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AMALGAMATIONS OF HOMOGENEOUS RIEMANNIAN SPACES

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0. INTRODUCTION

Amalgamation is a generalized product operation first introduced by O. Kowalski for s-manifolds in [5]. In this paper the operation of amalgamation is extended to affine reductive spaces and to homogeneous Riemannian spaces. Amalgamation enables us, under some assumptions, from two homogeneous Riemannian spaces (M_1, g_1), (M_2, g_2) to construct a new homogeneous Riemannian space (M, g) such that for every $i \in \{1, 2\}$

1) $\dim M_i < \dim M < \dim M_1 + \dim M_2$,

2) there is a totally geodesic foliation $\mathscr{F}_i$ on (M, g) such that every its leaf L_i is a homogeneous Riemannian space locally isometric to (M_i, g_i).

The main result of this paper is the following

Main Theorem. *Let (M_1, g_1), (M_2, g_2) be connected and simply connected irreducible homogeneous Riemannian spaces. Then any amalgamation (M, g) of (M_i, g_i) (if it exists) is an irreducible Riemannian space.*

Further, we show that any two Lie groups G, H of dimensions greater than 1 which do not coincide with their commutator groups can always be amalgamated (together with any invariant Riemannian metrics on them). And finally, we give some examples of amalgamations of the generalized symmetric Riemannian spaces from the classification list of [4].

1. AFFINE REDUCTIVE SPACES

We shall make use of the terminology of the book [5]. All differentiable manifolds, mappings, tensor fields are of the class C^∞, if not otherwise stated.

Let (M, ∇) be a manifold with an affine connection, let $u_o \in L(M)$ be a fixed frame at a point $o \in M$. Denote by $P(u_o)$ the holonomy bundle through u_o, i.e., the set of all $u \in L(M)$ which can be joined to u_o by a piece-wise differentiable horizontal curve. The group of all affine transformations of M preserving each holonomy bundle $P(u)$, $u \in M$, is called *the group of transvections* of (M, ∇) and will be denoted by $Tr(M, \nabla)$ or shortly $Tr(M)$. It is not difficult to show that for any affine transformation $f\colon M \to M$ we have: $f \in Tr(M)$ if and only if for every $x \in M$ there is a piece-wise differentiable curve c joining x to $f(x)$ such that tangent map $f_{*,x}\colon M_x \to M_{f(x)}$ coincides with the parallel transport along c.

A connected manifold with an affine connection $(M, \tilde{\nabla})$, such that the transvection group $Tr(M)$ acts transitively on each holonomy bundle $P(u) \subset L(M)$, is called *an affine reductive space.* Following [5], Theorem I.25, each affine reductive space $(M, \tilde{\nabla})$ can be represented as a reductive homogeneous space K/H with respect to the decomposition $\mathfrak{k} = \mathfrak{m} \dotplus \mathfrak{h}$, where the action of K on K/H is effective and $\tilde{\nabla}$ is the canonical connection of the reductive homogeneous space K/H (one such representation is $K = Tr(M)$, $H = K_o$, $o \in M$ is a fixed point). Conversely, each reductive homogeneous space K/H with the canonical connection $\tilde{\nabla}$ determines an affine reductive space $(M = K/H, \tilde{\nabla})$.

Let $(M, \tilde{\nabla})$ be an affine reductive space. According to [5], Corollary I.12, the connection $\tilde{\nabla}$ is complete and $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$.

2. AMALGAMATIONS OF AFFINE REDUCTIVE SPACES

2.1. Affine reductive spaces and infinitesimal ar-manifolds

Let $(M, \tilde{\nabla})$ be an affine reductive space. Choose a point $o \in M$ and denote $V = M_o$. Since $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$, we get according to [5], Proposition I.16:

Proposition 2.1. *Let $X, Y, Z \in V$. Then the following holds:*

a) *The endomorphism $\tilde{R}_o(X, Y)$ acting as a derivation on the tensor algebra $\mathcal{T}(V)$ satisfies*

$$\tilde{R}_o(X, Y)(\tilde{R}_o) = \tilde{R}_o(X, Y)(\tilde{T}_o) = 0\,;$$

b) $\tilde{R}_o(X, Y) = -\tilde{R}_o(Y, X)$, $\tilde{T}_o(X, Y) = -\tilde{T}_o(Y, X)$;

c) $\sigma[\tilde{R}_o(X, Y)Z - \tilde{T}_o(\tilde{T}_o(X, Y), Z)] = 0$ (*the first Bianchi identity*);

d) $\sigma[\tilde{R}_o(\tilde{T}_o(X, Y), Z] = 0$ (*the second Bianchi identity*);

σ denotes the cyclic sum with respect to X, Y, Z.

The conditions a)–d) from Proposition 2.1 completely characterize the local structure of the space $(M, \tilde{\nabla})$, as we shall see later. Therefore we introduce the following

Definition 2.2. *An infinitesimal affine reductive manifold* (shortly infinitesimal ar-manifold) is a collection $\mathscr{V} = (V, \tilde{R}, \tilde{T})$, where V is a real vector space and $\tilde{R}$, $\tilde{T}$

are tensors of the types $(1, 3)$, $(1, 2)$ on V such that the conditions a)–d) from Proposition 2.1 are satisfied (without the index o).

