Štefan Schwabik
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VARIATIONAL STABILITY FOR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

ŠTEFAN SCHWABIK, Praha
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INTRODUCTION

In this paper we study the generalized differential equation

$$\frac{dx}{d\tau} = DF(x, t) \tag{1}$$

for which the identically zero function is a solution. For $c > 0$ write $B_c = \{x \in R^n; |x| < c\}$ where $|\cdot|$ stands for some norm in the space R^n.

Let two functions be given: a function $h : [0, +\infty) \to R$, nondecreasing and continuous from the left on $[0, +\infty)$, and a function $\omega : [0, +\infty) \to R$, continuous, increasing and such that $\omega(0) = 0$; these functions are fixed for the rest of the paper. The following assumptions are imposed on the right hand side F of the generalized differential equation (1) throughout the paper:

$$\text{there is } \quad c > 0 \quad \text{such that} \quad F : B_c \times [0, +\infty) \to R^n ; \tag{2}$$

$$|F(x, t_2) - F(x, t_1)| \leqq |h(t_2) - h(t_1)| \tag{3}$$
$$\text{for every} \quad x \in B_c\,, \quad t_1, t_2 \in [0, +\infty) ;$$

$$|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)| \leqq \tag{4}$$
$$\leqq \omega(|x_2 - x_1|) \,.\, |h(t_2) - h(t_1)| \quad \text{for every} \quad x_1, x_2 \in B_c\,, \quad t_1, t_2 \in [0, +\infty) ;$$

$$F(0, t_2) - F(0, t_1) = 0 \quad \text{for every} \quad t_1, t_2 \in [0, +\infty) . \tag{5}$$

Generalized differential equations of the form (1) were introduced and extensively studied in detail by J. Kurzweil [2], [3]. The assumptions (2), (3) and (4) are given in [3] and the set of all functions F satisfying the assumptions (3) and (4) on $G = B_c \times [0, +\infty)$ is denoted by $\mathscr{F} = \mathscr{F}(G, h, \omega)$. Generalized differential equations of the form (1) with $F \in \mathscr{F}(G, h, \omega)$ represent a sufficiently wide class of equations, which includes e.g. the class of ordinary differential equations with right hand sides satisfying the known Carathéodory conditions.

Let us mention that a function $x : [a, b] \to R^n$ is a solution of (1) if

a) $$(x(t), t) \in B_c \times [0, +\infty) \quad \text{for every} \quad t \in [a, b]$$

and

b) $$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t) \quad \text{for every} \quad s_1, s_2 \in [a, b] .$$

The integral used here is the generalized Perron integral introduced by Kurzweil in [2]. More details on this integral can be found in [4].

The fundamental local existence result for solutions of (1) is given in [3]: *If $F \in \mathscr{F}(G, h, \omega)$ and $\tilde{x} \in B_c$, $t_0 \in [0, +\infty)$ where $\tilde{x}^+ = \tilde{x} + F(\tilde{x}, t_{0+}) - F(\tilde{x}, t_0) \in B_c$ then there exists $\delta > 0$ such that there is a solution $x : [t_0, t_0 + \delta] \to R^n$ of (1) with $x(t_0) = \tilde{x}$.*

Proposition 1. *Assume that $s_1, s_2 \in [0, +\infty)$, $s_1 \leqq s_2$. If $y : [s_1, s_2] \to B_c$ is such a function that the integral $\int_{s_1}^{s_2} DF(y(\tau), t)$ exists then*

(6) $$\left| \int_{s_1}^{s_2} DF(y(\tau), t) \right| \leqq h(s_2) - h(s_1) .$$

For the proof see [3].

By this proposition we can conclude that every solution $x : [a, b] \to R^n$ of the equation (1) is of bounded variation on $[a, b]$, $x \in BV[a, b]$. In fact, if $s_1 \leqq s_2$, $s_1, s_2 \in [a, b]$ then

(7) $$|x(s_2) - x(s_1)| = \left| \int_{s_1}^{s_2} DF(x(\tau), t) \right| \leqq h(s_2) - h(s_1)$$

and consequently, also $\operatorname{var}_a^b x \leqq h(b) - h(a)$. The continuity from the left of the function h together with (7) yields that every solution of the equation (1) is continuous from the left.

If $s_1, s_2 \in [0, +\infty)$, $s_1 \leqq s_2$ and $x, y : [s_1, s_2] \to B_c$ are such functions that the integrals $\int_{s_1}^{s_2} DF(x(\tau), t)$ and $\int_{s_1}^{s_2} DF(y(\tau), t)$ exist then

(8) $$\left| \int_{s_1}^{s_2} D[F(x(\tau), t) - F(y(\tau), t)] \right| \leqq \int_{s_1}^{s_2} D[\omega(|x(\tau) - y(\tau)|) h(t)] =$$
$$= \int_{s_1}^{s_2} \omega(|x(\tau) - y(\tau)|) \, dh(\tau) .$$

This statement follows from the assumption (4) and from Lemma 3.1 in [3] (cf. also [4]). The integral on the right hand side is the Perron-Stieltjes integral. Finally, let us mention that if $x : [s_1, s_2] \to B_c$, $0 \leqq s_1 \leqq s_2 < +\infty$ is a regulated function (having onesided limits at each point $s \in [s_1, s_2]$) then the integral $\int_{s_1}^{s_2} DF(x(\tau), t)$ exists. In our case it is important that the integral $\int_{s_1}^{s_2} DF(x(\tau), t)$ exists whenever $x \in BV[s_1, s_2]$ is a function with values in B_c. By the assumptions (5) we have

$$\int_{s_1}^{s_2} D[F(0, t)] = F(s_2) - F(s_1) = 0 \quad \text{for every} \quad s_1, s_2 \in [0, +\infty) .$$

Hence the function $x \equiv 0$ is a solution of the equation (1) on $[0, \infty)$.

The concept of the generalized differential equation with a right hand side satisfying conditions (3) and (4) includes the Carathéodory theory of ordinary differential equations as well as the concept of the measure differential equation (see e.g. [5]) and the theory of systems with impulses which has been developed intensively by A. M. Samoilenko and others (see e.g. [6]).

In the present paper we give some results concerning the concept of stability of a solution of the generalized differential equation. They generalize the results known for the Carathéodory concept of differential equation and they also cover the interesting case of systems with impulses. The starting point of our approach is the stimulative paper of I. Vrkoč [8] on integral stability and the improvements of his results given by S.-N. Chow and J. A. Yorke in [1]. Since the solutions of the equation (1) are functions of bounded variation it seems to be very reasonable to use the concept of variational stability which was mentioned by I. Vrkoč in [8] and which belongs to H. Okamura. The concept of variational stability in the case of Carathéodory equations is equivalent to the integral stability introduced by I. Vrkoč, see [8]. In the case of classical differential equations the concept of variational stability has some features of artificiality; in this case the solutions of the differential equations are absolutely continuous functions and the power of the concept of the variation of a function is not fully exploited. In the case of generalized differential equations we have to distinquish also the discontinuities of functions and this can be done in a satisfactory way in terms of the total variation of a function.

Let us introduce the basic definitions of stability and asymptotic stability used throughout the paper.

DEFINITIONS AND PRELIMINARY RESULTS

Definition 1. *The solution $x \equiv 0$ of the equation (1) is called variationally stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $y : [t_0, t_1] \to B_c$, $0 \leqq t_0 < t_1 < +\infty$, is a function of bounded variation, continuous from the left with*

$$|y(t_0)| < \delta \quad and \quad \operatorname{var}_{t_0}^{t_1}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) < \delta ,$$

then

$$|y(t)| < \varepsilon \quad \textit{for} \quad t \in [t_0, t_1] .$$

Definition 2. *The solution $x \equiv 0$ of the equation (1) is called variationally attracting if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ and $\gamma = \gamma(\varepsilon) > 0$ such that if $y : [t_0, t_1] \to B_c$, $0 \leqq t_0 \leqq t_1 < +\infty$, is a function of bounded variation, continuous from the left and*

$$|y(t_0)| < \delta_0 \quad \textit{and} \quad \operatorname{var}_{t_0}^{t_1}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) < \gamma ,$$

then

$$|y(t)| < \varepsilon \quad \textit{for all} \quad t \in [t_0, t_1] , \quad t \geqq t_0 + T(\varepsilon) , \quad t_0 \geqq 0 .$$

Definition 3. *The solution $x \equiv 0$ of the equation (1) is called variational-asymptotically stable if it is variationally stable and variationally attracting.*

The above concepts of variational stability and variational asymptotic stability are closely connected with a certain kind of stability with respect to perturbations of the generalized differential equation (1).

Together with the equation (1) we consider the perturbed equation

$$\frac{dx}{d\tau} = D[F(x, t) + P(t)] , \tag{9}$$

where $P : [0, +\infty) \to R^n$ is a function of locally bounded variation $P \in BV_{\text{loc}}[0, +\infty)$ (i.e. for every $[a, b] \subset [0, +\infty)$ we have $\operatorname{var}_a^b P < \infty$) which is continuous from the left at each point belonging to $[0, +\infty)$. It can be easily verified that if the function F satisfies (3) and (4) then the function $F(x, t) + P(t) = G(x, t)$ satisfies similar assumptions, i.e.

$$|G(x, t_2) - G(x, t_1)| = |F(x, t_2) + P(t_2) - F(x, t_1) - P(t_1)| \leqq$$
$$\leqq |h(t_2) + \operatorname{var}_0^{t_2} P - h(t_1) - \operatorname{var}_0^{t_1} P|$$

and

$$|G(x_2, t_2) - G(x_2, t_1) - G(x_1, t_2) + G(x_1, t_1)| \leqq$$
$$\leqq \omega(|x_2 - x_1|)\,|h(t_2) - h(t_1)| .$$

Hence the right hand side of the generalized differential equation (9) belongs to the set $\mathscr{F}(G, \tilde{h}, \omega)$ where $\tilde{h}(t) = h(t) + \operatorname{var}_0^t P$, $t \in [0, +\infty)$, and all the fundamental results are valid for the equation (9) as well; this concerns especially the local existence of solutions.

Proposition 2. *The solution $x \equiv 0$ of the equation (1) is variationally stable if and only if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $|y_0| < \delta$ and $P \in BV[t_0, t_1]$, continuous from the left and with $\operatorname{var}_{t_0}^{t_1} P < \delta$, then*

$$|y(t, t_0, y_0)| < \varepsilon \quad \textit{for every} \quad t \in [t_0, t_1],$$

where $y(t, t_0, y_0)$ *is a solution of the equation*

$$\frac{dx}{d\tau} = D[F(x, t) + P(t)] \tag{9}$$

with $y(t_0, t_0, y_0) = y_0$.

Proof. 1) Assume that $x \equiv 0$ is variationally stable, i.e. for $\varepsilon > 0$ there exists $\delta > 0$ according to Definition 1. Assume that $|y_0| < \delta$, $\operatorname{var}_{t_0}^{t_1} P < \delta$ and that $y(t) = y(t, t_0, y_0)$ is a solution of the equation (9) on $[t_0, t_1]$; then evidently $|y(t_0)| = |y_0| < \delta$ and for $s_1, s_2 \in [t_0, t_1]$ we have

$$y(s_2) - y(s_1) = \int_{s_1}^{s_2} DF(y(\tau), t) + P(s_2) - P(s_1),$$

i.e.

$$y(s_2) - \int_{t_0}^{s_2} DF(y(\tau), t) - y(s_1) - \int_{t_0}^{s_1} DF(y(\tau), t) = P(s_2) - P(s_1)$$

and this yields

$$\operatorname{var}_{t_0}^{t_1}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) = \operatorname{var}_{t_0}^{t_1} P < \delta.$$

Hence by the variational stability we have

$$|y(t)| = |y(t, t_0, y_0)| < \varepsilon$$

for $t \in [t_0, t_1]$ and the condition given in the proposition is satisfied.

