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Časopis pro pěstování matematiky, roč. 109 (1984), Praha

ON SOME REGULARITIES OF GRAPHS II

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§ 0

In this paper we continue the study of problems formulated by J. Płonka in [3].

Let $G = (V, E)$ be a simple graph. For $v \in V$ we denote: $\Gamma(v) = \{u \in V: \{u, v\} \in E\}$, $\varrho(v) = |\Gamma(v)|$, $\varrho_\Gamma(v) = \sum_{u \in \Gamma(v)} \varrho(u)$; $\varrho_\Gamma(v) = 0$ if $\Gamma(v) = \emptyset$.

Let m be a non-negative integer. A graph $G = (V, E)$ will be called *m-Γ^--regular* iff for any $v \in V$ we have $\varrho_\Gamma(v) - \varrho(v) = m$.

J. Płonka gave a complete characterization of m-Γ^--regular graphs for $m \leqq 5$. He also described all m-Γ^--regular graphs in which the degrees of vertices assume only two values. In [3] the following problem has been stated:

Describe all m-Γ^--regular graphs G in which $|D(G)| = 3$ ($D(G)$ *denotes the set of all degrees of vertices of G*).

It turns out that such graphs can be of various structure and it is rather difficult to characterize them in general. In [1] the following problem has been solved:

Problem 1. *Let $(m, d_1, \ldots, d_k)$ be a sequence of non-negative integers such that $d_1 > \ldots, > d_k$, $k > 2$. Decide whether there exists an m-Γ^--regular graph G such that $D(G) = \{d_1, \ldots, d_k\}$.*

If such a graph exists we say that G Γ^--represents the sequence $(m, d_1, \ldots, d_k)$.

Studying this problem we obtained more general results (see [2]), namely, we are able to decide if there exists a graph in which the degrees of the neighbours of any vertex are given numbers.

Let now $\varphi = (m, d_1, \ldots, d_k)$ be a sequence of non-negative integers such that $d_1 > \ldots > d_k$, $k > 2$, and let $F = \{v_1, v_2, \ldots\}$ be a fixed countable set. We denote by $G_{\Gamma^-}(\varphi)$ the set of all graphs G Γ^--representing the sequence φ such that $V(G) = \{v_1, v_2, \ldots, v_n\}$ for some $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers).

In this paper we consider the following

Problem 2. *Describe the set* $G_{\Gamma^-}(\varphi)$.

In Sec. 2 (Theorem 3) we obtain a certain characterization of $G_{\Gamma^-}(\varphi)$. We introduce some special constructions on graphs which preserve m-Γ^--regularity and show that by means of them one can find the relations between graphs Γ^--representing a given sequence (Theorem 4).

In Sec. 1 we recall some notions and results from [1, 2] which we need in Sec. 2.

The notations used in [1] and [2] are partially different. Here we adopt the one from [2].

§ 1

Let $G = (V, E)$ be a finite simple graph such that $D(G) = \{d_1, d_2, \dots, d_k\}$. For $i, j \in \{1, 2, \dots, k\}$ we define:

$V_i = \{v \in V: \varrho(v) = d_i\}$,

$E_{ij} = \{\{u, v\} \in E: u \in V_i, v \in V_j\}$,

$t^i(v) = |V_i \cap \Gamma(v)|$.

A function $\boldsymbol{t}_G: V \to \mathbb{N}^k$ such that

$$\boldsymbol{t}_G(v) = (t^1(v), t^2(v), \dots, t^k(v)) \quad \text{for} \quad v \in V$$

will be called the *distribution function of vertices of G.*

Let $V(G) = \{v_1, v_2, \dots, v_n\}$. A $(k \times n)$-matrix $\boldsymbol{M}_G$ of the form

$$\boldsymbol{M}_G = [\boldsymbol{t}_G(v_1), \boldsymbol{t}_G(v_2), \dots, \boldsymbol{t}_G(v_n)]$$

will be called the *distribution matrix of the graph G.* Then G will be called a *realization of the matrix* $\boldsymbol{M}_G$.