Definition 2.3. a) Let $\mathscr{V}_1 = (V_1, \tilde{R}_1, \tilde{T}_1)$, $\mathscr{V}_2 = (V_2, \tilde{R}_2, \tilde{T}_2)$ be infinitesimal ar-manifolds. An isomorphism $f: V_1 \to V_2$ of the vector spaces is called *an isomorphism* of the manifolds $\mathscr{V}_1$ and $\mathscr{V}_2$ if

$$(1) \qquad f(\tilde{R}_1) = \tilde{R}_2\,, \quad f(\tilde{T}_1) = \tilde{T}_2\,.$$

b) An infinitesimal ar-manifold $(V, \tilde{R}, \tilde{T})$ is called *the direct sum* of infinitesimal ar-manifolds $(V_1, \tilde{R}_1, \tilde{T})$, $(V_2, \tilde{R}_2, \tilde{T}_2)$ if

i) $V = V_1 \dotplus V_2$ (*direct sum*),

ii) for every $X, Y, Z \in V$ we have

$$(2) \qquad \begin{aligned} \tilde{R}(X, Y)\,Z &= \tilde{R}_1(\pi_1 X, \pi_1 Y)\,\pi_1 Z + \tilde{R}_2(\pi_2 X, \pi_2 Y)\,\pi_2 Z\,,\\ \tilde{T}(X, Y) &= \tilde{T}_1(\pi_1 X, \pi_1 Y) + \tilde{T}_2(\pi_2 X, \pi_2 Y)\,. \end{aligned}$$

c) An infinitesimal ar-manifold $(V', \tilde{R}', \tilde{T}')$ is called *an infinitesimal ar-submanifold of* $(V, \tilde{R}, \tilde{T})$ *if*

i) V' is a subspace of the vector space V,

(3) ii) $\tilde{R} \mid V' = \tilde{R}'$, $\tilde{T} \mid V' = \tilde{T}'$.

Let $(M, \tilde{\nabla})$ be an affine reductive space. Since the space $(M, \tilde{\nabla})$ is homogeneous, the infinitesimal ar-manifolds $(M_p, \tilde{R}_p, \tilde{T}_p)$, $(M_q, \tilde{R}_q, \tilde{T}_q)$ are isomorphic for any $p, q \in M$

Definition 2.4. The isomorphism class $(M_p, \tilde{R}_p, \tilde{T}_p)$, $p \in M$, of infinitesimal ar-manifolds is called the *infinitesimal model* of the affine reductive space $(M, \tilde{\nabla})$.

Let $(M_1, \tilde{\nabla}_1)$, $(M_2, \tilde{\nabla}_2)$ be affine reductive spaces. Theorem 7.4 in [2], Vol. I, implies that the spaces $(M_1, \tilde{\nabla}_1)$, $(M_2, \tilde{\nabla}_2)$ are locally isomorphic if and only if they have the same infinitesimal model.

The following proposition ([5], Theorem I.17) gives us a correspondence between the affine reductive spaces and the infinitesimal ar-manifolds.

Proposition 2.5. *Let* $\mathscr{V} = (V, \tilde{R}, \tilde{T})$ *be an infinitesimal ar-manifold. Then there is a unique (up to an isomorphism) simply connected affine reductive space* $(M, \tilde{\nabla})$ *with the infinitesimal model* $\mathscr{V}$.

2.2. Submanifolds and foliations on affine reductive spaces

The results of this part will be used to characterize some special types of submanifolds of amalgamated spaces. For this reason we shall limit ourselves only to simply connected spaces.

Let $(M, \tilde{\nabla})$ be a simply connected affine reductive space and $N \subset M$ its connected submanifold.

Definition 2.6. The submanifold N is said to be *weakly invariant* if for each $f \in Tr(M)$ we have: $f(N) \cap N \neq \emptyset \Rightarrow f(N) = N$.

Let N be a weakly invariant submanifold of the affine reductive space $(M, \tilde{\nabla})$. Since N is autoparallel, N is in a natural way an affine reductive space $(N, \tilde{\nabla}^N)$, where $\tilde{\nabla}^N = \tilde{\nabla} \mid N$.

The notion of the weakly invariant submanifold can be transferred to the class of infinitesimal ar-manifolds in the following way.

Definition 2.7. Let $\mathscr{V} = (V, \tilde{R}, \tilde{T})$ be an infinitesimal ar-manifold and let $\mathscr{W} = (W, \tilde{R}, \tilde{T})$ be its submanifold. The manifold $\mathscr{W}$ is said to be *weakly invariant* if $\tilde{R}(X, Y) Z \in W$ for every $X, Y \in V$ and $Z \in W$.

Proposition 2.8. *Let $(N, \tilde{\nabla}^N)$ be a weakly invariant submanifold of an affine reductive space $(M, \tilde{\nabla})$. Denote by $\mathscr{V} = (V, \tilde{R}, \tilde{T})$, $\mathscr{W} = (W, \tilde{R}, \tilde{T})$ the infinitesimal models of the spaces $(M, \tilde{\nabla})$, $(N, \tilde{\nabla}^N)$. Then $\mathscr{W}$ is a weakly invariant submanifold of $\mathscr{V}$.*

Let $\mathscr{V} = (V, \tilde{R}, \tilde{T})$ be an infinitesimal ar-manifold, $(M, \tilde{\nabla})$ its geometric realization. Let us choose a point $p \in M$ and identify $\mathscr{V}$ with the infinitesimal ar-manifold of the affine reductive space $(M, \tilde{\nabla})$ at the point p. We shall show that every weakly invariant submanifold $\mathscr{W} = (W, \tilde{R}, \tilde{T})$ of the manifold $\mathscr{V}$ determines a foliation $\mathscr{F}$ on $(M, \tilde{\nabla})$, the leaves of which have the infinitesimal model $\mathscr{W}$.

Let us define a distribution Δ on $(M, \tilde{\nabla})$ via $\Delta_q = \tau_c(W)$, where $\tau_c \colon V \to M_q$ is the parallel transport along some piecewise differentiable curve c connecting the points p and q. Since the subspace W is $\tilde{R}(V, V)$-invariant, Δ_q is independent of the choice of c, i.e., the definition of Δ is correct. The distribution Δ is $\tilde{\nabla}$-parallel and involutive. Let $\mathscr{F}$ be the foliation on $(M, \tilde{\nabla})$ consisting of all maximal integral manifolds of the distribution Δ.