2) Let us assume that the condition given in our statement is satisfied. Let $y : [t_0, t_1] \to R^n$ be of bounded variation, continuous from the left, and such that $|y(t_0)| < \delta$ and

$$\operatorname{var}_{t_0}^{t_1}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) < \delta,$$

where $\delta > 0$ corresponds to the given $\varepsilon > 0$ by the assumed condition. For all $s_1, s_2 \in [t_0, t_1]$ we have

$$y(s_2) - y(s_1) = \int_{s_1}^{s_2} DF(y(\tau), t) + y(s_2) - y(s_1) - \int_{s_1}^{s_2} DF(y(\tau), t) =$$

$$= \int_{s_1}^{s_2} DF(y(\tau), t) + y(s_2) - \int_{t_0}^{s_2} DF(y(\tau), t) - y(s_1) + \int_{t_0}^{s_1} DF(y(\tau), t) =$$

$$= \int_{s_1}^{s_2} DF(y(\tau), t) + P(s_2) - P(s_1),$$

where $P(s) = y(s) - \int_{t_0}^{s} DF(y(\tau), t)$, $s \in [t_0, t_1]$. Hence clearly, $P \in BV[t_0, t_1]$, P is continuous from the left and the function $y : [t_0, t_1] \to R^n$ is a solution of the equation (9), where

$$|y(t_0)| < \delta \quad \text{and} \quad \operatorname{var}_{t_0}^{t_1} P = \operatorname{var}_{t_0}^{t_1} \left(y(s) - \int_{t_0}^{s} DF(y(\tau), t) \right) < \delta \,.$$

Hence by the condition given in the statement we have

$$|y(t)| = |y(t, t_0, y(t_0))| < \varepsilon$$

for every $t \in [t_0, t_1]$, i.e. the solution $x \equiv 0$ of the equation (1) is variationally stable.

Proposition 3. *The solution $x \equiv 0$ of the equation (1) is variationally attracting if and only if there exists $\delta_0 > 0$ and for any $\varepsilon > 0$ there exist $T \geqq 0$ and $\gamma > 0$ such that if*

$$|y_0| < \delta_0 \quad \text{and} \quad \operatorname{var}_{t_0}^{t_1} P < \gamma$$

with $P \in BV[t_0, t_1]$ continuous from the left, then

$$|y(t, t_0, y_0)| < \varepsilon \quad \text{for all} \quad t \geqq t_0 + T\,, \qquad t \in [t_0, t_1] \quad \text{and} \quad t_0 \geqq 0\,,$$

where $y(t, t_0, y_0)$ is a solution of the equation (9) with $y(t_0, t_0, y_0) = y_0$.

Proof. 1) Assume that $x \equiv 0$ is variationally attracting, i.e. that there exists $\delta_0 > 0$ and for $\varepsilon > 0$ also $T > 0$ and $\gamma > 0$ by Definition 2. Assume further that $|y_0| < \delta_0$, $P \in BV[t_0, t_1]$ continuous from the left such that $\operatorname{var}_{t_0} P < \gamma$, and that $y(t) = y(t, t_0, y_0)$ is a solution of the equation (9) on $[t_0, t_1]$. Then $|y(t_0)| = |y_0| < \delta_0$ and

$$\operatorname{var}_{t_0}^{t_1} \left(y(s) - \int_{t_0}^{s} DF(y(\tau), t) \right) = \operatorname{var}_{t_0}^{t_1} P < \gamma$$

(cf. the proof of Proposition 2). Hence by Definition

$$|y(t, t_0, y_0)| = |y(t)| < \varepsilon \quad \text{for all} \quad t \geqq t_0 + T \quad \text{and} \quad t_0 \geqq 0\,.$$

2) If the condition given in Proposition 3 is satisfied then assume that $y : [t_0, t_1] \to R^n$ is of bounded variation, continuous from the left, such that $|y(t_0)| < \delta$ and

$$\operatorname{var}_{t_0}^{t_1} \left(y(s) - \int_{t_0}^{s} DF(y(\tau), t) \right) < \gamma\,.$$

Then it can be easily shown in the same way as in the proof of Proposition 2 that y is a solution of the equation (9) with

$$|y(t_0)| < \delta_0 \quad \text{and} \quad P(s) = y(s) - \int_{t_0}^{s} DF(y(\tau), t)\,, \quad s \in [t_0, t_1]\,.$$

$P \in BV[t_0, t_1]$ is continuous from the left, $\mathrm{var}_{t_0}^{t_1} P < \gamma$. Hence

$$|y(t)| < \varepsilon \quad \text{for all} \quad t \geqq t_0 + T \quad \text{and} \quad t_0 \geqq 0 .$$

Remark. Let us mention that the perturbation P in the equation (9) can be replaced by a more general perturbation $H(x, t)$ which satisfies

$$|H(x, t_2) - H(x, t_1)| \leqq |P(t_2) - P(t_1)|$$

for all x in a neighbourhood of 0 and

$$|H(x_2, t_2) - H(x_2, t_1) - H(x_1, t_2) + H(x_1, t_1)| \leqq \omega(|x_1 - x_2|) \, |P(t_2) - P(t_1)| .$$

These more general perturbations lead to results of the same kind and provide no new ideas for the object of the study; for this reason we consider the simpler case of perturbations independent of x.

The conditions given in Propositions 2 and 3 are equivalent to the notion of variational stability and variational attractivity. We use these conditions because they are more convenient in many situations.

Proposition 4. *Assume that* $-\infty < a < b < +\infty$ *and that* $f, g : [a, b] \to R$ *are two functions continuous from the left in* $(a, b]$, $f, g \in BV[a, b]$. *If for any* $t \in [a, b)$ *there exists* $\delta(t) > 0$ *such that for every* $h \in (0, \delta(t))$ *the inequality*

$$f(t + h) - f(t) \leqq g(t + h) - g(t)$$

holds, then

$$f(s) - f(a) \leqq g(s) - g(a)$$

for every $s \in [a, b]$.

Proof. Let us denote $M = \{s \in [a, b]; f(\sigma) - f(a) \leqq g(\sigma) - g(a) \text{ for } \sigma \in [a, s]\}$ and set $S = \sup M$. Since $f(a + \eta) - f(a) \leqq g(a + \eta) - g(a)$ for $\eta \in (0, \delta(a))$, we evidently have $S > a$ and $f(s) - f(a) \leqq g(s) - g(a)$ for every $s < S$. Since the functions f, g are continuous from the left we also have

$$f(s) - f(a) \leqq g(s) - g(a) .$$

If $S < b$ were valid then by the assumption

$$f(S + \eta) - f(S) \leqq g(S + \eta) - g(S)$$

for every $\eta \in (0, \delta(S))$ and consequently also

$$f(S + \eta) - f(a) = f(S + \eta) - f(S) + f(S) - f(a) \leqq g(S + \eta) - g(a) ,$$

i.e. for $\eta \in (0, \delta(S))$ we should have $S + \eta \in M$. This contradiction yields $S = b$ and $M = [a, b]$.

Lemma 1. *Let* $V : [0, +\infty) \times R \to R$ *be such that for every* $x \in R^n$ *the function* $V(\cdot, x)$ *is continuous from the left,* $V(\cdot, x) \in BV_{\mathrm{loc}}[0, +\infty)$ *and*

$$(10) \qquad |V(t,x) - V(t,y)| \leqq K|x-y| \quad \text{for} \quad x, y \in R^n, \quad t \in [0, +\infty),$$

where $K > 0$.

Assume that for every solution $x : (\alpha, \beta) \to R^n$ *of the equation* (1) *we have*

$$(11) \qquad \limsup_{\eta \to 0+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \leqq \Phi(x(t)),$$

where $\Phi : R^n \to R$ *is a real function,* $t \in (\alpha, \beta)$.

If $y : [t_0, t_1] \to R^n$ $0 \leqq t_0 < t_1 < +\infty$ *is a function continuous from the left on* $[t_0, t_1]$, $y \in BV[t_0, t_1]$, *then the inequality*

$$(12) \quad V(t_1, y(t_1)) \leqq V(t_0, y(t_0)) + K \operatorname{var}_{t_0}^{t_1}\left(y(s) - \int_{t_0}^{s} \mathrm{D}F(y(\tau), t)\right) + M(t_1 - t_0)$$

holds with $M = \sup_{t \in [t_0, t_1]} \Phi(y(t))$.

Proof. Assume that $y : [t_0, t_1] \to R^n$ is given as in the statement.

Let $\sigma \in [t_0, t_1]$ be arbitrary and let x be a solution of the equation (1) $\mathrm{d}x/\mathrm{d}\tau = \mathrm{D}F(x, t)$ such that $x(\sigma) = y(\sigma)$. By virtue of the local existence theorem for equations of the form (1) (see e.g. [3]) there exists $\eta_1(\sigma) > 0$ such that a solution x exists on $[\sigma, \sigma + \eta_1(\sigma)]$. For $\eta \in [0, \eta_1(\sigma)]$ we have by (10)

$$V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma+\eta, x(\sigma+\eta)) \leqq K|y(\sigma+\eta) - x(\sigma+\eta)| =$$
$$= K\left|y(\sigma+\eta) - y(\sigma) - \int_{\sigma}^{\sigma+\eta} \mathrm{D}F(x(\tau), t)\right|.$$

Further,

$$V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma, y(\sigma)) = V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma+\eta, x(\sigma+\eta)) +$$
$$+ V(\sigma+\eta, x(\sigma+\eta)) - V(\sigma, x(\sigma)).$$

By the assumption (11) for every $\varepsilon > 0$ there exists $\eta_2(\sigma) > 0$ such that $\eta_2(\sigma) \leqq \eta_1(\sigma)$ and the following inequality holds for $\eta \in [0, \eta_2(\sigma)]$:

$$V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma, y(\sigma)) \leqq$$
$$\leqq K\left|y(\sigma+\eta) - y(\sigma) - \int_{\sigma}^{\sigma+\eta} \mathrm{D}F(x(\tau), t)\right| + \eta\Phi(y(\sigma)) \leqq$$
$$\leqq K\left|y(\sigma+\eta) - y(\sigma) - \int_{\sigma}^{\sigma+\eta} \mathrm{D}F(x(\tau), t)\right| + \eta M + \eta\varepsilon.$$

Hence for $\eta \in [0, \eta_2(\sigma)]$ we also have

$$V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma, y(\sigma)) \leqq K\left|y(\sigma+\eta) - y(\sigma) - \int_{\sigma}^{\sigma+\eta} \mathrm{D}F(y(\tau), t)\right| +$$

$$+ K\left|\int_{\sigma}^{\sigma+\eta} \mathrm{D}[F(y(\tau), t) - F(x(\tau), t)]\right| + \eta M + \eta\varepsilon .$$

Let us set

$$P(s) = y(s) - \int_{t_0}^{s} \mathrm{D}F(y(\tau), t)$$

for $s \in [t_0, t_1]$. Evidently $P : [t_0, t_1] \to R^n$ is continuous from the left, $P \in BV[t_0, t_1]$ and

(13) $$V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma, y(\sigma)) \leqq K[\mathrm{var}_{t_0}^{\sigma+\eta} P\text{-}\mathrm{var}_{t_0}^{\sigma} P] + $$
$$+ M\eta + \varepsilon\eta + K\left|\int_{\sigma}^{\sigma+\eta} \mathrm{D}[F(y(\tau), t) - F(x(\tau), t]\right| .$$

Using (4) and Lemma 3,1 from [3] for estimating the last term on the right hand side of this inequality we obtain

(14) $$\left|\int_{\sigma}^{\sigma+\eta} \mathrm{D}[F(y(\tau), t) - F(x(\tau), t)]\right| \leqq \int_{\sigma}^{\sigma+\eta} \mathrm{D}[\omega(|y(\tau) - x(\tau)|)\, h(t)] =$$
$$= \int_{\sigma}^{\sigma+\eta} \omega(|y(\tau) - x(\tau)|)\, \mathrm{d}h(\tau) = \lim_{\alpha\to 0+}\left(\int_{\sigma}^{\sigma+\alpha} + \int_{\sigma+\alpha}^{\sigma+\eta}\right) \omega(|y(\tau) - x(\tau)|)\, \mathrm{d}h(\tau) =$$
$$= \lim_{\alpha\to 0+} \int_{\sigma}^{\sigma+\eta} \omega(|y(\tau) - x(\tau)|)\, \mathrm{d}h(\tau) \leqq$$
$$\leqq \sup_{\varrho\in(\sigma,\sigma+\eta]} \omega(|y(\varrho) - x(\varrho)|) \lim_{\alpha\to 0+} (h(\sigma + \eta) - h(\sigma + \alpha)) =$$
$$= \sup_{\varrho\in(\sigma,\sigma+\eta]} \omega(|y(\varrho) - x(\varrho)|)\, (h(\sigma + \eta) - h(\sigma +))$$

because

$$\lim_{\alpha\to 0+} \int_{\sigma}^{\sigma+\alpha} \omega(|y(\tau) - x(\tau)|)\, \mathrm{d}h(\tau) = \omega(|y(\sigma) - x(\sigma)|) \lim_{\alpha\to 0+} (h(\sigma + \alpha) - h(\sigma)) =$$
$$= \omega(0) \lim_{\alpha\to 0+} (h(\sigma + \alpha) - h(\sigma)) = 0$$

by Theorem 1.3.5 from [2].

