phi_M: FM \to GM$ is a base-preserving morphism for all $M \in \mathrm{Ob}\, \mathcal{M}$. We shall consider only projectable natural transformations.

Consider an action λ of the category L^r on $\mathcal{S} = \{S_1, S_2, \ldots\}$ and an action μ of L^r on $\mathcal{R} = \{R_1, R_2, \ldots\}$. A sequence of maps $\varphi_i: S_i \to R_i$ which satisfy

$$\varphi_n(\lambda_{m,n}(A, s)) = \mu_{m,n}(A, \varphi_m(s)) \tag{10}$$

for all $s \in S_m$, $A \in L^r(m, n)$ will be called a covariant map of the action λ into the action μ.

For any morphism $f: M \to N$ and a natural transformation of F into G we have the following commutative diagram:

$$\begin{array}{ccc} FM & \xrightarrow{\Phi_M} & GM \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ FN & \xrightarrow{\Phi_N} & GN \end{array}$$

If $M = \mathbf{R}^m$, $N = \mathbf{R}^n$, $f(0) = 0$, then using restriction to the fibres $S_m = F_0\mathbf{R}^m$, $R_m = G_0\mathbf{R}^m$, we have $(Gf \mid R_m) \circ (\Phi_{\mathbf{R}^m} \mid S_m) = (\Phi_{\mathbf{R}^n} \mid S_n) \circ (Ff \mid S_m)$ and from the definition of FA and GA, $A = j_0^r f$, using the notation $\varphi_m = \Phi_{\mathbf{R}^m} \mid S_m$ we have

$$GA \circ \varphi_m = \varphi_n \circ FA \,. \tag{11}$$

(3) defines the action λ (or μ) of L^r on $\mathscr{S} = \{F_0\mathbf{R}^1, F_0\mathbf{R}^2, \ldots\}$ (or $\mathscr{R} = \{G_0\mathbf{R}^1, G_0\mathbf{R}^2, \ldots\}$) by $\lambda_{m,n}(A, s) = FA(s)$ (or $\mu_{m,n}(A, r) = GA(r)$) for all $A \in L^r(m, n)$, $s \in S_m$ (or $r \in R_m$). Then rewritting (11) we have (10), which means that $\varphi_m = \Phi_{\mathbf{R}^m} \mid S_m$ defines a covariant map of λ into μ.

On the other hand, consider an action λ (or μ) of the category L^r on a system $\mathscr{S} = \{S_1, S_2, \ldots\}$ (or $\mathscr{R} = \{R_1, R_2, \ldots\}$). According to Theorem 2 we have the r-th order prolongation functor F (or G) given by the action λ (or μ). Consider a covariant map φ of λ into μ given by a sequence of maps $\varphi_i: S_i \to R_i$ satisfying (10). Then with any $M \in \mathrm{Ob}\,\mathscr{M}$ we asociate a morphism $\Phi_M: FM \to GM$ defined by

$$\Phi_M: (H^rM, S_m) \to (H^rM, R_m) \,, \quad \Phi_M := (\mathrm{id}_{H^rM}, \varphi_m) \,,$$

$m = \dim M$. Then it is easy to prove that Φ_M defines a natural transformation Φ of the functor F into the functor G.

Thus we have proved

Theorem 3. *There is a bijective correspondence between the set of natural transformations of two r-th order prolongation functors and the set of covariant maps of actions of L^r given by these prolongation functors.*

4. Let $\pi: Y \to X$ be a fibre manifold. $f: \bar{X} \to X$ is a map. Denote by $f^!Y$ the induced fibre manifold over $\bar{X}$, i.e. $f^!Y = \{(\bar{x}, y) \in \bar{X} \times Y, f(\bar{x}) = \pi(y)\}$. Then we have the canonical morphism of fibre manifolds $f_Y: f^!Y \to Y$ over r given by $f_Y(\bar{x}, y) = y \in Y_{f(\bar{x})}$. The restriction of f_Y to the fibre over $\bar{x}$ is a diffeomorphism. If $g: \tilde{X} \to \bar{X}$ is another map we have the well-known identity $(f \circ g)^! Y = g^! f^! Y$.

Consider two fibre manifolds $\pi_1: Y_1 \to X$, $\pi_2: Y_2 \to X$ with the same base and a base-preserving morphism $\varphi: Y_1 \to Y_2$. Let $f: \bar{X} \to X$ be a map. Then we define the induced morphism of fibre manifolds $f^!\varphi: f^!Y_1 \to f^!Y_2$ by the rule $(\bar{x}, y) \mapsto (\bar{x}, \varphi(y))$, where $\pi_1(y) = f(\bar{x})$. Then

$$f_{Y_2} \circ f^!\varphi = \varphi \circ f_{Y_1} \,. \tag{12}$$

Now, we describe some properties of prolongation cofunctors, [3].

Definition 3. *A prolongation cofunctor $F: M \to FM$ is a rule transforming any manifold M into a fibre manifold $p_M: FM \to M$ and any map $f: M \to N$ into a base-preserving morphism*

$$Ff: f^!FN \to FM$$

such that

$$F\,\mathrm{id}_M = \mathrm{id}_{FM} \quad \textit{for all } M \textit{ and } \quad F(g \circ f) = Ff \circ f^!Fg$$

for all $f: M \to N$ *and* $g: N \to P$.