Let us consider a $(k \times n)$-matrix of non-negative integers of the form

$$(1) \qquad \boldsymbol{M} = \begin{bmatrix} t_{11}^1 & \dots & t_{1s_1}^1 & \dots & t_{i1}^1 & \dots & t_{is_i}^1 & \dots & t_{k1}^1 & \dots & t_{ks_k}^1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ t_{11}^k & \dots & t_{1s_1}^k & \dots & t_{i1}^k & \dots & t_{is_i}^k & \dots & t_{k1}^k & \dots & t_{ks_k}^k \end{bmatrix},$$

where for any $i = 1, 2, \dots, k$, $q = 1, 2, \dots, s_i$ we have

$$t_{iq}^1 + t_{iq}^2 + \dots + t_{iq}^k = d_i, \quad d_1 > d_2 > \dots > d_k.$$

For $i, j \in \{1, 2, \dots, k\}$ denote

$$\bar{t}_j^i = (\bar{t}_{j1}^i, \bar{t}_{j2}^i, \dots, \bar{t}_{js_j}^i),$$

where $(\bar{t}_{j1}^i, \dots, \bar{t}_{js_j}^i)$ is a non-increasing permutation of the sequence t_j^i.

Theorem 1 [2]. *A $(k \times n)$-matrix of the form* (1) *has a graphic realization iff for any $i, j \in \{1, 2, \ldots, k\}$ the following conditions hold:*

(i) $\sum_{r=1}^{s_i} \bar{t}_{ir}^i \equiv 0 \pmod 2$,

(ii) $\sum_{r=1}^{s_i} \bar{t}_{ir}^i \leqq m(m-1) + \sum_{r=m+1}^{s_i} \min\{m, \bar{t}_{ir}^i\}$ for $m = 1, 2, \ldots, s_i$,

(iii) $\sum_{r=1}^{s_i} \bar{t}_{ir}^j = \sum_{r=1}^{s_j} \bar{t}_{jr}^i$,

(iv) $\sum_{r=1}^{m} \bar{t}_{ir}^j \leqq \sum_{r=1}^{s_j} \min\{m, \bar{t}_{jr}^i\}$ for $m = 1, 2, \ldots, s_i$; $i < j$.

Theorem 1 and the definition of an m-Γ^--regular graph imply

Corollary 1. *If a graph G is a relatization of a matrix M of the form* (1), *then G is m-Γ^--regular iff for any $i = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, s_i$ the following formula holds:*

(v) $d_1 t_{iq}^1 + d_2 t_{iq}^2 + \ldots + d_k t_{iq}^k = m + d_i$.

Proof. Assume that a column $(t_{iq}^1, \ldots, t_{iq}^k)$ is the distribution of the vertex $v \in V(G)$. Then $\varrho_\Gamma(v) = d_1 t_{iq}^1 + \ldots + d_k t_{iq}^k$ and $\varrho(v) = d_i$. It is obvious that (v) is equivalent to the formula $\varrho_\Gamma(v) = m + \varrho(v)$.

Let $R_V(\boldsymbol{M})$ be the set of all graphs G which are realizations of the matrix $\boldsymbol{M}$ of the form (1) and defined on a given vertex-set $V = \{v_1, v_2, \ldots, v_n\}$.

Let $G \in R_V(\boldsymbol{M})$, $G = (V, E)$ and let (v_p, v_q, v_r, v_s) be a sequence of different vertices from V such that

1° $v_p, v_r \in V_i$, $v_q, v_s \in V_j$,

2° $\{v_p, v_q\}$, $\{v_r, v_s\} \in E$,

3° $\{v_p, v_s\}$, $\{v_q, v_r\} \notin E$.

The graph $G(v_p, v_q, v_r, v_s) = (V, E')$, where

$$E' = (E \setminus \{\{v_p, v_q\}, \{v_r, v_s\}\}) \cup \{\{v_p, v_s\}, \{v_q, v_r\}\},$$

will be called a *$(*)$-switching of G.*

Theorem 2 [2]. *The set $R_V(\boldsymbol{M})$ can be generated by one of its elements by a finite number of* $(*)$-switching operations.

Now we present a solution of Problem 1 (see [1]), but in terms of the notions of this paper. The solution reduces to finding the matrix $\boldsymbol{M}$ of the form (1) satisfying the conditions (i)–(v). We do this in two steps.

Step 1. For $i = 1, 2, \ldots, k$ we solve the system of two equations

$$\sum_{r=1}^{k} d_r z_i^r = m + d_i \,, \quad \sum_{r=1}^{k} z_i^r = d_i \,. \tag{2}$$

Let

$$\mathscr{P}_i = \{(p_{i1}^1, \ldots, p_{i1}^k), \ldots, (p_{ic_i}^1, \ldots, p_{ic_i}^k)\}$$

be the set of all solutions of (2). If $\mathscr{P}_i = \emptyset$ for some i, then the sequence $\varphi = (m, d_1, \ldots, d_k)$ is not Γ^--representable. Otherwise every solution of the equations (2) can be given as a column of $\boldsymbol{M}$. Other columns cannot occur in $\boldsymbol{M}$.