For $\varrho \in (\sigma, \sigma + \eta_2(\sigma))$ we have $x(\varrho) = y(\sigma) + \int_{\sigma}^{\varrho} \mathrm{D}F(x(\tau), t)$ (x is a solution of (1) with $x(\sigma) = y(\sigma)$) and

$$y(\varrho) - x(\varrho) = y(\varrho) - y(\sigma) - \int_{\sigma}^{\varrho} \mathrm{D}F(x(\tau), t) .$$

Hence

$$\lim_{\varrho\to\sigma+}(y(\varrho)-x(\varrho))=\lim_{\varrho\to\sigma+}(y(\varrho)-y(\sigma)-\int_\sigma^\varrho DF(x(\tau),t))=$$

$$=\lim_{\varrho\to\sigma+}[y(\varrho)-y(\sigma)-[(F(x(\sigma),\varrho)-F(x(\sigma),\sigma))]=$$

$$=y(\sigma+)-y(\sigma)-F(y(\sigma),\sigma+)+F(y(\sigma),\sigma)=$$

$$=y(\sigma+)-y(\sigma)-\lim_{\varrho\to\sigma+}\int_\sigma^\varrho DF(y(\tau),t)=$$

$$=y(\sigma+)-y(\sigma)-\lim_{\varrho\to\sigma+}\left(\int_{t_0}^\varrho DF(y(\tau),t)-\int_{t_0}^\sigma DF(y(\tau),t)\right)=$$

$$=\lim_{\varrho\to\sigma+}P(\varrho)-P(\sigma)=P(\sigma+)-P(\sigma)$$

and

$$\lim_{\varrho\to\sigma+}|y(\varrho)-x(\varrho)|=|P(\sigma+)-P(\sigma)|\,. \tag{15}$$

Let $\varepsilon>0$ be arbitrary and let us set

$$\alpha=\frac{\varepsilon}{K[h(t_1)-h(t_0)+1]}>0\,. \tag{16}$$

Assume that $\hat{\varrho}(\alpha)>0$ is such that for $\varrho\in(0,\hat{\varrho}(\alpha)]$ we have $\omega(\varrho)<\alpha$ and let us set $\gamma\in(0,\hat{\varrho}(\alpha)/2)$. By (15) there exists $\eta_3(\sigma)>0$, $\eta_3(\sigma)\leqq\eta_2(\sigma)$, such that for $\varrho\in(\sigma,\sigma+ +\eta_3(\sigma)]$ we have

$$|y(\varrho)-x(\varrho)|\leqq|P(\sigma+)-P(\sigma)|+\gamma\,. \tag{17}$$

Denote

$$N(\alpha)=\left\{\sigma\in[t_0,t_1];\ |P(\sigma+)-P(\sigma)|\geqq\frac{\hat{\varrho}(\alpha)}{2}\right\};$$

since $P\in BV[t_0,t_1]$, the set $N(\alpha)$ is finite; let $l(\alpha)$ be the number of elements of the set $N(\alpha)$. For $\sigma\in[t_0,t_1]\setminus N(\alpha)$ and $\varrho\in(\sigma,\sigma+\eta_3(\sigma)]$ we have

$$\omega(|y(\varrho)-x(\varrho)|)\leqq\omega(|P(\sigma+)-P(\sigma)|+\gamma)<\omega(\hat{\varrho}(\alpha)/2+\hat{\varrho}(\alpha)/2)=\omega(\hat{\varrho}(\alpha))<\alpha$$

and by (14) also

$$\left|\int_\sigma^{\sigma+\eta}D[F(y(\tau),t)-F(x(\tau),t)]\right|\leqq\alpha(h(\sigma+\eta)-h(\sigma+)) \tag{18}$$

for $\eta\in(0,\eta_3(\sigma))$.

Assume now that $\sigma\in[t_0,t_1]\cap N(\alpha)$. Since the limit

$$\lim_{\eta \to 0+} h(\sigma + \eta) = h(\sigma+)$$

exists, there exists $\eta_4(\sigma) > 0$, $\eta_4(\sigma) \leqq \eta_3(\sigma)$ such that for $0 < \eta < \eta_4(\sigma)$ we have

$$h(\sigma + \eta) - h(\sigma +) < \frac{\alpha}{(l(\alpha) + 1) \, . \, \omega(|P(\sigma+) - P(\sigma)| + \gamma)}$$

and $(\sigma, \sigma + \eta_4(\sigma)] \cap N(\alpha) = \emptyset$. By (14) and (17) we then have for $\eta \in (\sigma, \sigma + \eta_4(\sigma)]$

$$(19) \qquad \left| \int_\sigma^{\sigma+\eta} \mathrm{D}[F(y(\tau), t) - F(x(\tau), t)] \right| \leqq \sup_{\varrho \in (\sigma, \sigma+\eta)} \omega(|y(\varrho) - x(\varrho)|) \, .$$

$$\cdot \frac{\alpha}{(l(\alpha) + 1)(\omega(|P(\sigma+) - P(\sigma)| + \gamma)} \leqq$$

$$\leqq \omega(|P(\sigma+) - P(\sigma)| + \gamma) \, . \, \frac{\alpha}{l(\alpha) + 1} \, . \, \frac{1}{\omega(|P(\sigma+) - P(\sigma)| + \gamma)} = \frac{\alpha}{l(\alpha) + 1} \, .$$

Let us set

$$h_\alpha(t) = \frac{\alpha}{l(\alpha) + 1} \sum_{\sigma \in N(\alpha)} H_\sigma(t) \, , \quad t \in [t_0, t_1] \, ,$$

where $H_\sigma(t) = 0$ for $t \leqq \sigma$ and $H_\sigma(t) = 1$ for $t > \sigma$. The function $h_\alpha : [t_0, t_1] \to R$ is evidently nondecreasing and continuous from the left, $\mathrm{var}_{t_0}^{t_1} h_\alpha = h_\alpha(t_1) - h_\alpha(t_0) = l(\alpha) \, . \, \alpha/(l(\alpha) + 1) < \alpha$, the points of discontinuity of h_α are only the points belonging to $N(\alpha)$ and for every $t \in N(\alpha)$ we have $h_\alpha(t+) - h_\alpha(t) = \alpha/(l(\alpha) + 1)$.

Define further

$$\tilde{h}_\alpha(t) = \alpha \, h_c(t) + h_\alpha(t) \, . \quad t \in [t_0, t_1] \, ,$$

where h_c is the continuous part of the function h. The function $\tilde{h}_\alpha$ is nondecreasing and continuous from the left on $[t_0, t_1]$ and

$$\tilde{h}_\alpha(t_1) - \tilde{h}_\alpha(t_0) = \alpha[h_c(t_1) - h_c(t_0)] + h_\alpha(t_1) - h_\alpha(t_0) \leqq \alpha(h(t_1) - h(t_0) + 1) \, .$$

If we set $\eta(\sigma) = \eta_3(\sigma)$ for $\sigma \in [t_0, t_1] \setminus N(\alpha)$ and $\eta(\sigma) = \eta_4(\sigma)$ for $\sigma \in [t_0, t_1] \cap N(\alpha)$, then by (18), (19) and by the construction of $\tilde{h}_\alpha : [t_0, t_1] \to R$ we have

$$\left| \int_\sigma^{\sigma+\eta} [\mathrm{D}F(y(\tau), t) - F(x(\tau), t)] \right| \leqq \tilde{h}_\alpha(\sigma + \eta) - \tilde{h}_\alpha(\sigma)$$

for every $\eta \in [0, \eta(\sigma)]$ and by (13) also

$$V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma, y(\sigma)) \leqq K[\mathrm{var}_{t_0}^{\sigma+\eta} P - \mathrm{var}_{t_0}^{\sigma} P] + M\eta + \varepsilon\eta +$$
$$+ K(\tilde{h}_\alpha(\sigma + \eta) - \tilde{h}_\alpha(\sigma)) = g(\sigma + \eta) - g(\sigma)$$

for every $\sigma \in [t_0, t_1)$ and $\eta \in [0, \eta(\sigma)]$ where $\eta(\sigma) > 0$ and $g(t) = K \, \mathrm{var}_{t_0}^{t} P + Mt + \varepsilon t +$

$+ K\, \tilde{h}_\alpha(t)$ for $t \in [t_0, t_1]$ is a function continuous from the left on $[t_0, t_1]$, $g \in \in BV[t_0, t_1]$.

Using Proposition 4 we immediately conclude from this inequality that

$$
\begin{aligned}
(20) \qquad & V(t_1, y(t_1)) - V(t_0, y(t_0)) \leqq g(t_1) - g(t_0) = \\
& = K \operatorname{var}_{t_0}^{t_1} P + M(t_1 - t_0) + K(\tilde{h}_\alpha(t_1) - \tilde{h}_\alpha(t_0)) + \varepsilon(t_1 - t_0) \leqq \\
& \leqq K \operatorname{var}_{t_0}^{t_1} y(s) - \int_{t_0}^{s} \mathrm{D}F(y(\tau), t) + M(t_1 - t_0) + \varepsilon + \varepsilon(t_1 - t_0)
\end{aligned}
$$

because by (16) we have

$$\tilde{h}_\alpha(t_1) - \tilde{h}_\alpha(t_0) \leqq \alpha(h(t_1) - h(t_0) + 1) = \varepsilon/K .$$

Since $\varepsilon > 0$ was given arbitrarily, the inequality (12) follows directly from (20).

LJAPUNOV THEOREMS ON STABILITY

Now we give sufficient conditions for variational stability and asymptotic variational stability of the solution $x \equiv 0$ of the generalized differential equation (1). These sufficient conditions are formulated in terms of a certain kind of Ljapunov functions, which are suitable for the case of generalized differential equations.