Proposition 2.9. *Every leaf L of the foliation $\mathscr{F}$ is a weakly invariant autoparallel submanifold of the space $(M, \tilde{\nabla})$. $(L, \tilde{\nabla}^L)$ is an affine reductive space with the infinitesimal model $\mathscr{W}$.*

Proof of Proposition 2.9 is a modification of the proof of Theorem IV.5 in [5].

2.3. Amalgamations of infinitesimal ar-manifolds

Amalgamation of two infinitesimal ar-manifolds is an operation which enables us, under some assumptions, from two infinitesimal ar-manifolds $\mathscr{V}_1, \mathscr{V}_2$ to construct a new infinitesimal ar-manifold $\mathscr{V}$ such that

(4) 1) $\dim V_i < \dim V < \dim V_1 + \dim V_2$, $i = 1, 2$,

(5) 2) the infinitesimal ar-manifold $\mathscr{V}_i$ can be embedded in a natural way into $\mathscr{V}$, $i = 1, 2$.

The infinitesimal ar-manifold $\mathscr{V}$ is constructed by glueing together the vector spaces V_1, V_2 along suitable subspaces $A_1 \subset V_1$, $A_2 \subset V_2$, $\dim A_1 = \dim A_2$.

Definition 2.10. Let $\mathscr{V} = (V, \tilde{R}, \tilde{T})$ be an infinitesimal ar-manifold. The ve ctor space decomposition $V = W \dotplus A$ is said to be *amalgamating* if the following conditions are satisfied:

a) The subspaces W and A determine non-trivial submanifolds $\mathscr{W}$ and $\mathscr{A}$ of the infinitesimal ar-manifold $\mathscr{V}$,

b) $\mathscr{A}$ is weakly invariant submanifold of $\mathscr{V}$,

c) the subspace W is an ideal in V, i.e., $\tilde{T}(X, Y) \in W$ whenever $X \in W$ or $Y \in W$, and $\tilde{R}(X, Y) Z \in W$ whenever $X \in W$ or $Y \in W$ or $Z \in W$.

Let $\mathscr{V}_i = (V_i, \tilde{R}_i, \tilde{T}_i)$, $i = 1, 2$, be infinitesimal ar-manifolds and let $V_i = W_i \dotplus A_i$, $i = 1, 2$, some amalgamating decompositions. Denote by $\mathscr{A}_1, \mathscr{A}_2$ the infinitesimal ar-submanifolds of $\mathscr{V}_1, \mathscr{V}_2$ corresponding to the subspaces A_1, A_2. Let us suppose that there exists an isomorphism $f: \mathscr{A}_1 \to \mathscr{A}_2$.

Definition 2.11. The infinitesimal ar-submanifold of the manifold $\mathscr{V}_1 \dotplus \mathscr{V}_2$, generated by the subspace $V_1 \cup_f V_2 = \{w_1 + w_2 + a_1 + f(a_1);\ w_1 \in W_1,\ w_2 \in W_2,\ a_1 \in A_1\}$, is called *the amalgamation* of $\mathscr{V}_1$ and $\mathscr{V}_2$ with respect to the map f and is denoted by $\mathscr{V}_1 \cup_f \mathscr{V}_2$.

It is evident that the condition (4) is satisfied. Let us define linear maps

$$(6)\qquad f_1: V_1 \to V_1 \cup_f V_2\,,\quad f_1(x) = \begin{cases} x & \text{for } x \in W_1 \\ x + f(x) & \text{for } x \in A_1\,, \end{cases}$$

$$f_2: V_2 \to V_1 \cup_f V_2\,,\quad f_2(x) = \begin{cases} x & \text{for } x \in W_2 \\ f^{-1}(x) + x & \text{for } x \in A_2\,. \end{cases}$$

The maps f_1, f_2 are injective morphisms of infinitesimal ar-manifolds and the condition (5) is satisfied. The infinitesimal ar-manifold $f_i(\mathscr{V}_i)$ will be denoted briefly by $\overline{\mathscr{V}}_i$, $i = 1, 2$.

Let us remark that the amalgamation of two infinitesimal ar-manifolds $\mathscr{V}_1, \mathscr{V}_2$ need not exist. The condition for its existence is the existence of suitable amalgamating decompositions on V_1 and V_2.

Now the operation of amalgamation can be generalized for any finite number of infinitesimal ar-manifolds. Let $\mathscr{V}_i = (V_i, \tilde{R}_i, \tilde{T}_i)$ be an infinitesimal ar-manifold, $V_i = W_i \dotplus A_i$ an amalgamating decomposition, $i = 1, 2, \ldots, n$, and let $\mathscr{F} = \{f_j: \mathscr{A}_1 \to \mathscr{A}_j;\ j = 2, \ldots, n\}$ be a set of isomorphisms of infinitesimal ar-manifolds. *The amalgamation* $\mathscr{V}$ of the manifolds $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_n$ with respect to the system $\mathscr{F}$ is defined as the infinitesimal ar-submanifold of $\bigoplus_{i=1}^{n} \mathscr{V}_i$ generated by the subspace $\{w_1 + \ldots + w_n + a_1 + f_2(a_1) + \ldots + f_n(a_1);\ w_i \in W_i,\ a_1 \in A_1,\ i = 1, 2, \ldots, n\}$. In the case when $\mathscr{V}_1 = \mathscr{V}_2 = \ldots = \mathscr{V}_n = \mathscr{U}$, $U = W \dotplus A$ is an amalgamating decomposition and $f_2 = \ldots = f_n = id$, we speak about *a selfamalgamation*. The selfamalgamation of n copies of $\mathscr{U}$ is denoted shortly by $\mathscr{U}^{(n)}$ and is called *the n-th amalgamation power* of $\mathscr{U}$ (with respect to the decomposition $U = W \dotplus A$).