Step 2. We solve the following system of equations and inequalities (3)–(5):

$$\sum_{r=1}^{c_i} p_{ir}^j x_{ir} = \sum_{r=1}^{c_j} p_{jr}^i x_{jr} \quad \text{where} \quad i, j = 1, 2, \ldots, k \,, \tag{3}$$

$$\sum_{r=1}^{c_i} p_{ir}^i x_{ir} \equiv 0 \pmod 2 \quad \text{where} \quad i = 1, 2, \ldots, k \,, \tag{4}$$

$$\sum_{r=1}^{c_i} x_{ir} > 0 \qquad\qquad \text{where} \quad i = 1, 2, \ldots, k \,. \tag{5}$$

If no solution exists, then the sequence φ is not Γ^--representable. Otherwise, from solutions of (3)–(5) we can choose a solution

$$(n_{11}, \ldots, n_{1c_1}, \ldots, n_{i1}, \ldots, n_{ic_i}, \ldots, n_{k1}, \ldots, n_{kc_k}) \tag{6}$$

which satisfies the following conditions (7)–(9):

$$\min_{r \in \{1, \ldots, c_i\}} n_{ir} \geqq \max_{s \in \{1, \ldots, c_j\}} p_{js}^i \quad \text{for} \quad i < j \,, \quad i, j \in \{1, 2, \ldots, k\} \,, \tag{7}$$

$$\min_{r \in \{1, \ldots, c_i\}} n_{ir} \geqq 2 \max_{r \in \{1, \ldots, c_i\}} p_{ir}^i \quad \text{for} \quad i \in \{1, 2, \ldots, k\} \,, \tag{8}$$

$$n_{ir} \textit{ is an even number} \quad \text{for} \quad i \in \{1, 2, \ldots, k\} \,, \quad r \in \{1, 2, \ldots, c_i\} \,. \tag{9}$$

We form a matrix $\boldsymbol{M}$ such that for $r = 1, 2, \ldots, c_i$ and $i = 1, 2, \ldots, k$ the column $(p_{ir}^1, \ldots, p_{ir}^k)$ occurs n_{ir} times. The matrix $\boldsymbol{M}$ is the distribution matrix of some simple graph. A method of constructing this graph is presented in [1].

§ 2

Now let us consider Problem 2.

Let

$$\boldsymbol{P} = \begin{bmatrix} p_{11}^1 & \cdots & p_{1c_1}^1 & \cdots & p_{i1}^1 & \cdots & p_{ic_i}^1 & \cdots & p_{k1}^1 & \cdots & p_{kc_k}^1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ p_{11}^k & \cdots & p_{1c_1}^k & \cdots & p_{i1}^k & \cdots & p_{ic_i}^k & \cdots & p_{k1}^k & \cdots & p_{kc_k}^k \end{bmatrix} \tag{10}$$

be the matrix of all solutions of the systems (2) for $i = 1, 2, \ldots, k$. Obviously all columns in $\boldsymbol{P}$ are different.

Let $\mathcal{Q}$ be the set of all solutions of the systems (3)–(5). Every element of $\mathcal{Q}$ is a sequence of the form (6). For the matrix $\boldsymbol{P}$ and for $\alpha \in \mathcal{Q}$ denote by $\boldsymbol{P}_\alpha$ the matrix obtained from $\boldsymbol{P}$ by repeating the column $(p_{ir}^1, \ldots, p_{ir}^k)$ side by side n_{ir} times for $i = 1, 2, \ldots, k$ and $r = 1, 2, \ldots, c_i$.

We denote by $\boldsymbol{R}_V(\boldsymbol{P}_\alpha)$ the set of all realizations G of $\boldsymbol{P}_\alpha$ such that $V(G) = \{v_1, \ldots, v_{s(\alpha)}\}$, where $s(\alpha)$ is the sum of all elements of α.