1. Theorem *Assume that $V: [0, \infty) \times \bar{B}_a \to R$, $0 < a < c$, is such that for every $x \in \bar{B}_a$, $V(\cdot, x) \in BV[0, +\infty)$, $V(\cdot, x)$ is continuous from the left. Moreover, let $V(t, x)$ be positive definite, i.e. there exists a continuous increasing real function $b : [0, +\infty) \to R$ such that $b(\varrho) = 0$ if and only if $\varrho = 0$* *) *and*

$$(21) \qquad V(t, x) \geqq b(|x|) \quad \textit{for all} \quad (t, x) \in [0, +\infty) \times \bar{B}_a ,$$

$$(22) \qquad V(t, 0) = 0$$

and

$$(23) \qquad |V(t, x) - V(t, y)| \leqq K|x - y| ,$$

$K > 0$ being a constant.

If the function $V(t, x(t))$ is nonincreasing along every solution $x(t)$ of the equation (1) then the solution $x \equiv 0$ of the equation (1) is variationally stable.

Proof. Since for a solution $x : [a, b] \to R^n$ of (1) the function $V(t, x(t))$ is nonincreasing, we have

---

*) Let us note that if b^*: $[0, +\infty) \to R$ is continuous nondecreasing and such that $b^*(\varrho) = 0$ iff $\varrho = 0$ then there exists b: $[0, +\infty) \to R$, $b(\varrho) \leqq b^*(\varrho)$, $\varrho \geqq 0$ with the properties given in Theorem 1, i.e. b is increasing.

$$(24) \qquad \limsup_{\eta \to 0+} \frac{V(t + \eta, x(t + \eta)) - V(t. x(t))}{\eta} \leqq 0 .$$

Let $\varepsilon > 0$ and let $y : [t_0, t_1] \to R^n$ be such that $y \in BV[t_0, t_1]$, y continuous from the left on $[t_0, t_1]$.

By (12), from Lemma 1 and by (22), (23) we have ($\Phi \equiv 0$ in our case)

$$(25) \qquad V(r, y(r)) \leqq V(t_0, y(t_0)) + K \operatorname{var}_{t_0}^r \left(y(s) - \int_{t_0}^s DF(y(\tau), t) \right) \leqq$$

$$\leqq K|y(t_0)| + K \operatorname{var}_{t_0}^r \left(y(s) - \int_{t_0}^s DF(y(\tau), t) \right) \quad \text{for} \quad r \in [t_0, t_1] .$$

Let us set $\alpha(\varepsilon) = \inf_{r \geqq \varepsilon} b(r)$; then $\lim_{\varepsilon \to 0+} \alpha(\varepsilon) = 0$ and $\alpha(\varepsilon) > 0$ for $\varepsilon > 0$. Further, choose $\delta(\varepsilon) > 0$ such that $2K\, \delta(\varepsilon) < \alpha(\varepsilon)$. If $|y(t_0)| < \delta(\varepsilon)$ and $\operatorname{var}_{t_0}^{t_1} \left(y(s) - \int_{t_0}^s DF(y(\tau), t) \right) < \delta(\varepsilon)$ then by (25) we have

$$(26) \qquad V(r, y(r)) \leqq 2K\, \delta(\varepsilon) < \alpha(\varepsilon) \quad \text{for any} \quad r \in [t_0, t_1] .$$

If there existed $t^* \in [t_0, t_1]$ such that $|y(t^*)| \geqq \varepsilon$, then

$$\alpha(\varepsilon) = \inf_{r \geqq \varepsilon} b(r) \leqq b(|y(t^*)|) \leqq V(t^*, y(t^*))$$

would also hold and this contradicts (26). Hence $|y(t)| < \varepsilon$ for all $t \in [t_0, t_1]$ and $x \equiv 0$ is a variationally stable solution of (1) by Definition 1.

Remark. In the proof of Theorem 1 we use Lemma 1. In Lemma 1 the function V is given for $(t, x) \in [0, +\infty) \times R^n$. It is evident that it is possible to extend the function $V : [0, +\infty) \times \bar{B}_a \to R$ given in the assumptions of the theorem to the whole halfspace $[0, +\infty) \times R^n$ such that all requirements of Lemma 1 hold. The same is true also for the following theorem.

Theorem 2. *Assume that a function $V : [0, +\infty) \times \bar{B}_a \to R$, $0 < a \leqq c$ with the properties stated in Theorem 1 is given.*

If for every solution $x : [t_0, t_1] \to B_a$ of the equation (1) the inequality

$$(27) \qquad \limsup_{\eta \to 0+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leqq -\Phi(x(t)), \quad t_0 \leqq t < t_1$$

holds where $\Phi : R^n \to R$ is continuous, $\Phi(0) = 0$, $\Phi(x) > 0$ for $x \neq 0$, then the solution $x \equiv 0$ of the equation (1) is variational-asymptotically stable.

Proof. Since the function V satisfies all the assumptions given in Theorem 1 the solution $x \equiv 0$ of the equation (1) is variationally stable. It remains to show that this solution is also variationally attracting.

Let δ_0 be given such that $0 < \delta_0 < a$ and if $y \in BV[t_0, t_1]$, $0 \leqq t_0 < t_1 < +\infty$, y continuous from the left, $|y(t_0)| < \delta_0$ and $\operatorname{var}_{t_0}^{t_1}(y(s) - \int_{t_0}^{s} DF(y(\tau), t)) < \delta_0$ then $|y(t)| < a$, $t \in [t_0, t_1]$.

Further, let $\varepsilon > 0$ be arbitrary. Since $x \equiv 0$ is variationally stable there exists $\gamma^*(\varepsilon) > 0$ such that for every $y : [t_2, t_3] \to R^n$, $y \in BV[t_2, t_3]$, continuous from the left and such that

$$|y(t_2)| < \gamma^*(\varepsilon)$$

and

$$\operatorname{var}_{t_2}^{t_3}\left(y(s) - \int_{t_2}^{s} DF(y(\tau), t)\right) < \gamma^*(\varepsilon), \tag{28}$$

the inequality

$$|y(t)| < \varepsilon \tag{29}$$

holds for $t \in [t_2, t_3]$.

Let us set $\gamma(\varepsilon) = \min(\delta_0, \gamma^*(\varepsilon)) > 0$ and

$$T(\varepsilon) = -K(\delta_0 + \gamma(\varepsilon))/M > 0,$$

where $M = \sup_{\gamma(\varepsilon) \leqq |x| < \varepsilon} (-\Phi(x)) = -\inf_{\gamma(\varepsilon) \leqq |x| < \varepsilon} \Phi(x) < 0$, and assume that we are given $y : [t_0, t_1] \to R^n$, $y \in BV[t_0, t_1]$, y continuous from the left on $[t_0, t_1]$ and such that $|y(t_0)| < \delta_0$ and

$$\operatorname{var}_{t_0}^{t_1}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) < \gamma(\varepsilon).$$

Assume that $|y(t)| \geqq \gamma(\varepsilon)$ for every $t \in [t_0, t_0 \in T(\varepsilon)]$. Using Lemma 1 we obtain

$$\begin{aligned}
V(t, y(t)) - V(t_0, y(t_0)) &= V(t, y(t)) - V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon)) + \\
&\quad + V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon)) - V(t_0, y(t_0)) \leqq \\
&\leqq K \operatorname{var}_{t_0}^{t_0 + T(\varepsilon)}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) + M \,.\, T(\varepsilon) + \\
&\quad + K \operatorname{var}_{t_0 + T(\varepsilon)}^{t}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) + \sup_{\tau \in [t_0 + T(\varepsilon), t]} [-\Phi(x(\tau))] \leqq \\
&\leqq K \operatorname{var}_{t_0}^{t}\left(y(s) - \int_{t_0}^{s} DF(y(\tau), t)\right) + M \,.\, T(\varepsilon) < \\
&< K \,.\, \gamma(\varepsilon) + M \,.\, \frac{-K(\delta_0 + \gamma(\varepsilon))}{M} = -K\delta_0 \,.
\end{aligned}$$

Hence

$$V(t, y(t)) < V(t_0, y(t_0)) - K\delta_0 \leqq K|y(t_0)| - K\delta_0 < K\delta_0 - K\delta_0 = 0$$

and this contradicts the fact that V is assumed positive definite. Hence there necessarily exists $t^* \in [t_0, t_0 + T(\varepsilon)]$ such that $|y(t^*)| < \gamma(\varepsilon) \leqq \gamma^*(\varepsilon)$ and at the same time also

$$\operatorname{var}_{t^*}^{t_1}\left(y(s) - \int_{t^*}^{s} DF(y(\tau), t)\right) < \gamma(\varepsilon) \leqq \gamma^*(\varepsilon).$$

Hence by (29) we have $|y(t)| < \varepsilon$ for $t \in [t^*, t_1]$. This in particular yields $|y(t)| < \varepsilon$ for $t \in [t_0 + T(\varepsilon), t_1]$.

SOME REMARKS ON LINEAR SYSTEMS

Let us make some remarks on the concept of variational stability for equations with a special linear form of the function $F(x, t)$. We use the notation $L(R^n)$ for the linear space of $n \times n$ – matrices (linear operators on R^n) endowed with the operator norm corresponding to the norm given on R^n. Assume that $A : [0, +\infty) \to L(R^n)$ is continuous from the left, i.e. $A(t-) = A(t)$ for every $t \in (0, +\infty)$, and locally of bounded variation, i.e. $\operatorname{var}_a^b A < \infty$ for every compact interval $[a, b] \subset [0, +\infty)$.

For $(x, t) \in R^n \times [0, +\infty)$ define $F(x, t) = A(t)\,x$. It can be easily checked that the function $F(x, t)$ satisfies the assumptions (3), (4) and (5) with $h(t) = \operatorname{var}_0^t A$, $t \in [0, +\infty)$ and $\omega(r) = r$ for $r \geqq 0$.

The generalized differential equation corresponding to this linear function $F(x, t)$ was studied in [7]. It is clear that a function $x : [a, b] \to R^n$ is a solution of the equation

$$\frac{dx}{d\tau} = D[A(t)\,x] \tag{30}$$

if for every $s_1, s_2 \in [a, b]$ the equality

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} D[A(t)\,x(\tau)]$$

holds or (since the integral on the right hand side of this relation is the Perron-Stieltjes integral)

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} d[A(\tau)]\,x(\tau).$$

For the initial value problem

$$\frac{dx}{d\tau} = D[A(t)\,x], \quad x(t_0) = x_0 \in R^n$$

the solution satisfies the Stieltjes integral equation

$$x(t) = x_0 + \int_{t_0}^{s} \mathrm{d}[A(\tau)]\, x(\tau)\,.$$

In some situations there exists a uniquely determined matrix $X(t)$: $[0, +\infty) \to L(R^n)$ such that

$$X(t) = I + \int_0^t \mathrm{d}[A(r)]\, X(r) \quad \text{for} \quad t \geqq 0$$

(I is the $n \times n$ identity matrix) and the solution of the initial value problem can be written in the form

$$x(t) = U(t, t_0)\, x_0$$

where $U(t, t_0) = X(t)\, X^{-1}(t_0)$, $0 \leqq t_0 \leqq t < +\infty$.

It is known (see [7]) that *the matrix* $X : [0, +\infty) \to L(R^n)$ *exists and is regular for every* $t \geqq 0$ *only if and if the matrix* $I + \Delta^+ A(t) = I + A(t+) - A(t)$ *is regular for every* $t \in [0, +\infty)$.