2.4. Amalgamations of affine reductive spaces

Affine reductive spaces will be amalgamated through their infinitesimal models making use of Proposition 2.5.

Let $\mathscr{M}_1 = (M_1, \tilde{\nabla}_1)$, $\mathscr{M}_2 = (M_2, \tilde{\nabla}_2)$ be affine reductive spaces and let $\mathscr{V}_1 = (V_1, \tilde{R}_1, \tilde{T}_1)$, $\mathscr{V}_2 = (V_2, \tilde{R}_2, \tilde{T}_2)$ be their infinitesimal models. Suppose that there are amalgamating decompositions $V_1 = W_1 \dotplus A_1$, $V_2 = W_2 \dotplus A_2$ and an amalgamation $\mathscr{V}_1 \cup_f \mathscr{V}_2$ with respect to an isomorphism $f: \mathscr{A}_1 \to \mathscr{A}_2$.

Definition 2.12. The simply connected affine reductive space $(M, \tilde{\nabla})$ with the infinitesimal model $\mathscr{V}_1 \cup_f \mathscr{V}_2$ is called *the amalgamation* of the spaces $\mathscr{M}_1$ and $\mathscr{M}_2$ with respect to the map f and is denoted by $\mathscr{M}_1 \cup_f \mathscr{M}_2 = (M_1 \cup_f M_2, \tilde{\nabla})$.

The infinitesimal ar-manifold $\mathscr{V}_i$ $(i = 1, 2)$ can be identified with the submanifold $\overline{\mathscr{V}}_i$ of the manifold $\mathscr{V} = \mathscr{V}_1 \cup_f \mathscr{V}_2$ (the embeddings are given by (6)). Let $o \in M = M_1 \cup_f M_2$. Let us identify the tangent space M_o with $V_1 \cup_f V_2$. Since the subspace $\overline{V}_i$ is $\tilde{R}(V, V)$-invariant, $\overline{V}_i$ determines the foliation $\mathscr{F}_i$ on M. Directly from Proposition 2.9 we get

Proposition 2.13. *Every leaf L_i of the foliation $\mathscr{F}_i$ is a weakly invariant autoparallel submanifold of the space $\mathscr{M}_1 \cup_f \mathscr{M}_2 = (M, \tilde{\nabla})$. $(L_i, \tilde{\nabla}^{L_i})$ is an affine reductive space with the infinitesimal model $\mathscr{V}_i$ and thus locally isomorphic to the space $\mathscr{M}_i$.*

Remark 2.14. One can define an amalgamation of more than two affine reductive spaces through the amalgamation of their infinitesimal models.

2.5. Examples of amalgamations

Denote by $\mathscr{T}$ the class of all simply connected Lie groups G which are not equal to their commutator groups, i.e., such that $[\mathfrak{g}, \mathfrak{g}] \subsetneqq \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. Let $G \in \mathscr{T}$ be a group of dimension at least 2 and $\tilde{\nabla}$ its canonical connection (i.e. the Cartan $(-)$-connection). Then $\tilde{R} = 0$, $\tilde{T} = -[\ ,\ ]$. Let $\mathscr{V} = (\mathfrak{g}, 0, \tilde{T})$

be the infinitesimal model of the group G. We show that there is at least one amalgamating decomposition of the vector space $\mathfrak{g}$. Since $[\mathfrak{g}, \mathfrak{g}] \subsetneqq \mathfrak{g}$, there is a vector $a \in \mathfrak{g}$, $a \notin [\mathfrak{g}, \mathfrak{g}]$. Then we can choose vectors $w_1, w_2, \ldots, w_{n-1} \in \mathfrak{g}$ in such a way that the set $\{a, w_1, \ldots, w_{n-1}\}$ is a vector basis in $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] \subset \langle w_1, \ldots, w_{n-1} \rangle$. The decomposition $\mathfrak{g} = \langle w_1, \ldots, w_{n-1} \rangle \dotplus \langle a \rangle$ is amalgamating. Since the decomposition of this type exists on each group $G \in \mathscr{T}$, every two groups $G, H \in \mathscr{T}$ of dimension > 1 can be amalgamated and the result of this operation is a group from the class $\mathscr{T}$ again. A similar result is true for the class $\mathscr{S}$ of all solvable Lie groups.

Example. Let us consider the Lie groups

$$G = \left\{ \begin{Vmatrix} e^{-p} & q \\ 0 & 1 \end{Vmatrix} ; \; p, q \in \mathbf{R} \right\}, \quad H = \left\{ \begin{Vmatrix} 1 & s & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{Vmatrix} ; \; r, s, t \in \mathbf{R} \right\}, \quad G, H \in S \,.$$

The Lie algebras $\mathfrak{g}, \mathfrak{h}$ of the groups G, H can be represented in the following way:

$$\mathfrak{g} = \langle w, a \rangle \,, \quad [w, a] = w \,,$$

$$\mathfrak{h} = \langle w_1, w_2, b \rangle \,, \quad [w_1, b] = w_2 \,, \quad [w_1, w_2] = [w_2, b] = 0 \,.$$

The decompositions $\mathfrak{g} = \langle w \rangle \dotplus \langle a \rangle$, $\mathfrak{h} = \langle w_1, w_2 \rangle \dotplus \langle b \rangle$ are amalgamating. The amalgamation of the groups G, H with respect to the map $f: \langle a \rangle \to \langle b \rangle$, $f(a) = b$, is the solvable group K of all matrices of the form

$$\begin{Vmatrix} 1 & 0 & p & q \\ 0 & e^{-r} & 0 & s \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

$p, q, r, s \in \mathbf{R}$. Further, the n-th amalgamation power of the group G or H (with respect to the described amalgamating decomposition) is a solvable Lie group $G^{(n)}$ or $H^{(n)}$ of all matrices of the form

$$\begin{Vmatrix} e^{-p} & & & q_1 \\ & \cdot & 0 & \cdot \\ & & \cdot & \cdot \\ & & \cdot & \cdot \\ 0 & & e^{-p} & q_n \\ & & & 1 \end{Vmatrix} \quad \text{or} \quad \begin{Vmatrix} 1 & & & s_1 & r_1 \\ & \cdot & 0 & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ 0 & & & s_n & r_n \\ & & & 1 & t \\ & & & & 1 \end{Vmatrix},$$

respectively, where $p, t, q_i, r_i, s_i \in \mathbf{R}$, $i = 1, 2, \ldots, n$.