Theorem 3. *Let $\varphi = (m, d_1, \ldots, d_k)$ be a Γ^--representable sequence. Then we have*

$$\boldsymbol{G}_{\Gamma^-}(\varphi) = \bigcup_{\alpha \in \mathcal{Q}} \boldsymbol{R}_V(\boldsymbol{P}_\alpha)\,. \tag{11}$$

Proof. Let $G \in \bigcup_{\alpha \in \mathcal{Q}} \boldsymbol{R}_V(\boldsymbol{P}_\alpha)$. Then $G \in \boldsymbol{R}_V(\boldsymbol{P}_{\alpha_0})$ for some $\alpha_0 \in \mathcal{Q}$. Since $\boldsymbol{P}$ is the matrix of all solutions of (2) for $i = 1, 2, \ldots, k$, so by Corollary 1, G is an m-Γ^--regular graph. As the sequence α_0 satisfies (3)–(5), so $D(G) = \{d_1, d_2, \ldots, d_k\}$. Thus $G \in \boldsymbol{G}_{\Gamma^-}(\varphi)$.

Let now $G \in \boldsymbol{G}_{\Gamma^-}(\varphi)$. G is an m-Γ^--regular graph such that $D(G) = \{d_1, d_2, \ldots, d_k\}$. Let a matrix $\boldsymbol{M}$ of the form (1) be the distribution matrix of G. We have to show that $\boldsymbol{M} = \boldsymbol{P}_\alpha$ for some $\alpha \in \mathcal{Q}$.

Assume that $\{\gamma_{11}, \ldots, \gamma_{1r_1}, \ldots, \gamma_{k1}, \ldots, \gamma_{kr_k}\}$ is the set of all different columns of $\boldsymbol{M}$ and that for $i = 1, 2, \ldots, k$, $q = 1, 2, \ldots, r_i$ we have $\gamma_{iq} = (c_{iq}^1, \ldots, c_{iq}^k)$ where $\sum_{s=1}^{k} c_{iq}^s = d_i$. By Corollary 1, any column of the matrix $\boldsymbol{M}$ satisfies (v), so for any $i = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, r_i$ we get $\sum_{s=1}^{k} d_s c_{iq}^s = m + d_i$. Hence it follows that any column of $\boldsymbol{M}$ satisfies the system (2), so it is a column of $\boldsymbol{P}$. Obviously $\boldsymbol{M}$ need not contain all columns of $\boldsymbol{P}$.

Assume that for $i = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, r_i$ the column γ_{iq} occurs m_{iq} times in $\boldsymbol{M}$. By Theorem 1 the elements of the matrix $\boldsymbol{M}$ satisfy (iii) and (i), hence we get

$$\sum_{q=1}^{r_i} c_{iq}^j m_{iq} = \sum_{q=1}^{r_j} c_{jq}^i m_{jq} \quad \text{for} \quad i, j = 1, 2, \ldots, k\,,$$

$$\sum_{q=1}^{r} c_{iq}^i m_{iq} \equiv 0 \pmod 2 \quad \text{for} \quad i = 1, 2, \ldots, k\,.$$

Let us form a sequence

$$\alpha = (n_{11}, \ldots, n_{1c_1}, \ldots, n_{k1}, \ldots, n_{kc_k})\,,$$

where for $i = 1, 2, \ldots, k$ and $r = 1, 2, \ldots, c_i$ we have $n_{ir} = m_{iq}$ if $(p_{ir}^1, \ldots, p_{ir}^k) = \gamma_{iq}$ and $n_{ir} = 0$ if the column $(p_{ir}^1, \ldots, p_{ir}^k)$ does not occur in $\boldsymbol{M}$. The sequence α constructed in this way satisfies (3) and (4). Since $D(G) = \{d_1, d_2, \ldots, d_k\}$, so the sequence α satisfies also (5). Thus $\alpha \in \mathcal{Q}$ and $\boldsymbol{M} = \boldsymbol{P}_\alpha$.

For some sequences φ one can give a simpler characterization of the set G_{Γ}-(φ). To show this we introduce some additional notions.

Let $G_1, G_2 \in G_{\Gamma}$-(φ), $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and $V_1 = \{v_1, v_2, \ldots, v_r\}$, $V_2 = \{v_1, v_2, \ldots, v_s\}$. We form a graph $G_2' = (V_2', E_2')$ such that $V_2' = \{v_{r+1}, v_{r+2}, \ldots \ldots, v_{r+s}\}$ and G_2 is isomorphic to G_2'.