We restrict ourselves to this case and consider the initial value problem for the nonhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A(t)\, x + P(t)]\,, \; x(t_0) = x_0\,, \tag{31}$$

where $P : [0, \infty) \to R^n$ is a function of locally bounded variation which is continuous from the left; then the solution to this problem satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} \mathrm{d}[A(r)]\, x(r) + P(t) - P(t_0)\,.$$

Using the variation of constants formula (see e.g. Proposition III.2.15 in [7]) we can write the explicit form of the solution $y(t, t_0, y_0)$ of the initial value problem for the nonhomogeneous equation:

$$y(t, t_0, y_0) = X(t)\, X^{-1}(t_0)\, y_0 + P(t) - P(t_0) - X(t) \int_{t_0}^{t} \mathrm{d}[X^{-1}(s)]\, (P(s) - P(t_0)) =$$

$$= X(t)\, [X^{-1}(t_0)\, y_0 + X^{-1}(t)\, (P(t) - P(t_0)) - \int_{t_0}^{t} \mathrm{d}[X^{-1}(s)]\, (P(s) - P(t_0))]\,.$$

Using the integration by parts formula for the Perron-Stieltjes integral (see [7], Theorem I.4.33) we obtain

$$X^{-1}(t)\, (P(t) - P(t_0)) - \int_{t_0}^{t} \mathrm{d}_s[X^{-1}(s)]\, (P(s) - P(t_0)) =$$

$$= \int_{t_0}^{t} X^{-1}(s)\, \mathrm{d}[P(s) - P(t_0)] + \sum_{t_0 \leqq \sigma < t} \Delta^+ X^{-1}(\sigma)\, \Delta^+ P(\sigma)\,,$$

since it can be shown (see Theorem III.2.10 in [7]) that the function $X^{-1}(t)$: $: [0, +\infty) \to L(R^n)$ is locally of bounded variation. Hence we have the following expression for the solution of the initial value problem (31):

$$(32) \qquad y(t, t_0, y_0) = X(t) \left\{ X^{-1}(t_0)\, y_0 + \int_{t_0}^{t} X^{-1}(s)\, \mathrm{d}[P(s) - P(t_0)] + \right.$$
$$\left. + \sum_{t_0 \leqq \sigma < t} \Delta^+ X^{-1}(\sigma) \Delta^+ P(\sigma) \right\}, \quad t \geqq t_0 .$$

This formula for the solution of (31) leads to the following result.

If $A : [0, +\infty) \to L(R^n)$ is an $n \times n$ – matrix valued function which is continuous from the left on $(0, +\infty)$, locally of bounded variation and such that the matrix $I + \Delta^+ A(t) = I + A(t+) - A(t)$ is regular for every $t \geqq 0$ then the zero solution of the generalized linear differential equation (30) is variationally stable if and only if the fundamental matrix $U(t, t_0) = X(t) X^{-1}(t_0)$ is bounded for $0 \leqq t_0 \leqq t$.

In fact, if $|U(t, t_0)| = |X(t) X^{-1}(t_0)| \leqq M$ for $0 \leqq t_0 \leqq t$ then by (32) we easily obtain the estimate

$$(33) \qquad |y(t, t_0, y_0)| \leqq M|y_0| + \left| \int_{t_0}^{t} U(t, s)\, \mathrm{d}[P(s) - P(t_0)] \right| +$$
$$+ \left| \sum_{t_0 \leqq \sigma < t} [U(t, \sigma^+) - U(t, \sigma)] \,.\, \Delta^+ P(\sigma) \right| \leqq$$
$$\leqq M|y_0| + M \,.\, \mathrm{var}_{t_0}^t P + 2M \,.\, \mathrm{var}_{t_0}^t P =$$
$$= M|y_0| + 3M \,.\, \mathrm{var}_{t_0}^t P , \quad t \geqq t_0 .$$

Hence if for every $\varepsilon > 0$ we take $\delta = \varepsilon/(4M + 1) > 0$ and if $|y_0| < \delta$, $\mathrm{var}_{t_0}^{t_1} P < \delta$ then $y(t, t_0, y_0) \leqq 4M \,.\, \delta < \varepsilon$ and the zero solution of (30) is evidently variationally stable by Proposition 2.

If, conversely, the zero solution of the equation (30) is variationally stable then by Definition 1 there exists $\delta > 0$ such that if $x : [t_0, +\infty) \to R^n$ is a solution of (30) i.e.

$$\mathrm{var}_{t_0}^{t_1} \left(x(s) - \int_{t_0}^{s} \mathrm{D}[A(t)\, x(\tau)] \right) = 0 \quad \text{for every} \quad t_1 \geqq t_0$$

and $|x(t_0)| < \delta$ then $|x(t)| < 1$ for every $t \geqq t_0$. Let us set $y(t) = X(t) X^{-1}(t_0)\, z$, $t \geqq t_0$ and define $x(t) = (\delta/2)\, y(t)$, $t \geqq t_0$. It is assumed that $z \in R^n$ is arbitrary and such that $|z| \leqq 1$. Then $|x(t_0)| = (\delta/2) |y(t_0)| = (\delta/2) |z| \leqq \delta/2 < \delta$ and $x(t)$ is a solution of (30). Hence $|x(t)| < 1$, i.e. $(\delta/2) |y(t)| < 1$ and consequently

$$|X(t) X^{-1}(t_0)\, z| = |y(t)| < 2/\delta .$$

Hence also $|X(t) X^{-1}(t_0)| = \sup_{|z| \leqq 1} |X(t) X^{-1}(t_0)\, z| \leqq 2/\delta = M$ for $t \geqq t_0$ and the fundamental matrix $U(t, t_0)$ is bounded.

The subsequent parts of the paper are devoted to the conversion of the Ljapunov type stability Theorems 1 and 2. We will show that the variational stability and the asymptotic variational stability imply the existence of Ljapunov functions of the type given in Theorems 1 and 2.

The methods used here for generalized differential equations are strongly influenced by those of Chow and Yorke from [1]. The character of the generalized differential equations and their solutions, which are in general functions of locally bounded variation, forces us to use different devices for obtaining the corresponding results.

FURTHER AUXILIARY STATEMENTS

Let us introduce a slightly modified notion of the variation of a function.

Definition 4. *Assume that $-\infty < a < b < +\infty$ and let $G : [a, b] \to R^n$ is given. For a given decomposition*

$$D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$$

of the interval $[a, b]$ and for every $\lambda \geqq 0$ define

$$\sum_{j=1}^{k} \mathrm{e}^{-\lambda(b-\alpha_{j-1})}|G(\alpha_j) - G(\alpha_{j-1})| = v_\lambda(G, D)$$

and set

$$\mathrm{e}_\lambda \operatorname{var}_a^b G = \sup_D v_\lambda(G, D),$$

where the supremum is taken over all finite decompositions D of the interval $[a, b]$. The number $\mathrm{e}_\lambda \operatorname{var}_a^b G$ will be called the e_λ-variation of the function G over the interval $[a, b]$.

Lemma 2. *If $-\infty < a < b < +\infty$ and $G : [a, b] \to R^n$ then for every $\lambda \geqq 0$ we have*

$$\mathrm{e}^{-\lambda(b-a)} \operatorname{var}_a^b G \leqq \mathrm{e}_\lambda \operatorname{var}_a^b G \leqq \operatorname{var}_a^b G. \tag{34}$$

If $a \leqq c \leqq b$ then for $\lambda \geqq 0$ the identity

$$\mathrm{e}_\lambda \operatorname{var}_a^b G = \mathrm{e}^{-\lambda(b-c)}\, \mathrm{e}_\lambda \operatorname{var}_a^c G + \mathrm{e}_\lambda \operatorname{var}_c^b G \tag{35}$$

holds.

Proof. For every $\lambda \geqq 0$ and every decomposition $D : a = \alpha_0 < \alpha_1 < \ldots < a_k = b$ we have

$$\mathrm{e}^{-\lambda(b-a)} \leqq \mathrm{e}^{-\lambda(b-\alpha_{j-1})} \leqq \mathrm{e}^0 = 1\,, \quad j = 1, 2, \ldots, k\,.$$

Hence

$$\mathrm{e}^{-\lambda(b-a)} v_0(G, D) \leqq v_\lambda(G, D) \leqq v_0(G, D) = \sum_{j=1}^{k} |G(\alpha_j) - G(\alpha_{j-1})|$$

and passing to the supremum over all finite decompositions D of the interval $[a, b]$ we obtain (34).

For second statement it is easy to see that we can restrict ourselves to decompositions D which contain the point c as a node, i.e.

$$D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_{l-1} < \alpha_l = c < \alpha_{l+1} < \ldots < \alpha_k = b\,.$$

Then we have

$$\begin{aligned}(36) \qquad v_\lambda(G, D) &= \sum_{j=1}^{k} e^{-\lambda(b-\alpha_{j-1})}|G(\alpha_j) - G(\alpha_{j-1})| = \\ &= \sum_{j=1}^{l} e^{-\lambda(b-\alpha_{j-1})}|G(\alpha_j) - G(\alpha_{j-1})| + \sum_{j=l+1}^{k} e^{-\lambda(b-\alpha_{j-1})}|G(\alpha_j) - G(\alpha_{j-1})| = \\ &= e^{-\lambda(b-c)} \sum_{j=1}^{l} e^{-\lambda(c-\alpha_{j-1})}|G(\alpha_j) - G(\alpha_{j-1})| + \sum_{j=l+1}^{k} e^{-\lambda(b-\alpha_{j-1})}|G(\alpha_j) - G(\alpha_{j-1})| = \\ &= e^{-\lambda(b-c)} v_\lambda(G, D_1) + v_\lambda(G, D_2)\,,\end{aligned}$$

where

$$D_1 : a = \alpha_0 < \alpha_1 < \ldots < \alpha_l = c \quad \text{and}$$

$$D_2 : c = \alpha_1 < \alpha_{l+1} < \ldots < \alpha_k = b$$

are decompositions of $[a, c]$ and $[c, b]$, respectively. On the other hand, any two such decompositions D_1 and D_2 form a decomposition D of the interval $[a, b]$. The equality (35) now easily follows from (36) when we pass to the corresponding suprema.

Corollary 1. *If $a \leqq c \leqq b$ and $\lambda \geqq 0$ then*

$$(37) \qquad e_\lambda \operatorname{var}_c^b G \leqq e_\lambda \operatorname{var}_a^b G\,.$$

For $a > 0$, $t > 0$, $x \in B_a$ let us denote

$$A_a(t, x) = \{\varphi : [0, +\infty) \to R^n;\ \varphi \in BV_{\mathrm{loc}}[0, +\infty)\,,\ \ \varphi(0) = 0\,,$$
$$\varphi(t) = x,\ \varphi \text{ is continuous from the left and } \sup_{s\in[0,t]} |\varphi(s)| < a\}\,.$$

Moreover, for $\lambda \geqq 0$, $s \geqq 0$ and $x \in B_a$ define

$$(38) \qquad V_\lambda(s, x) \begin{cases} = \inf\limits_{\varphi\in A_a(s,x)} \left\{e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right)\right\} & \text{for } s > 0 \\ = |x| \quad \text{for } s = 0\,. \end{cases}$$

Let us mention that this definition makes sense because for $\varphi \in A_a(s, x)$ the integral $\int_0^\sigma DF(\varphi(\tau), t)$ is of bounded variation as a function of the variable σ and consequently, the function $\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)$ is of bounded variation on $[0, s]$ as well and the e_λ-variation of this function is bounded.

It is evident that the function $\varphi = 0$ belongs to $A_a(s, 0)$ and consequently we have

$$V_\lambda(s, 0) = 0 \tag{39}$$

for every $s \geqq 0$ and $\lambda \geqq 0$, because $\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \equiv 0$ for every $\sigma \geqq 0$.

Since $e_\lambda \operatorname{var}_0^s(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)) \geqq 0$ for every $\varphi \in A_a(s, x)$ we have by the definition (38) also

$$V_\lambda(s, x) \geqq 0 \quad \text{for every} \quad s \geqq 0 \quad \text{and} \quad x \in R^n . \tag{40}$$

Lemma 3. *For $x, y \in B_a = \{x \in R^n;\ |x| < a\}$, $s \in [0, +\infty)$ and $\lambda \geqq 0$ the inequality*

$$|V_\lambda(s, x) - V_\lambda(s, y)| \leqq |x - y| \tag{41}$$

holds.