3. AMALGAMATIONS OF HOMOGENEOUS RIEMANNIAN SPACES

This chapter is divided into three parts and is devoted mainly to the proof of Main Theorem.

3.1. Amalgamations of affine reductive spaces with parallel metrics

Definition 3.1. Let $(M, \tilde{\nabla})$ be an affine reductive space, g such a Riemannian metric on M that $\tilde{\nabla} g = 0$. The triple $\mathcal{M} = (M, \tilde{\nabla}, g)$ is called *the affine reductive space with a parallel metric.* The class of all affine reductive spaces with parallel metrics will be denoted by Ω.

Now some basic notions for affine reductive spaces with parallel metrics will be introduced.

The local isomorphisms and *the isomorphisms* of spaces of the class Ω are defined similarly to affine reductive spaces, but the maps are required to be also isometric. *The direct product* of the spaces $\mathcal{M}_1 = (M_1, \tilde{\nabla}_1, g_1)$, $\mathcal{M}_2 = (M_2, \tilde{\nabla}_2, g_2)$ from Ω is the space $\mathcal{M}_1 \times \mathcal{M}_2 = (M_1 \times M_2, \tilde{\nabla}_1 \times \tilde{\nabla}_2, g_1 \times g_2) \in \Omega$.

Let $(M, \tilde{\nabla}, g) \in \Omega$, $o \in M$ be a fixed point and $V = M_0$. Denote by $\tilde{R}$, $\tilde{T}$ the curvature and the torsion tensor of the connection $\tilde{\nabla}$, respectively. Then the conditions b), c), d) from Proposition 2.1 and the condition

a′) $\quad \forall X, Y \in V\colon \tilde{R}_o(X, Y)(\tilde{R}_o) = \tilde{R}_o(X, Y)(\tilde{T}_o) = \tilde{R}_o(X, Y)(g_o) = 0$

are satisfied.

Definition 3.2. *A Riemannian infinitesimal ar-manifold* is a collection $\mathcal{V} = (V, g, \tilde{R}, \tilde{T})$, where $\mathcal{V}$ is a real vector space, g is a scalar product on $\mathcal{V}$ and $\tilde{R}$, $\tilde{T}$ are respectively tensors of type $(1, 3)$, $(1, 2)$ on $\mathcal{V}$ such that the condition a′) and the conditions b), c), d) from Proposition 2.1 are satisfied (without the index o).

The isomorphisms, direct products and submanifolds of Riemannian infinitesimal ar-manifolds are defined as in Definition 2.3; we only add to (1) the condition $f(g_1) = g_2$, to (2) the condition $g(X, Y) = g_1(\pi_1 X, \pi_1 Y) + g_2(\pi_2 X, \pi_2 Y)$ and to (3) the condition $g \mid V' = g'$.

The infinitesimal model of an affine reductive space with a parallel metric is defined in the same way as in the non-metric case. It again determines the space uniquely up to a local isomorphism.

The next proposition is the analogue of Proposition 2.5 for Riemannian infinitesimal ar-manifolds:

Proposition 3.3. *Let $\mathcal{V} = (V, g, \tilde{R}, \tilde{T})$ be a Riemannian infinitesimal ar-manifold. Then there is a unique (up to an isomorphism) simply connected space $\mathcal{M} = (M, \tilde{\nabla}, g) \in \Omega$ with the infinitesimal model $\mathcal{V}$.*

Now the operation of amalgamation will be specified for Riemannian infinitesimal ar-manifolds.

Definition 3.4. Let $\mathcal{V} = (V, g, \tilde{R}, \tilde{T})$ be a Riemannian infinitesimal ar-manifold. An orthogonal vector space decomposition $V = W \dotplus A$ is called *amalgamating* if the conditions a), b), c) from Definition 2.10 are satisfied.

Let $\mathscr{V}_1 = (V_1, g_1, \tilde{R}_1, \tilde{T}_1)$, $\mathscr{V}_2 = (V_2, g_2, \tilde{R}_2, \tilde{T}_2)$ be two Riemannian infinitesimal ar-manifolds, $V_1 = W_1 \dotplus A_1$, $V_2 = W_2 \dotplus A_2$ amalgamating decompositions and $f: \mathscr{A}_1 \to \mathscr{A}_2$ an isomorphism.

Definition 3.5. *The amalgamation* of Riemannian infinitesimal ar-manifolds $\mathscr{V}_1$ and $\mathscr{V}_2$ with respect to the isomorphism f is a Riemannian infinitesimal ar-manifold $\mathscr{V}_1 \cup_f \mathscr{V}_2 = (V, g, \tilde{R}, \tilde{T})$ such that $(V, \tilde{R}, \tilde{T}) = (V_1, \tilde{R}_1, \tilde{T}_1) \cup_f (V_2, \tilde{R}_2, \tilde{T}_2)$ and the metric g on V is defined via

$$(7) \qquad g(w_1 + w_2 + a_1 + f(a_1), w'_1 + w'_2 + a'_1 + f(a'_1)) =$$
$$= g_1(w_1, w'_1) + g_2(w_2, w'_2) + \tfrac{1}{2}(g_1(a_1, a'_1) + g_2(f(a_1), f(a'_1))),$$
$$w_i, w'_i \in W_i, \quad a_1, a'_1 \in A_1, \quad i = 1, 2.$$

It is not difficult to verify that the conditions (4), (5) from Section 2.3 are satisfied for $\mathscr{V} = \mathscr{V}_1 \cup_f \mathscr{V}_2$.