A graph $G_1 \uplus G_2 = (V_1 \cup V_2', E_1 \cup E_2')$ will be called the disjoint union of the graphs G_1 and G_2.

Let $G = (V, E)$, $V(G) = \{v_1, v_2, \ldots, v_n\}$. We define:

$$1G = G\,, \quad (n + 1)\,G = nG \uplus G \quad \text{for} \quad n \in \mathbb{N}\,.$$

We shall write $H = (*)_n$-$sw(G)$ to express that the graph H can be obtained from G by applying $(*)$-switching operations n times for $n \in \mathbb{N} \cup \{0\}$.

If $\alpha = (a_1, a_2, \ldots, a_s)$, $\beta = (b_1, b_2, \ldots, b_s)$ are two sequences of non-negative integers and $n \in \mathbb{N} \cup \{0\}$, then we denote as usual

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_s + b_s)\,,$$
$$n\alpha = (na_1, na_2, \ldots, na_s)\,.$$

Lemma 1.

(a) *If* $G \in \boldsymbol{R}_V(\boldsymbol{P}_\alpha)$ *and* $G' = (*)_n$-$sw(G)$, *then* $G' \in \boldsymbol{R}_V(\boldsymbol{P}_\alpha)$;

(b) *if* $G_1 \in \boldsymbol{R}_V(\boldsymbol{P}_\alpha)$, $G_2 \in \boldsymbol{R}_V(\boldsymbol{P}_\beta)$, *then* $G_1 \uplus G_2 \in \boldsymbol{R}_V(\boldsymbol{P}_{\alpha+\beta})$;

(c) *if* $G \in \boldsymbol{R}_V(\boldsymbol{P}_{\alpha+\beta})$, *then for any one of the graphs* $G_1 \in \boldsymbol{R}_V(\boldsymbol{P}_\alpha)$ *and* $G_2 \in \boldsymbol{R}_V(\boldsymbol{P}_\beta)$ *there exists* $n \in \mathbb{N} \cup \{0\}$ *such that* $G = (*)_n$-$sw(G_1 \uplus G_2)$.

Proof.

(a) follows from the fact that the operation of $(*)$-switching preserves the distribution of any vertex

(b) follows from the definition of theset $\boldsymbol{R}_V(\boldsymbol{P}_\xi)$ for $\xi \in \mathcal{Q}$ and the definition of the union $\uplus$.

To prove (c) assume that $G \in \boldsymbol{R}_V(\boldsymbol{P}_{\alpha+\beta})$ and G_1, G_2 are such graphs that $G_1 \in \boldsymbol{R}_V(\boldsymbol{P}_\alpha)$, $G_2 \in \boldsymbol{R}_V(\boldsymbol{P}_\beta)$. Then $G_1 \uplus G_2 \in \boldsymbol{R}_V(\boldsymbol{P}_{\alpha+\beta})$. From Theorem 2 we infer that any graph from the set $\boldsymbol{R}_V(\boldsymbol{P}_{\alpha+\beta})$ can be obtained from the graph $G_1 \uplus G_2$ by applying the operation of $(*)$-switching finitely many times. Thus there exists $n \in \mathbb{N} \cup \{0\}$ such that $G = (*)_n$-$sw(G_1 \uplus G_2)$.

Let $\mathcal{Q}$ denote as before the set of all sequences of non-negative integers satisfying (3)–(5). A finite subset $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_r\}$ of $\mathcal{Q}$ will be called a *base of* $\mathcal{Q}$ iff the following conditions 1° and 2° are satisfied:

1° For any $\alpha \in \mathcal{Q}$ there exist $n_1, n_2, \ldots, n_r \in \mathbb{N} \cup \{0\}$ such that

$$\alpha = \sum_{i=1}^{r} n_i \beta_i\,;$$

2° if $\mathcal{B}' \subsetneq \mathcal{B}$, then $\mathcal{B}'$ does not satisfy 1°.