Proof. Assume that $s > 0$ and $0 < \eta < s$. Let $\varphi \in A_a(s, x)$ be arbitrary. Define

$$\varphi_\eta(\sigma) = \varphi(\sigma) \quad \text{for} \quad \sigma \in [0, s - \eta] ,$$

$$\varphi_\eta(\sigma) = \varphi(s - \eta) + \frac{1}{\eta}(y - \varphi(s - \eta))(\sigma - s + \eta) \quad \text{for} \quad \sigma \in [s - \eta, s] .$$

The function φ_η coincides with the function φ on $[0, s - \eta]$ and is linear with $\varphi_\eta(s) = y$ on $[s - \eta, s]$. By the definition we clearly have $\varphi_\eta \in A_a(s, y)$ and using (35) from Lemma 2 we get

$$V_\lambda(s, y) \leqq e_\lambda \operatorname{var}_0^s \left((\varphi_\eta(\sigma) - \int_0^\sigma DF(\varphi_\eta(\tau), t)\right) =$$

$$= e^{-\lambda\eta} e_\lambda \operatorname{var}_0^{s-\eta} \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) + e_\lambda \operatorname{var}_{s-\eta}^s \left(\varphi_\eta(\sigma) - \int_0^\sigma DF(\varphi_\eta(\tau), t)\right) \leqq$$

$$\leqq e^{-\lambda\eta} e_\lambda \operatorname{var}_0^{s-\eta}\left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) + \operatorname{var}_{s-\eta}^s \varphi_\eta + \operatorname{var}_{s-\eta}^s \left(\int_0^\sigma DF(\varphi_\eta(\tau), t)\right) \leqq$$

$$\leqq e^{-\lambda\eta} e_\lambda \operatorname{var}_0^{s-\eta} \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) + |y - \varphi(s - \eta)| + h(s) - h(s - \eta) .$$

Since for every $\eta > 0$ we have

$$e^{-\lambda\eta} e_\lambda \operatorname{var}_0^{s-\eta} \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) = e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) -$$

$$-\mathrm{e}_\lambda \operatorname{var}_{s-\eta}^{s}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) \leqq \mathrm{e}_\lambda \operatorname{var}_0^{s}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right),$$

we obtain for every $\eta > 0$ the inequality

$$V_\lambda(s, y) \leqq \mathrm{e}_\lambda \operatorname{var}_0^{s}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) + |y - \varphi(s - \eta)| + h(s) - h(s - \eta).$$

Since φ and h are assumed to be continuous from the left: $\lim_{\tau \to s-} \varphi(\tau) = \varphi(s) = x$, and the last inequality holds for every $\eta > 0$ we can pass to the limit $\eta \to 0+$ in order to obtain the inequality

$$V_\lambda(s, y) \leqq \mathrm{e}_\lambda \operatorname{var}_0^{s}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) + |y - x|,$$

which holds for every $\varphi \in A_a(s, x)$. Taking the infimum for all $\varphi \in A_a(s, x)$ on the right hand side of this inequality we get the inequality

$$V_\lambda(s, y) \leqq V_\lambda(s, x) + |y - x|. \tag{42}$$

Since all is symmetric in x and y we similarly obtain the inequality

$$V_\lambda(s, x) \leqq V_\lambda(s, y) + |y - x|$$

and this together with (42) yields the inequality (41) for $s > 0$.

If $s = 0$, then we have by definition

$$|V_\lambda(0, y) - V_\lambda(0, x)| = \big| |y| - |x| \big| \leqq |y - x|.$$

Hence the statement of Lemma 3 is proved.

Corollary 2. *Since $V_\lambda(s, 0) = 0$ for every $s \geqq 0$, we have by (39) and (41)*

$$0 \leqq V_\lambda(s, x) \leqq |x|. \tag{43}$$

Lemma 4. *For $y \in B_a$, $s, r \in [0, +\infty)$ and $\lambda \geqq 0$ the inequality*

$$|V_\lambda(r, y) - V_\lambda(s, y)| \leqq (1 - \mathrm{e}^{-\lambda|r-s|})\, a + |h(r) - h(s)| \tag{44}$$

holds.

Proof. Suppose that $0 < s \leqq r$ and let $\varphi \in A_a(r, y)$ be given. Then by Lemma 2 we have

$$\mathrm{e}_\lambda \operatorname{var}_0^{r}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) = \mathrm{e}^{-\lambda(r-s)} \mathrm{e}_\lambda \operatorname{var}_0^{s}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) + \tag{45}$$

$$+ \mathrm{e}_\lambda \operatorname{var}_s^{r}\left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) \geqq$$

$$\geqq e^{-\lambda(r-s)}V_\lambda(s, \varphi(s)) + e^{-\lambda(r-s)} \operatorname{var}_s^r \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) \geqq$$

$$\geqq e^{-\lambda(r-s)}\left[V_\lambda(s, \varphi(s)) + \operatorname{var}_s^r \varphi - \operatorname{var}_s^r \int_0^\sigma DF(\varphi(\tau), t)\right] \geqq$$

$$\geqq e^{-\lambda(r-s)}[V_\lambda(s, \varphi(s)) + |y - \varphi(s)| - (h(r) - h(s))] \geqq$$

$$\geqq e^{-\lambda(r-s)}[V_\lambda(s, y) - (h(r) - h(s))] .$$

By (41) from Lemma 3, we have

$$V_\lambda(s, \varphi(s)) + |y - \varphi(s)| \geqq V_\lambda(s, y) .$$

Hence passing to the infimum over $\varphi \in A_a(r, y)$ on the left hand side of (45) we obtain

$$V_\lambda(r, y) \geqq e^{-\lambda(r-s)}[V_\lambda(s, y) - (h(r) - h(s))] \geqq \tag{46}$$

$$\geqq e^{-\lambda(r-s)}V_\lambda(s, y) - (h(r) - h(s)) .$$

Now let $\varphi \in A_a(s, y)$ be arbitrary; define

$$\varphi^*(\sigma) = \begin{cases} \varphi(\sigma) & \text{for} \quad \sigma \in [0, s] , \\ y & \text{for} \quad \sigma \in [s, r] . \end{cases}$$

Evidently $\varphi^*(s) = \varphi(s) = y$, $\varphi^* \in A_a(r, y)$ and by (35), (34)

$$V_\lambda(r, y) \leqq e_\lambda \operatorname{var}_0^r \left(\varphi^*(\sigma) - \int_0^\sigma DF(\varphi^*(\tau), t)\right) =$$

$$= e^{-\lambda(r-s)}e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) + e_\lambda \operatorname{var}_s^r \left(\varphi^*(\sigma) - \int_0^\sigma DF(\varphi^*(\tau), t)\right) \leqq$$

$$\leqq e^{-\lambda(r-s)}e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) + \operatorname{var}_s^r \varphi^* + \operatorname{var}_s^r \left(\int_0^\sigma DF(\varphi^*(\tau), t)\right) \leqq$$

$$\leqq e^{-\lambda(r-s)}e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) + h(r) - h(s) .$$

Taking the infimum over all $\varphi \in A^a(s, y)$ on the right hand side of this inequality we obtain

$$V_\lambda(r, y) \leqq e^{-\lambda(r-s)}V_\lambda(s, y) + h(r) - h(s) .$$

Together with (46) we have

$$|V_\lambda(r, y) - e^{-\lambda(r-s)}V_\lambda(s, y)| \leqq h(r) - h(s) .$$

Hence by (43) we get the inequality

$$|V_\lambda(r, y) - V_\lambda(s, y)| \leqq |V_\lambda(r, y) - e^{-\lambda(r-s)}V_\lambda(s, y)| + |1 - e^{-\lambda(r-s)}| \, |V_\lambda(s, y)| \leqq$$

$$\leqq h(r) - h(s) + (1 - e^{-\lambda(r-s)})\,|y| \leqq h(r) - h(s) + (1 - e^{-\lambda(r-s)})\,a$$

because $|y| \leqq a$, i.e. we have obtained (44).

Assume that $s = 0$ and $r > 0$. Then by (43) and from the definition (38),

$$V_\lambda(r, y) - V_\lambda(s, y) = V_\lambda(r, y) - V_\lambda(0, y) = V_\lambda(r, y) - |y| \leqq 0\,. \tag{47}$$

Let us derive an estimate from below. Assume that $\varphi \in A_a(r, y)$. We have

$$e_\lambda \operatorname{var}_0^r \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)\right) \geqq e_\lambda \operatorname{var}_0^r \varphi(\sigma) - e_\lambda \operatorname{var}_0^r \int_0^\sigma DF(\varphi(\tau), t) \geqq$$

$$\geqq e^{-\lambda r} \operatorname{var}_0^r \varphi - \operatorname{var}_0^r \int_0^\sigma DF(\varphi(\tau), t) \geqq e^{-\lambda r}|\varphi(r) - \varphi(0)| - (h(r) - h(0)) =$$

$$= e^{-\lambda r}|y| - (h(r) - h(0))$$

by (34), Lemma 2 and Proposition 1. Passing again to the infimum for $\varphi \in A_a(r, y)$ on the left hand side of this inequality we get

$$V_\lambda(r, y) \geqq e^{-\lambda r}|y| - (h(r) - h(0))$$

and

$$V_\lambda(r, y) - V_\lambda(0, y) = V_\lambda(r, y) - |y| \geqq (e^{-\lambda r} - 1)\,|y| - (h(r) - h(0)) =$$

$$= -(1 - e^{-\lambda r})\,|y| - (h(r) - h(0))\,.$$

Together with (47) we obtain

$$|V_\lambda(r, y) - V_\lambda(0, y)| \leqq (1 - e^{-\lambda r})\,a + (h(r) - h(0))\,,$$

i.e. the inequality (44) holds in this case, too. The remaining case $s = r = 0$ is evident. Finally, let us mention that the case $r < s$ can be dealt with in the same way because the situation is symmetric in s and r.

By Lemmas 3 and 4 we immediately conclude that the following holds.

Corollary 3. *For $x, y \in B_a = \{x \in R^n;\ |x| < a\}$, $r, s \in [0, +\infty)$ and $\lambda \geqq 0$ the inequality*

$$|V_\lambda(s, x) - V_\lambda(r, y)| \leqq |x - y| + (1 - e^{-\lambda|r-s|})\,a + |h(r) - h(s)| \tag{48}$$

holds.

Let us now discuss the behaviour of the function $V_\lambda(t, x)$ along the solutions of the generalized differential equation

$$\frac{dx}{d\tau} = DF(x, t)\,.$$

We still assume that the function $F(x, t)$ satisfies the usual assumptions (2), (3), (4) and (5). The following statement is important for the forthcoming considerations.

Lemma 5. *If $\psi : [s, s + \eta(s)] \to R^n$ is a solution of the generalized differential equation* (1), $s \geqq 0$, $\eta(s) > 0$, *then for every $\lambda \geqq 0$ the inequality*

$$(49) \qquad \limsup_{\eta \to 0+} \frac{V_\lambda(s + \eta, \psi(s + \eta)) - V_\lambda(s, \psi(s))}{\eta} \leqq -\lambda V_\lambda(s, \psi(s))$$

holds.