Remark 3.6. Similarly as in the case of infinitesimal ar-manifolds it is possible to define amalgamations of more than two Riemannian infinitesimal ar-manifolds.

Let $\mathscr{M}_1 = (M_1, \tilde{\nabla}_1, g_1)$, $\mathscr{M}_2 = (M_2, \tilde{\nabla}_2, g_2)$ be affine reductive spaces with parallel metrics and $\mathscr{V}_1 = (V_1, g_1, \tilde{R}_1, \tilde{T}_1)$, $\mathscr{V}_2 = (V_2, g_2, \tilde{R}_2, \tilde{T}_2)$ their infinitesimal models. Suppose that there are amalgamating decompositions $V_1 = W_1 \dotplus A_1$, $V_2 = W_2 \dotplus A_2$ such that there is an amalgamation $\mathscr{V}_1 \cup_f \mathscr{V}_2$ with respect to some isomorphism $f: \mathscr{A}_1 \to \mathscr{A}_2$.

Definition 3.7. A simply connected space $\mathscr{M} = (M, \tilde{\nabla}, g) \in \Omega$ with the infinitesimal model $\mathscr{V}_1 \cup_f \mathscr{V}_2$ is called *the amalgamation* of $\mathscr{M}_1$ and $\mathscr{M}_2$ with respect to the isomorphism f and is denoted by $\mathscr{M}_1 \cup_f \mathscr{M}_2$.

Remark 3.8. According to Remark 3.6 it is possible to amalgamate more than two spaces from the class Ω.

Similarly as in the case of affine reductive spaces we can construct the foliations $\mathscr{F}_1, \mathscr{F}_2$ on $\mathscr{M}_1 \cup_f \mathscr{M}_2 = (M, \tilde{\nabla}, g)$, the properties of which are characterized in the following two propositions (which are only slight modifications of the results of [5], Chapter IV).

Proposition 3.9. *Let us choose $i \in \{1, 2\}$. Then every leaf L_i of the foliation $\mathscr{F}_i$ is an autoparallel submanifold of the space $(M, \tilde{\nabla})$. The space $(L_i, \tilde{\nabla}^{L_i}, g^{L_i})$ is an affine reductive space with a parallel metric with the infinitesimal model $\mathscr{V}_i$ and thus locally isomorphic to the space $(M_i, \tilde{\nabla}_i, g_i)$.*

Proposition 3.10. *The Riemannian manifold (L_i, g^{L_i}) is a totally geodesic submanifold of the manifold (M, g) (notation as in Proposition 3.9).*

3.2. Irreducibility Theorem

Here we shall prove that any amalgamation of two affine reductive spaces with parallel metrics preserves irreducibility.

Let us consider a space $\mathcal{M} = (M, \tilde{\nabla}, g) \in \Omega$. The symbol ∇ will always denote the Riemannian connection on (M, g). Further, denote by R the curvature tensor of the connection ∇. The symbol D will be used for the difference tensor $D_X Y = \nabla_X Y - \tilde{\nabla}_X Y$.

Making use of the methods introduced in [5], Chapter III, we get :

Proposition 3.11. *Let a space $(M, \tilde{\nabla}, g) \in \Omega$ and $o \in \Omega$ be given. Then for every $X, Y, Z \in M_o$ we have:*

(8) $$\tilde{T}(X, Y) = D_Y X - D_X Y,$$

(9) $$\tilde{R}(X, Y) = R(X, Y) - [D_X, D_Y] - D_{\tilde{T}(X,Y)},$$

(10) $$2\, g(D_Y X, Z) = g(\tilde{T}(X, Y), Z) + g(\tilde{T}(X, Z), Y) + g(\tilde{T}(Y, Z), X).$$

Let us consider a space $\mathcal{M} = (M, \tilde{\nabla}, g) \in \Omega$ and a point $o \in M$. Denote by $\mathcal{V} = (V, g, \tilde{R}, \tilde{T})$ the Riemannian infinitesimal ar-manifold of the space M at the point o. Further, suppose that there exists an amalgamating decomposition $V = W \dotplus A$.

Proposition 3.12. *For the difference tensor D and the Riemannian curvature tensor R we have:*

a) $D_A A \subset A,\ D_W A \subset W,\ D_A W \subset W,$

b) $R(A, A)\, A \subset A,\ R(A, A)\, W \subset W,\ R(A, W)\, A \subset W.$

Proof. a) Apply (9). b) Apply (10).

Later on we shall need the following lemma which makes use of some basic properties of the exponential map $\exp\colon \mathfrak{gl}(V) \to GL(V)$.

Lemma 1. *Let V be a finite dimensional vector space over the field $\mathbf{R}$ of real numbers, G a connected subgroup of the group $GL(V)$ and $\mathfrak{g}$ its Lie algebra. Then for every vector subspace $U \subset V$ we have:*

$$G(U) \subset U \quad \textit{if and only if} \quad \mathfrak{g}(U) \subset U.$$

If V is a vector space with a scalar product and G consists of orthogonal transformations, then for every vector subspace $U \subset V$ we have:

$$\mathfrak{g}(U) \subset U \quad \textit{if and only if} \quad \mathfrak{g}(U^\perp) \subset U^\perp.$$

Let (M, g) be a connected and simply connected analytic Riemannian space and $x \in M$. According to [2], Vol. I, Chapter III, the Lie algebra $\mathfrak{g}(x)$ of the holonomy group with the reference point x is generated by all endomorphisms of the vector space M_x of the form

$$(\nabla^k R)(X, Y; U_1, \ldots, U_k),$$

where $X, Y, U_1, \ldots, U_k \in M_x$, $k = 0, 1, \ldots$.