Theorem 4. *Let $\varphi = (m, d_1, \ldots, d_k)$ be a Γ^--representable sequence. Let $\{\beta_1, \beta_2, \ldots, \beta_r\}$ be a base of the set $\mathcal{Q}$ and let $\mathbf{P}_{\beta_1}, \mathbf{P}_{\beta_2}, \ldots, \mathbf{P}_{\beta_r}$ be the distribution matrices of the graphs $G_1, G_2, \ldots, G_r$, respectively. Then any graph belonging to G_{Γ}-(φ) can be obtained from the graphs $G_1 G_2, \ldots, G_r$ by using first the operation $\uplus$ finitely many times and then using $(*)$-switching operations finitely many times.*

Proof. Let $G \in \boldsymbol{G}_{\Gamma}$-$(\varphi)$. Then by Theorem 3 there exists a sequence $\alpha \in \mathcal{Q}$ such that $G \in \boldsymbol{R}_V(\mathbf{P}_\alpha)$. Assume $\alpha = \sum_{i=1}^{r} n_i \beta_i$, where $n_i \in \mathbb{N} \cup \{0\}$. By (c) from Lemma 1 there exists $s \in \mathbb{N} \cup \{0\}$ such that $G = (*)_s$-$sw(G')$, where $G' = n_1 G_1 \uplus n_2 G_2 \uplus \ldots \uplus n_r G_r$.

Example 1. Let $\varphi = (10,8,4,2)$.
Then

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 7 & 1 & 3 & 0 \end{bmatrix}, \quad \mathcal{Q} = \{\alpha \colon \alpha = (p, 4p, p, 7p), \quad p \in \mathbb{N}\}.$$

It is easy to see that the set $\{\alpha\}$ where $\alpha = (1, 4, 1, 7)$ is the only base of $\mathcal{Q}$. The graph G in Fig. 1 is a realization of the matrix $\mathbf{P}_\alpha$. By Theorem 4 we have:
$H \in \boldsymbol{G}_{\Gamma}$-$(\varphi)$ iff $H = (*)_r$-$sw(sG)$ for some $r \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N}$.

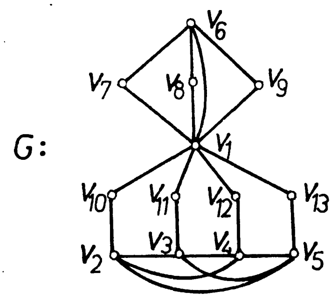

Fig. 1

Example 2. Let $\varphi = (10, 5, 4, 3)$.

Then

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 5 & 2 & 3 & 0 & 1 \end{bmatrix}, \quad \mathcal{Q} = \{\alpha \in (\mathbb{N} \cup \{0\})^5 : \alpha = (\tfrac{1}{5}p + \tfrac{4}{5}q, p, 0, p, 2q)\}.$$

One can prove that any sequence from the set $\mathcal{Q}$ is a sum of sequences $\alpha_1, \alpha_2, \alpha_3$, where $\alpha_1 = (1, 1, 0, 1, 2)$, $\alpha_2 = (1, 5, 0, 5, 0)$, $\alpha_3 = (4, 0, 0, 0, 10)$. However, the sequence α_3 is not an element of $\mathcal{Q}$ since it does not satisfy (5), whereas the matrix $\mathbf{P}_{\alpha_1}$ is the distribution matrix of a pseudograph and of no simple graph.

To obtain a result analogous to that of Theorem 4 we need three generalizations, namely:

1) a generalization of the base on the set $\mathscr{Q}'$ of all solutions of (3) and (4),

2) a definition of the disjoint union of pseudographs analogous to that of simple graphs,

3) a generalization of the operation of $(*)$-switching consisting in neglecting the condition 3° and the assumption that the vertices v_p, v_q, v_r, v_s are different.

From results of [2] it follows that a matrix M of the form (1) is representable by a pseudograph iff for any $i, j \in \{1, 2, \ldots, k\}$ the conditions (i) and (ii) hold.

Then in our Example 2 we get:

Any simple graph Γ^--representing the sequence $\varphi = (10, 5, 4, 3)$ can be obtained from the graphs G_1, G_2, G_3 (Fig. 2) by using the operation of disjoint union finite many times and then using operations of $(*)$-switchings finite many times.