Proof. Let $s \in [0, +\infty)$ and $x \in R^n$ be given. Let us choose $a > 0$ such that $a > |x| + h(s + 1) - h(s)$. Assume that $\varphi \in A_a(s, x)$ is given and let ψ be a solution of the equation (1) with $\psi(s) = x$ defined for $\sigma \in [s, s + \eta(s)]$; $0 < \eta(s) < 1$. The existence of such a solution ψ is guaranteed by the local existence theorem for the equation (1), see e.g. Theorem 2,1 in [3]. For $0 < \eta < \eta(s)$ define

$$\varphi_\eta(\sigma) = \varphi(\sigma) \quad \text{for} \quad \sigma \in [0, s],$$
$$\varphi_\eta(\sigma) = \psi(\sigma) \quad \text{for} \quad \sigma \in [s, s + \eta];$$

because we have $\varphi(s) = \psi(s) = x = \psi_\eta(s)$. Evidently $\varphi_\eta \in A_a(s + \eta, \psi(s + \eta))$ because $\psi(\sigma)$ is continuous from the left and by the definition of a solution we have

$$|\psi(\sigma)| = \left| x + \int_s^\sigma DF(\psi(\tau), t) \right| \leqq |x| + h(\sigma) - h(s) \leqq$$
$$\leqq |x| + h(s + 1) - h(s) < a$$

for $\sigma \in [s, s + \eta]$ and

$$V_\lambda(s + \eta, \psi(s + \eta)) \leqq e_\lambda \operatorname{var}_0^{s+\eta} \left(\varphi_\eta(\sigma) - \int_0^\sigma DF(\varphi_\eta(\tau), t) \right) =$$
$$= e^{-\lambda\eta} e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \right) + e_\lambda \operatorname{var}_s^{s+\eta} \left(\psi(\sigma) - \int_0^s DF(\varphi(\tau), t) - \right.$$
$$\left. - \int_s^\sigma DF(\psi(\tau), t) \right) =$$
$$= e^{-\lambda\eta} e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \right) + e_\lambda \operatorname{var}_s^{s+\eta} \left(x + \int_0^s DF(\varphi(\tau), t) \right) =$$
$$= e^{-\lambda\eta} e_\lambda \operatorname{var}_0^s \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \right).$$

Taking the infimum for all $\varphi \in A_a(s, x)$ on the right hand side of this inequality we obtain

$$V_\lambda(s + \eta, \psi(s + \eta)) \leqq e^{-\lambda\eta} V_\lambda(s, x) = e^{-\lambda\eta} V_\lambda(s, \psi(s)).$$

This inequality yields

$$V_\lambda(s + \eta, \psi(s + \eta)) - V_\lambda(s, \psi(s)) \leqq (e^{-\lambda\eta} - 1) V_\lambda(s, \psi(s))$$

and also

$$\frac{V_\lambda(s + \eta, \psi(s + \eta)) - V_\lambda(s, \psi(s))}{\eta} \leqq \frac{e^{-\lambda\eta} - 1}{\eta} V_\lambda(s, \psi(s))$$

for every $0 < \eta < \eta(s)$.

Since $\lim_{\eta \to 0} (e^{-\lambda\eta} - 1)/\eta = -\lambda$ we immediately obtain the inequality (49).

CONVERSE LJAPUNOV THEOREMS

Now we are in position when we can formulate and prove converse theorems to Theorems 1 and 2. The first of them concerns the case of variational stability.

Theorem 3. *If the solution $x \equiv 0$ of the generalized differential equation* (1) *is variationally stable then for every $0 < a < c$ there exists a function $V: [0, +\infty) \times B_a \to R$ satisfying the following conditions*:

1) *for every $x \in B_a$ the function $V(\cdot, x)$ is continuous from the left and $V(\cdot, x) \in BV_{\mathrm{loc}}[0, +\infty)$.*

2) $V(t, 0) = 0$ *and* $|V(t, x) - V(t, y)| \leqq |x - y|$ *for* $x, y \in B_a$, $t \in [0, +\infty)$,

3) *the function V is nonincreasing along the solutions of the equation* (1),

4) *the function $V(t, x)$ is positive definite, i.e., there exists a continuous nondecreasing real-valued function $b : [0, +\infty) \to R$ such that $b(\varrho) = 0$ if and only if $\varrho = 0$ and*

$$b(|x|) \leqq V(t, x)$$

for every $t \in [0, +\infty)$, $x \in B_a$.

Proof. A candidate for the function V is the function $V_0(s, x)$ defined by (38) for $\lambda = 0$, i.e., we take $V(s, x) = V_0(s, x)$. The properties stated in 1) are easy consequences of Corollary 3. 2) follows from (39) and from Lemma 3. By Lemma 5, for every solution $\psi : [s, s + \delta] \to R^n$ of (1) we have

$$\limsup_{\eta \to 0+} \frac{V(s + \eta, \psi(s + \eta)) - V(s, \psi(s))}{\eta} \leqq 0\,.$$

Hence 3) is also satisfied. It remains to prove that the function $V(t, x)$ is positive definite; this is the only point where the variational stability of the solution $x \equiv 0$ of the equation (1) is used. Assume that there is an ε, $0 < \varepsilon < a$, and a sequence (t_k, x_k), $k = 1, 2, \ldots$, $\varepsilon \leqq |x_k| < a$, $t_k \to \infty$ for $k \to \infty$ such that $V(t_k, x_k) \to 0$ for $k \to \infty$.

Let $\delta(\varepsilon) > 0$ (cf. Proposition 2) be such that for every $t_0 \geqq 0$, $P \in BV_{\mathrm{loc}}[0, +\infty)$ continuous from the left we have $|y(t, t_0, y_0)| < \varepsilon$ for $t \geqq t_0$, where $y(t, t_0, y_0)$ is a solution of

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \mathrm{D}[F(y, t) + P(t)]\,, \quad y(t_0, t_0, y_0) = y_0$$

with $|y_0| < \delta(\varepsilon)$, $\mathrm{var}_{t_0}^{\infty} P < \delta(\varepsilon)$.

Assume that $k_0 \in N$ is such that for $k > k_0$ we have $V(t_k, x_k) < \delta(\varepsilon)$. Then there exists $\varphi_k \in A_a(t_k, x_k)$ such that

$$\mathrm{var}_0^{t_k}\left(\varphi_k(\sigma) - \int_0^{\sigma} \mathrm{D}F(\varphi_k(\tau), t)\right) < \delta(\varepsilon)\,.$$

Let us set

$$P(\sigma) = \varphi_k(\sigma) - \int_0^{\sigma} \mathrm{D}F(\varphi_k(\tau), t) \quad \text{for} \quad \sigma \in [0, t_k]\,,$$

$$P(\sigma) = x_k - \int_0^{t_k} \mathrm{D}F(\varphi_k(\tau), t) \quad \text{for} \quad \sigma \in [t_k, +\infty)\,.$$

We evidently have

$$\mathrm{var}_0^{\infty} P = \mathrm{var}_0^{t_k}\left(\varphi_k(\sigma) - \int_0^{\sigma} \mathrm{D}F(\varphi_k(\tau), t)\right) < \delta(\varepsilon)$$

and P is continuous from the left. For $\sigma \in [0, t]$ we have

$$\varphi_k(\sigma) = \int_0^{\sigma} \mathrm{D}F(\varphi_k(\tau), t) + \varphi_k(\sigma) - \int_0^{\sigma} \mathrm{D}F(\varphi_k(\tau), t) =$$

$$= \int_0^{\sigma} \mathrm{D}F(\varphi_k(\tau), t) + P(\sigma) - P(0) = \varphi_k(0) + \int_0^{\sigma} \mathrm{D}[F(\varphi_k(\tau), t) + P(t)]$$

since $\varphi_k(0) = 0$. Hence φ_k is a solution of the equation $\mathrm{d}y/\mathrm{d}\tau = \mathrm{D}[F(y, t) + P(t)]$ and consequently, by the variational stability we have $|\varphi_k(s)| < \varepsilon$ for every $s \in [0, t_k]$. Hence we also have $|\varphi_k(t_k)| = |x_k| < \varepsilon$ but this contradicts our assumption. In this way we obtain that the function $V(t, x)$ is positive definite and 4) is also satisfied.

The following result is a converse theorem for the case of asymptotic variational stability.

Theorem 4. *If the solution $x \equiv 0$ of the generalized differential equation* (1) *is variational–asymptotically stable then for every $a > 0$, $a < c$ there exists a function $U : [0, +\infty) \times B_a \to R$ satisfying the following conditions:*

1) *For every $x \in B_a$ the function $U(\cdot, x)$ is continuous from the left and $U(\cdot, x) \in BV_{\mathrm{loc}}(0, +\infty)$.*

2) $U(t, 0) = 0$ *and*

$$|U(t, x) - U(t, y)| \leqq |x - y| \quad \text{for} \quad x, y \in B_a, \quad t \in [0, +\infty).$$

3) *For every solution* $\psi(\sigma)$ *of the equation* (1) *defined for* $\sigma \geqq t$ *and satisfying* $\psi(t) = x \in B_a$ *the relation*

$$\limsup_{\eta \to 0+} \frac{U(t + \eta, \psi(t + \eta)) - U(t, x)}{\eta} \leqq -U(t, x)$$

holds.

4) *The function* $U(t, x)$ *is positive definite.*

Proof. For $x \in B_a$, $s \geqq 0$ let us set

$$U(s, x) = V_1(s, x)$$

where V_1 is given by (38) for $\lambda = 1$. In the same way as in the proof of Theorem 3 we can easily see that the function U satisfies 1), 2) and 3). (Let us mention that 3) is exactly the fact stated in Lemma 5.) Hence it remains to show that 4) is also satisfied for our choice of the function U.

Since the solution $x \equiv 0$ of the equation (1) is variationally attracting (see Proposition 3) there exists $\delta_0 > 0$ and for any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $\gamma(\varepsilon) > 0$ such that if $|y_0| < \delta_0$ and $\text{var}_{t_0}^{t_1} P < \gamma(\varepsilon)$, $P \in BV[t_0, t_1]$, P continuous from the left, then

$$|y(t, t_0, y_0)| < \varepsilon$$

for all $t \in [t_0, t_1]$, $t \geqq t_0 + T(\varepsilon)$ and $t_0 \geqq 0$. The function $y(t, t_0, y_0)$ is a solution of $\mathrm{d}x/\mathrm{d}\tau = \mathrm{D}[F(x, t) + P(t)]$ with $y(t_0, t_0, y_0) = y_0$.

Assume that U is not positive definite. Then there exists ε, $0 < \varepsilon < a = \delta_0$, and sequences t_k, x_k, $k = 1, 2, \ldots$ such that $\varepsilon \leqq |x_k| < a$ for $k = 1, 2, \ldots$ and $t_k \to +\infty$, $U(t_k, x_k) \to 0$ for $k \to \infty$.

Let us choose $k_0 \in N$ such that for $k \in N$, $k > k_0$ wo have $t_k > T(\varepsilon) + 1$ and

$$U(t_k, x_k) < \gamma(\varepsilon)\, \mathrm{e}^{-(T(\varepsilon)+1)}.$$

According to the definition of U let us choose $\varphi \in A_a(t_k, x_k)$ such that

$$\mathrm{e}_1 \operatorname{var}_0^{t_k} \left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) < \gamma(\varepsilon)\, \mathrm{e}^{-(T(\varepsilon)+1)}.$$

Let us set $t_0 = t_k - (T(\varepsilon) + 1)$; we have $t_0 > 0$ because $t_k > T(\varepsilon) + 1$ and also $t_k = t_0 + T(\varepsilon) + 1 > t_0 + T(\varepsilon)$. Further, evidently

$$\mathrm{e}_1 \operatorname{var}_{t_0}^{t_k} \left(\varphi(\sigma) - \int_0^\sigma \mathrm{D}F(\varphi(\tau), t)\right) < \gamma(\varepsilon)\, \mathrm{e}^{-(T(\varepsilon)+1)}$$

and by (34) from Lemma 2 also

$$e^{-(T(\varepsilon)+1)} \operatorname{var}_{t_0}^{t_k} \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \right) =$$

$$= e^{-(t_k - t_0)} \operatorname{var}_{t_0}^{t_k} \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \right) < \gamma(\varepsilon)\, e^{-(T(\varepsilon)+1)}$$

and consequently

$$\operatorname{var}_{t_0}^{t_k} \left(\varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) \right) < \gamma(\varepsilon) . \tag{50}$$

For $\sigma \in [t_0, t_k]$ define

$$P(\sigma) = \varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t) .$$

$P : [t_0, t_k] \to R^n$ is continuous from the left and by (50),

$$\operatorname{var}_{t_0}^{t_k} P < \gamma(\varepsilon) .$$

Moreover, for $\sigma \in [t_0, t_k]$ we have

$$\varphi(\sigma) = \int_0^\sigma DF(\varphi(\tau), t) + \varphi(\sigma) - \int_0^\sigma DF(\varphi(\tau), t)$$

and also

$$\varphi(s) - \varphi(t_0) = \int_{t_0}^s DF(\varphi(\tau), t) + P(s) - P(t_0) = \int_{t_0}^s D[F(y(\tau), t) + P(t)] ,$$

i.e. the function $\varphi : [t_0, t_k] \to R^n$ is a solution of the equation $dx/d\tau = D[F(x, t) + P(t)]$ with $|\varphi(t_0)| \leqq a = \delta_0$ because $\varphi \in A_a(t_k, x_k)$. By the definition of variational attractivity the inequality $|\varphi(t)| < \varepsilon$ holds for every $t > t_0 + T(\varepsilon)$. This is valid also for the value $t = t_k > t_0 + T(\varepsilon)$, i.e. $|\varphi(t_k)| = |x_k| < \varepsilon$, which contradicts our assumption $|x_k| \geqq \varepsilon$. This yields the positive definiteness of U.