Let us consider a simply connected space $(M, \tilde{\nabla}, g) \in \Omega$ and a point $x \in M$. Denote by $\mathscr{L}$ the Lie algebra of all endomorphisms of the vector space M_x generated by the set

$$\{(D^k R)(X, Y; U_1, \ldots, U_k);\ X, Y, U_1, \ldots, U_k \in M_x,\ k = 0, 1, \ldots\}.$$

Since $\tilde{\nabla} R = 0$, we have $D^k R = \nabla^k R$ for all $k = 0, 1, \ldots$, hence $\mathscr{L} = \mathfrak{g}(x)$. From Lemma 1 we see that the underlying Riemannian manifold (M, g) is irreducible if and only if the Lie algebra $\mathscr{L}$ acts irreducibly on M_x.

The following theorem is one of the main results of this paper.

Theorem 3.13. *Let $\mathscr{M}_1, \mathscr{M}_2$ be simply connected affine reductive spaces with parallel metrics which are irreducible as Riemannian spaces. If an amalgamation $\mathscr{M}_1 \cup_f \mathscr{M}_2$ exists, then it is irreducible as a Riemannian space.*

Before we start the proof of Theorem 3.13, some suitable notation will be introduced. We also prove some formulas which characterize the Riemannian curvature tensor and its covariant derivations $\nabla^k R = D^k R$, $k = 1, 2, \ldots$, on the space $\mathscr{M}_1 \cup_f \mathscr{M}_2$.

Let $\mathscr{V}_1 = (V_1, g_1, \tilde{R}_1, \tilde{T}_1)$, $\mathscr{V}_2 = (V_2, g_2, \tilde{R}_2, \tilde{T}_2)$ be the infinitesimal models of the spaces $\mathscr{M}_1 = (M_1, \tilde{\nabla}_1, g_1)$, $\mathscr{M}_2 = (M_2, \tilde{\nabla}_2, g_2)$, respectively. Let $V_1 = W_1 \dotplus A_1$, $V_2 = W_2 \dotplus A_2$ be amalgamating decompositions and $f\colon \mathscr{A}_1 \to \mathscr{A}_2$ an isomorphism (see Definition 3.5). The embeddings $f_i\colon V_i \to V_1 \cup_f V_2$, $i = 1, 2$, are given by (6). We shall use the following notation:

$$\bar{W}_i = f_i(W_i), \quad A = \bar{A}_i = f_i(A_i), \quad \bar{V}_i = f_i(V_i), \quad V = V_1 \cup_f V_2, \quad i = 1, 2.$$

Proposition 3.14. *Let $X, Y, U_1, \ldots, U_k \in V_i$ be given, $i = 1, 2$, $k = 1, 2, \ldots$. Then the following formulas are satisfied:*

a) $f_i(D_{iX} Y) = D_{f_i(X)} f_i(Y)$,

b) $f_i(R_i(X, Y) Z) = R(f_i(X), f_i(Y)) f_i(Z)$,

c) $f_i((D_i^k R_i)(X, Y, Z;\ U_1, \ldots, U_k)) = (D^k R)(f_i(X), f_i(Y), f_i(Z); f_i(U_1), \ldots, f_i(U_k))$,

where D_i, R_i denote the difference tensor and the Riemannian curvature tensor, respectively, on the space $(M_i, \tilde{\nabla}_i, g_i)$, $i = 1, 2$.

Proof. It is sufficient to notice that the maps $f_i\colon V_i \to \bar{V}_i \subset V$, $i = 1, 2$, are isomorphisms.

Lemma 2. *Let $r, s \in \{1, 2\}$, $r \neq s$, be given. Then we have:*

a) $D_{\overline{W}_r} \overline{W}_s = 0$,

b) $D_{\overline{W}_r} \overline{V}_s \subset \overline{W}_r$, $D_{\overline{V}_s} \overline{W}_r \subset \overline{W}_r$,

c) $R(\overline{V}_r, \overline{V}_r)\, \overline{W}_s \subset \overline{W}_s$, $R(\overline{V}_r, \overline{W}_s)\, \overline{V}_r \subset \overline{W}_s$.

Proof. Let us consider $Y \in \overline{W}_r$, $X \in \overline{W}_s$, $Z \in V$. From (10) we get $2g(D_Y X, Z) = 0$. Hence $D_{\overline{W}_r} \overline{W}_s = 0$, because $Z \in V$ is arbitrary. Part b) is a consequence of a). Making use of Proposition 3.12 and part a), we get assertion c).

Lemma 3. *Let $k \geqq 0$ be an integer, $r, s \in \{1, 2\}$, $r \neq s$ and let $j \in \{1, 2, \dots, k + 3\}$ Then for each $U_j \in \overline{W}_s$ and for every $U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_{k+3} \in \overline{V}_r$* we have

$$(D^k R)(U_1, \dots, U_{k+3}) \in \overline{W}_s .$$

Proof. We shall proceed by induction with respect to k. The case $k = 0$ follows from c), Lemma 2. Let us suppose that the assertion is true for $k - 1 \geqq 0$. Then

$$(D^k R)(U_1, \dots, U_{k+3}) = (D(D^{k-1}R))(U_1, \dots, U_{k+3}) =$$
$$= D_{U_{k+3}}((D^{k-1}R)(U_1, \dots, U_{k+2}) - (D^{k-1}R)(D_{U_{k+3}} U_1, U_2, \dots, U_{k+2}) - \dots$$
$$\dots - (D^{k-1}R)(U_1, \dots, U_{k+1}, D_{U_{k+3}} U_{k+2}) .$$

Using part b) of Lemma 2 and the induction assumption we see that the last sum is contained in $\overline{W}_s$.