A prolongation cofunctor is said to be of order r if $j^r_x f = j^r_x g$ implies $Ff \mid (f^!FN)_x = Fg \mid (f^!FN)_x$.

For any r-jet $A \in J^r_x(M, N)_y$, $A = j^r_x f$, we can define a map $FA: F_yN \to F_xM$ by

$$FA := Ff \circ (f_{FN} \mid (f^!FN)_x)^{-1} . \tag{13}$$

If $B \in J^r_y(N, P)_z$, $B = j^r_y g$, we have

Lemma 2. $F(B \circ A) = FA \circ FB$.

Proof. From (13) we obtain

$$F(B \circ A) = Ff \circ (f^!Fg) \circ (f_{g^!FP} \mid (f^!g^!FP)_x)^{-1} \circ (g_{FP} \mid (g^!FP)_y)^{-1}$$

and using (12) we have Lemma 2, QED.

With any $A \in J^r_x(M, N)_y$ and $q \in F_yN$ we can associate $FA(q) \in F_xM$. Hence we can define an associated map

$$F_{M,N}: FN \oplus J^r(M, N) \to FM , \tag{14}$$

where $\oplus$ is Whitney's sum with respect to the projections p_N and β. Using the same methods as in Theorem 1 we can prove that (14) is smooth.

Let $(L^r)^{op}$ denote the dual category of L^r. Denote $\mathcal{S} = \{F_0R^1, F_0R^2, \ldots\}$, $F_0R^i = S_i$, then (14) defines an action of the category $(L^r)^{op}$ on $\mathcal{S}$

$$\varphi_{m,n}: S_n \times L^r(m, n) \to S_m$$

given by $\varphi_{m,n}(s, A) = FA(s)$, $A \in L^r(m, n)$, $s \in S_n$. It is easy to see that

$$\varphi_{m,p}(s, B \circ A) = \varphi_{m,n}(\varphi_{n,p}(s, B), A) \tag{15}$$

for $B \in L^r(n, p)$, $s \in S_p$.

On the other hand, let φ be an action of the category $(L^r)^{op}$ on some system $\mathcal{S} = \{S_1, S_2, \ldots\}$. Hence we have a system of maps $\varphi_{m,n}(-, A): S_n \to S_m$, $A \in L^r(m, n)$, satisfying (15). Denote by $\tilde{\varphi}_m$ the left action of L^r_m on S_m defined by $\tilde{\varphi}_m(A, s) = \varphi_{m,m}(s, A^{-1})$ and by $FM = (H^rM, S_m)$ a fibre manifold associated with H^rM with respect to the action $\tilde{\varphi}_m$. The equivalence on FM is given by

$$(u, s) \sim (u \circ A,\ \tilde{\varphi}_m(A^{-1}, s)) = (u \circ A, \varphi_{m,m}(s, A)) . \tag{16}$$

Now, let $A \in J^r_x(M, N)_y$, $A = j^r_x f$, $u \in H^r_xM$, $v \in H^r_yN$, then $v^{-1} \circ A \circ u \in L^r(m, n)$ and we can define the map

$$FA: F_yN \to F_xM$$

by

(17) $$FA(v, s) = (u, \varphi_{m,n}(s, v^{-1} \circ A \circ u)),$$

where $s \in S_n$. It is easy to prove that this map is correct and satisfies $F(B \circ A) = FA \circ FB$.

Now, we define $Ff := FA \circ f_{FN}: f^!FN \to FM$, and from (12) we obtain

$$F(g \circ f) = Ff \circ f^!Fg$$

for all $B = j^r_y g$ and $F(\mathrm{id}_M) = \mathrm{id}_{FM}$. The correspondence $M \mapsto FM$, $(f: M \to N) \mapsto (Ff: f^!FN \to FM)$ is an r-th order prolongation cofunctor.

Thus we have proved

Theorem 4. *There is a bijective correspondence between the set of all r-th order prolongation cofunctors and the set of all actions of the category* $(L^r)^{op}$.

5. In this section we shall deal with the composition of prolongation functors. To this end we need the concept of fibre jets defined by Kolář [6]. Let $Y \to X$ be a fibre manifold and N a manifold. We say that two maps $f, g: Y \to N$ have the r-th order fibre contact at $x \in X$, if $j^r_y f = j^r_y g$ for all $y \in Y_x$. Such an equivalence class $\mathbf{j}^r_x f$ will be called a fibre r-jet of Y into N. Given other fibre manifolds $W \to Z$ and $U \to V$, and provided $f: Y \to W$ and $g: W \to U$ are morphisms of fibre manifolds, then we can define

$$(\mathbf{j}^r_z g) \circ (\mathbf{j}^r_x f) = \mathbf{j}^r_x(g \circ f), \quad f(Y_x) \subset W_z.$$

Lemma 3. *Let F be an r-th order prolongation functor. If* $j^{r+s}_x f = j^{r+s}_x g$ *then* $\mathbf{j}^s_x Ff = \mathbf{j}^s_x Fg$ *for any* $f, g \in \operatorname{Hom} \mathcal{M}$.