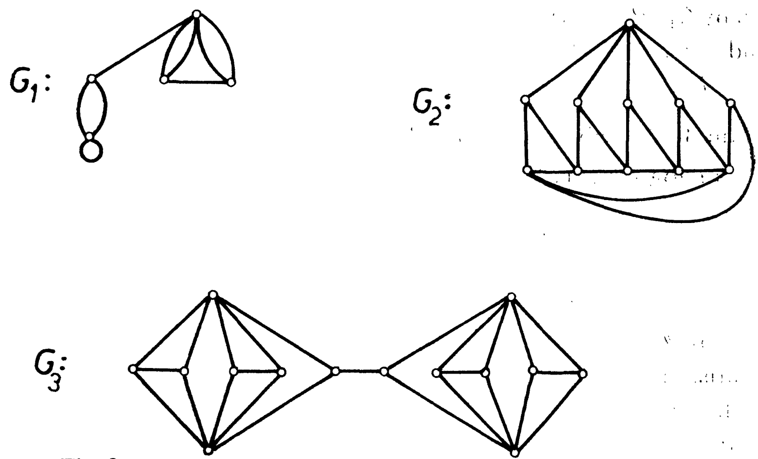

Fig. 2

Example 3.

a) Let $\varphi_1 = (12, 8, 4, 2)$. Then for $i = 3$ there are no solutions of (2).

b) Let $\varphi_2 = (14, 8, 4, 2)$.

Then

$$\boldsymbol{P}_2 = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 3 & 0 & 2 & 0 \\ 5 & 7 & 1 & 0 \end{bmatrix}, \quad \text{however} \quad \mathscr{Q}_2 = \emptyset .$$

c) For a sequence $\varphi_3 = (8, 8, 4, 2)$ we have:

$$\boldsymbol{P}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 8 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{Q}_3 = \emptyset .$$

Moreover let us observe that using the criteria of solutions of (2) given in [1] we can conclude that for the set $\{8, 4, 2\}$ there exists a unique m such that the sequence

$(m, 8, 4, 2)$ is Γ^--representable. Namely, $m = 10$. The set of graphs Γ^--representing $(10, 8, 4, 2)$ was described in Example 1.

Remark 1. In the considerations of Sec. 2 of this paper one can neglect the assumption of m-Γ^--regularity of graphs and generalize the problems as follows.

Let k be a positive integer and $\{d_1, d_2, \ldots, d_k\}$ a set of non-negative integers such that $d_1 > d_2 > \ldots > d_k$. For $i = 1, 2, \ldots, k$ let

$$\mathscr{P}_i = \{(p_{i1}^1, \ldots, p_{i1}^k), \ldots, (p_{ic_i}^1, \ldots, p_{ic_i}^k)\}$$

be a set of sequences of non-negative integers such that $p_{iq}^1 + p_{iq}^2 + \ldots + p_{iq}^k = d_i$ for $q = 1, 2, \ldots, c_i$. Let further

$$\mathscr{P} = \bigcup_{i \in \{1, \ldots, k\}} \mathscr{P}_i . \tag{12}$$

Denote by $\boldsymbol{G}_V(\mathscr{P})$ the set of all graphs G such that $V(G) = \{v_1, v_2, \ldots, v_n\}$ for some $n \in \mathbb{N}$ and such that $\mathscr{P} = \mathscr{D}(G)$, where $\mathscr{D}(G)$ is the set of all distributions of the vertices of G.

Problem 3. Describe the set $\boldsymbol{G}_V(\mathscr{P})$.

Denote analogously as before

$$P = \begin{bmatrix} p_{11}^1 & \cdots & p_{1c_1}^1 & \cdots & p_{i1}^1 & \cdots & p_{ic_i}^1 & \cdots & p_{k1}^1 & \cdots & p_{kc_k}^1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ p_{11}^k & \cdots & p_{1c_1}^k & \cdots & p_{i1}^k & \cdots & p_{ic_i}^k & \cdots & p_{k1}^k & \cdots & p_{kc_k}^k \end{bmatrix}$$

and denote by $\mathscr{Q}$ the set of all sequences of the form (7) whose coordinates are positive integers satisfying (3) and (4).

Using the same argument we can formulate theorems analogous to Theorem 3 and Theorem 4 as follows:

1° replace the first sentence both in Theorem 3 and Theorem 4 by the sentence "Let $\mathscr{P}$ be a set of k-tuples of the form (12) nad $\boldsymbol{G}_V(\mathscr{P}) \neq \emptyset$".

2° replace in Theorem 3 and Theorem 4 the symbol $\boldsymbol{G}_{\Gamma^-}(\varphi)$ by $\boldsymbol{G}_V(\mathscr{P})$.

It is known that for every non-empty set $\{d_1, d_2, \ldots, d_k\}$ of non-negative integers there exists a graph G such that $D(G) = \{d_1, d_2, \ldots, d_k\}$. However, there exist non-empty sets $\mathscr{P}$ of the form (12) which are distribution sets of no graph (see Example 3b) and 3c)).

References

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