INTERVAL BOUNDED PERTURBATIONS

In [1] Chow and Yorke proved that in the case of ordinary differential equations the integral asymptotic stability of the solution $x \equiv 0$ is maintained if the system is perturbed by the larger class of interval bounded functions. This result can be transfered to the case of generalized differential equations, too.

Definition 5. *A function $P : [0, +\infty) \to R^n$ is said to be of interval bounded variation if*

$$\sup_{t \geqq 0} \operatorname{var}_t^{t+1} P < \infty .$$

The space of functions with interval bounded variation will be denoted by $IBV[0, \infty) = IBV$. It is evident that every function $P : [0, \infty) \to R^n$ which is locally of bounded variation is also of interval bounded variation, i.e. $BV_{\mathrm{loc}} \subset IBV$.

We can use the general scheme for the definition of the concept of stability under perturbations in the class of functions of interval bounded variation which are continuous from the left. We say that $x \equiv 0$ is *stable under pertubations in the space of functions of interval bounded variation* if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|y_0| < \delta$ and $P \in IBV$, P continuous from the left such that

$$\sup_{t \geqq 0} \operatorname{var}_t^{t+1} P < \delta ,$$

then

$$|y(t, t_0, y_0)| < \varepsilon$$

for every $t \geqq t_0$, where $y(t, t_0, y_0)$ is a solution of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[F(x, t) + P(t)] \tag{9}$$

with $y(t_0, t_0, y_0) = y_0$.

The solution $x \equiv 0$ is said to be *asymptotically stable under perturbations* in the space of functions with interval bounded variation if it is stable under perturbations from this class and if it is also attracting under perturbations of this kind, i.e. if there exists a $\delta_0 > 0$ and for each $\varepsilon > 0$ there exists $T = T(\varepsilon) \geqq 0$ and $\gamma = \gamma(\varepsilon) > 0$ such that if $|y_0| < \delta_0$ and

$$\sup_{t \geqq 0} \operatorname{var}_t^{t+1} P < \gamma(\varepsilon) , \quad \text{then} \quad |y(t, t_0, y_0)| < \varepsilon$$

for all $t \geqq t_0 + T(\varepsilon)$ and $t_0 \geqq 0$ (here $y(t, t_0, y_0)$ is again a solution of the generalized differential equation (9) which satisfies the initial condition $y(t_0, t_0, y_0) = y_0$).

Since $BV_{\mathrm{loc}} \subset IBV$ it can be easily shown that if $x \equiv 0$ is asymptotically stable under perturbations in the space of functions with interval bounded variation then $x \equiv 0$ is also variational-asymptotically stable. The equivalent form of variational stability and variational attractivity stated in Propositions 1 and 2 has to be used for the proof of this fact.

Similarly as in [1] for the case of integral asymptotic stability, also in our case the converse is true, i.e. the following theorem is valid.

Theorem 5. *The solution $x \equiv 0$ of the equation (1) is variational-asymptotically stable if and only if $x \equiv 0$ is asymptotically stable under perturbations in the space of functions with interval bounded variation.*

Proof. It remains to prove that if $x \equiv 0$ is variational-asymptotically stable then $x \equiv 0$ is also asymptotically stable under perturbations in the space IBV. By Theorem 4 there exists a Ljapunov function $U : [0, +\infty) \times B_c \to R$ such that for every $x \in B_c$

the function $U(\cdot, x)$ is continuous from the left, $U(\cdot, x) \in BV_{\mathrm{loc}}[0, +\infty)$, $U(t, 0) = 0$, $t \geqq 0$,

$$|U(t, x) - U(t, y)| \leqq |x - y| \quad \text{for} \quad x, y \in B_c\,, \quad t \in [0, +\infty)\,.$$

$U(t, x)$ is positive definite, i.e. there is $b : [0, +\infty) \to R$, b continuous, increasing, $b(0) = 0$, $b(r) < r$ for $r > 0$ such that

$$b(|x|) \leqq U(t, x) \quad \text{for all} \quad t \in [0, +\infty)\,, \quad x \in B_c$$

and for every solution $\psi(\sigma)$ of the equation (1) defined for $\sigma \geqq t$ satisfying $\psi(t) = x \in B_c$ the relation

$$\limsup_{\eta \to 0+} \frac{U(t + \eta, \psi(t + \eta)) - U(t, x)}{\eta} \leqq -U(t, x)$$

holds.

Assume first that $x \equiv 0$ is not stable under perturbations in IBV. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there exist $P : [0, +\infty) \to R^n$ continuous from the left with

$$\sup_{t \geqq 0} \operatorname{var}_t^{t+1} P < \delta\,,$$

$y_0 \in R^n$; $|y_0| < \delta$, $t_0 \geqq 0$, and a solution $y(t) = y(t, t_0, y_0)$ of the equation (9) such that $|y(t_2)| \geqq \varepsilon$ for some $t_2 > t_0$. y is a solution of the generalized differential equation (9), hence y is continuous from the left and of bounded variation on every compact interval.

Assume now that $\delta > 0$ is so small that

$$\delta < b(b(\varepsilon/2)) < b(\varepsilon/2) < b(\varepsilon) < \varepsilon$$

and

$$b(\varepsilon/2) + \delta < b(\varepsilon)\,.$$

Using the continuity from the left of y we get the existence of $t_1 \in (t_0, t_2)$ such that $|y(t)| > b(\varepsilon/2)$ for $t \in (t_1, t_2)$ and $|y(t_1)| \leqq b(\varepsilon/2)$. (Let us remember that the function y is of bounded variation and consequently it has possibly a discontinuity at t_1.)

Using Lemma 1 for the function U and for y we obtain the inequality

$$U(t_2, y(t_2)) - U(t_1, y(t_1)) \leqq \operatorname{var}_{t_1}^{t_2} \left(y(s) - \int_{t_0}^{s} DF(y(\tau), t) \right) + M(t_2 - t_1) =$$

$$= \operatorname{var}_{t_1}^{t_2} P + M(t_2 - t_1)\,,$$

where $M = \sup_{t \in [t_1, t_2]} (-b(|y(t)|) = - \inf_{t \in [t_1, t_2]} b(|y(t)|) \leqq -b(b(\varepsilon/2)) < -\delta$ and evidently

$$\operatorname{var}_{t_1}^{t_2} P \leqq (t_2 - t_1 + 1) \sup_{t \geqq 0} \operatorname{var}_t^{t+1} P < (t_2 - t_1 + 1)\, \delta\,.$$

Hence we get the inequalities

$$b(\varepsilon) \leqq b(|y(t_2)|) \leqq U(t_2, y(t_2)) < U(t_1, y(t_1)) +$$
$$+ \delta(t_2 - t_1 + 1) - \delta(t_2 - t_1) =$$
$$= U(t_1, y(t_1)) + \delta \leqq |y(t_1)| + \delta \leqq b(\varepsilon/2) + \delta < b(\varepsilon)$$

and this evidently leads to a contradiction. Hence the solution $x \equiv 0$ is stable under perturbations in IBV.

Let us set $\delta_0 = \delta(c) > 0$ where $\delta(c)$ corresponds to $c > 0$ by the definition of the variational stability of the solution $x \equiv 0$; hence if $|y_0| < \delta_0$ then $|y(t)| < c$ for all $t \geqq t_0$, where $y(t) = y(t, t_0, y_0)$ is a solution of (9) with $y(t_0) = y(t_0, t_0, y_0) = y_0$.

Let $\varepsilon > 0$ be given and let us set

$$\gamma(\varepsilon) = \min\left(\delta(\varepsilon), \tfrac{1}{2}b(\delta(\varepsilon))\right), \quad T(\varepsilon) = \frac{\delta_0 + \frac{1}{2}b(\delta(\varepsilon))}{\frac{1}{2}b(\delta(\varepsilon))}$$

$\delta(\varepsilon) > 0$ corresponds to $\varepsilon > 0$ by the definition of the stability of $x \equiv 0$. Assume that for every $t_2 \in [t_0, t_0 + T(\varepsilon)]$ we have

$$c > |y(t_2)| \geqq \delta(\varepsilon).$$

Using again Lemma 1 we get the inequality

$$0 < b(\delta(\varepsilon)) \leqq U(t_2, y(t_2)) \leqq U(t_0, y(t_0)) + \operatorname{var}_{t_0}^{t_2} P +$$
$$+ \sup_{t \in [t_0, t_2]} (-b(|y(t)|))(t_2 - t_0) \leqq U(t_0, y(t_0)) + \operatorname{var}_{t_0}^{t_2} P - b(\delta(\varepsilon))(t_2 - t_0).$$

If we assume

$$\sup_{t \geqq 0} \operatorname{var}_t^{t+1} P < \gamma(\varepsilon) \leqq b(\delta(\varepsilon))/2$$

then $\operatorname{var}_{t_0}^{t_2} P \leqq b(\delta(\varepsilon))(t_2 - t_0 + 1)/2$ and

$$0 \leqq U(t_0, y(t_0)) + b(\delta(\varepsilon))(t_2 - t_0 + 1)/2 - b(\delta(\varepsilon))(t_2 - t_0) \leqq$$
$$\leqq |y(t_0)| + b(\delta(\varepsilon))/2 - b(\delta(\varepsilon))(t_2 - t_0)/2 <$$
$$< \delta_0 + b(\delta(\varepsilon))/2 - b(\delta(\varepsilon))(t_2 - t_0)/2.$$

If now $t_2 = t_0 + T(\varepsilon)$, i.e. $t_2 - t_0 = (\delta_0 + b(\delta(\varepsilon))/2)/(b(\delta(\varepsilon))/2)$, then we obtain the contradictory inequality

$$0 < \delta_0 + b(\delta(\varepsilon))/2 - b(\delta(\varepsilon))\frac{\delta_0 + b(\delta(\varepsilon))/2}{2b(\delta(\varepsilon))/2} = 0.$$

Hence for every solution $y(t)$ of the equation (9) with $y(t_0) = y_0$, $|y_0| < \delta_0$ there exists a point $t_1 \geqq t_0$ such that $|y(t_1)| < \delta(\varepsilon)$, and by the properties of $\delta(\varepsilon) > 0$ given in the definition of the stability we obtain that $|y(t)| < \varepsilon$ for all $t \geqq t_1$ and, in particular, for $t \geqq t_0 + T(\varepsilon)$. This together with the stability under perturbations in IBV yields also the asymptotic stability under perturbations in IBV and our theorem is proved.

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Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).