Proof. From (5) we have, in adapted coordinates (x^i, y^p) on FM and (z^α, w^λ) on FN, the following expression for Ff, Fg:

$$Ff: z^\alpha = f^\alpha(x, y), \quad w^\lambda = F^\lambda\left(\frac{\partial f^\alpha}{\partial x^i}, \ldots, \frac{\partial^r f^\alpha}{\partial x^{i_1} \ldots \partial x^{i_r}}, y^p\right)$$

$$Fg: z^\alpha = g^\alpha(x, y), \quad w^\lambda = F^\lambda\left(\frac{\partial g^\alpha}{\partial x^i}, \ldots, \frac{\partial^r g^\alpha}{\partial x^{i_1} \ldots \partial x^{i_r}}, y^p\right).$$

$\mathbf{j}^s_x Ff = \mathbf{j}^s_x Fg$ if and only if $j^s_y Ff = j^s_y Fg$ for all $y \in F_x M$. The coordinate expression of $j^s_y Ff$ (or $j^s_y Fg$) is given by partial derivatives of Ff (or Fg) with respect to x^i, y^p up to order s and from the coordinate expression it follows that if $j^{r+s}_x f = j^{r+s}_x g$, then $j^s_y Ff = j^s_y Fg$ for all $y \in F_x M$, QED.

Theorem 5. *Let F be an r-th order prolongation functor and G an s-th order prolongation functor. Then the composed functor GF is an* $(r + s)$*-th order prolongation functor.*

Proof. It is easy to prove that GF is a prolongation functor. We shall prove that GF is of the $(r + s)$-th order. F is of order r and according to Lemma 3, $j_x^{r+s}f = j_x^{r+s}g$ implies $j_x^s Ff = j_x^s Fg$ for all $f, g \in \text{Hom}\, \mathcal{M}, f, g: M \to N$. G is an s-th order prolongation functor, hence $j_y^s Ff = j_y^s Fg$ implies $G(Ff) \mid G_y FM = G(Fg) \mid G_y FM$ for all $y \in \in FM$. But from $j_x^s Ff = j_x^s Fg$ we have $j_y^s Ff = j_y^s Fg$ and hence $G(Ff) \mid G_y FM = = G(Fg) \mid G_y FM$ for all $y \in F_x M$. Consequently $GFf \mid GF_x M = GFg \mid GF_x M$, QED.

Example. 5. J^s is an s-th order prolongation functor on $\mathcal{FM} \subset \mathcal{M}$ which associates $J^s Y \to Y$ with a fibre manifold $Y \to X$ and $J^s f: J^s Y \to J^s Y$ over (f, f_0) given by $J^s f(j_x^s \gamma) = j_{f_0(x)}^s (f \circ \gamma \circ f_0^{-1})$ with a morphism of fibre manifolds $f: Y \to \bar{Y}$ over the diffeomorphism $f_0: X \to \bar{X}$. If F is an arbitrary r-th order lifting functor, then $J^s F$ is an $(r + s)$-th order lifting functor.

References

[1] *D. B. A. Epstein:* Natural vector bundles. Lecture Notes in Mathematics, N. 99, 171—195.

[2] *D. B. A. Epstein, W. Thurston:* Transformation groups and natural bundles. Proc. London Math. Soc. (*3*) *38*, 1979, N. 2, 219—236.

[3] *T. Klein:* Connections on higher order tangent bundles. Čas. pro pěst. mat. *106* (1981), 414—421.

[4] *I. Kolář:* On the prolongations of geometric object fields. Analele stiintifice ale Universitatii "Al. I. Cuza", T XVII, 1971, 437—446.

[5] *I. Kolář:* Prolongations of generalized connections. Colloquia Math. Soc. János Bolay, 31. diff. geom., Budapest, 1979, 317—325.

[6] *I. Kolář:* Higher order absolute differentiation with respect to generalized connections. (To appear in Banach Center Publications)

[7] *D. Krupka:* Differential invariants. Lecture Notes, Brno 1979.

[8] *D. Krupka:* Reducibility theorems for differentiable liftings in fiber bundles. Arch. Math. 2, Scripta fac. sci. nat. UJEP Brunensis, XV: 93—106, 1979.

[9] *A. Nijenhuis:* Natural bundles and their general properties. Diff. Geom., Kinokuniya, Tokyo 1972, 317—334.

[10] *R. S. Palais, Chuu-Lian Terng:* Natural bundles have finite order. Topology, Vol. *16*, 1977, 271—277.

[11] *S. E. Salvioli:* On the theory of geometric objects. J. Diff. Geom., *7* (1972), 257—278.

[12] *Chuu-Lian Terng:* Natural vector bundles and natural differential operators. Am. J. Math., Vol *100* (1978), N. 4, 775—826.

Author's address: 662 95 Brno, Janáčkovo nám. 2a (Přírodovědecká fakulta UJEP).