Let us define linear maps τ_1, τ_2 as follows:

$$\tau_1 \colon V \to \overline{V}_1 , \quad \tau_1(x) = x \text{ for } x \in \overline{V}_1 , \quad \tau_1(x) = 0 \text{ for } x \in \overline{W}_2 ,$$
$$\tau_2 \colon V \to \overline{V}_2 , \quad \tau_2(x) = x \text{ for } x \in \overline{V}_2 , \quad \tau_2(x) = 0 \text{ for } x \in \overline{W}_1 .$$

Proposition 3.15. *Let $A \colon V \to V$ be an endomorphism, $k \geqq 0$ an integer and $U_1, \dots, U_{k+3} \in \overline{V}_i$. Choose $r \in \{1, 2, \dots, k + 3\}$. Then*

$$\tau_i((D^k R)(U_1, \dots, U_{r-1}, A(U_r), U_{r+1}, \dots, U_{k+3})) =$$
$$= (D^k R)(U_1, \dots, U_{r-1}, \tau_i(A(U_r)), U_{r+1}, \dots, U_{k+3}) , \quad i = 1, 2 .$$

Proof. Let $U_1, \dots, U_{k+3} \in \overline{V}_i$ be given, $i \in \{1, 2\}$. Then $\tau_i((D^k R)(U_1, \dots, U_{r-1}, A(U_r), U_{r+1}, \dots, U_{k+3})) = \tau_i((D^k R)(U_1, \dots, U_{r-1}, A(U_r) - \tau_i(A(U_r)), U_{r+1}, \dots, U_{k+3})) + \tau_i((D^k R)(U_1, \dots, U_{r-1}, \tau_i(A(U_r)), U_{r+1}, \dots, U_{k+3})) = \tau_i((D^k R)(U_1, \dots, U_{r-1}, \tau_i(A(U_r)), U_{r+1}, \dots, U_{k+3})) = (D^k R)(U_1, \dots, U_{r-1}, \tau_i(A(U_r)), U_{r+1}, \dots, U_{k+3})$, where we have used also Lemma 3.

Corollary. *For each integer $k \geqq 0$ and for all vectors $X, Y, U_1, U_2, \dots, U_k$ from $\overline{V}_i$ we have:*

$$\tau_i \circ ((D^k R)(X, Y; U_1, \ldots, U_k)) = ((D^k R)(X, Y; U_1, \ldots, U_k)) \circ \tau_i, \; i = 1, 2.$$

Proof. It is sufficient to take $r = 3$ and $A = id$ in Proposition 3.15.

Let us denote by $\mathscr{L}_i$ the Lie algebra generated by the set of endomorphisms

$$\{(D^k R)(X, Y; U_1, \ldots, U_k); \; X, Y, U_1, \ldots, U_k \in \bar{V}_i, \; k = 0, 1, \ldots\}$$

of the vector space V, $i = 1, 2$.

Proposition 3.16. *Let U be an $\mathscr{L}_i$-invariant subspace of the vector space V. Then $\tau_i(U)$ is an $\mathscr{L}_i$-invariant subspace of V as well $(i = 1, 2)$.*

Proof. Let us consider $f \in \mathscr{L}_i$. Applying Corollary of Proposition 3.15 we get:

$$f(\tau_i(U)) = (f \circ \tau_i)(U) = (\tau_i \circ f)(U) = \tau_i(f(U)) \subset \tau_i(U).$$

Proof of Theorem 3.13. Let $(V, g, \tilde{R}, \tilde{T})$ be the infinitesimal model of the amalgamation $M_1 \cup_f M_2$. Let $\mathscr{L}$ be the Lie algebra of endomorphisms of the vector space V generated by the set $\{(D^k R)(X, Y; U_1, \ldots, U_k); \; X, Y, U_1, \ldots, U_k \in V, \; k = 0, 1, \ldots\}$. Further, suppose that U is an $\mathscr{L}$-invariant subspace of V, $U \neq 0$. Now it is sufficient to prove that $U = V$.

First we show that $\dim U \geqq \dim V_i$, $i = 1, 2$. Let us suppose that $\tau_1(U) \neq 0$. The subspace U is also $\mathscr{L}_1$-invariant. Proposition 3.16 implies the $\mathscr{L}_1$-invariance of the subspace $\tau_1(U)$. Since the Riemannian space (M_1, g_1) is irreducible, we get $\tau_1(U) = V_1$. Then $\tau_2(U) \neq 0$ and similarly $\tau_2(U) = V_2$.

Now the equality $U = V$ will be proved. Suppose that $\dim U < \dim V$. Since the holonomy group with the reference point x consists of orthogonal transformations of the vector space V (for any $x \in M$), we see from Lemma 1 that $U^\perp$ is also an $\mathscr{L}$-invariant subspace of V. Let us denote $s = \dim U$, $k = \dim \bar{A}_1 = \dim \bar{A}_2$, $m = \dim \bar{V}_1$, $n = \dim \bar{V}_2$. Then we have:

$$s \geqq \max\{m, n\} \geqq m, \tag{11}$$

$$m + n - k - s \geqq \max\{m, n\} \geqq n, \tag{12}$$

because $U, U^\perp$ are non-trvial $\mathscr{L}$-invariant subspaces of V. From (11) and (12) we get $-k \geqq 0$, i.e. $k \leqq 0$, a contradiction. Hence $\dim U = \dim V$ and $U = V$.

Remark 3.17. Theorem 3.13 can be generalized to the amalgamation of more than two spaces.

3.3. Amalgamations of homogeneous Riemannian spaces

To prove our Main Theorem we only have to define properly the amalgamation of two homogeneous Riemannian spaces.

First we prove that every homogeneous Riemannian space admits a suitable reductive structure, i.e. the structure of a space from the class Ω (Ω-structure).

Proposition 3.18. *Let (M, g), $M = K/H$, be a connected homogeneous Riemannian space, where K is some transitive group of isometries on M and H is its isotropy subgroup. Then there is at least one structure of a reductive homogeneous space on K/H.*

Proof. Since the group H is compact, there is at least one $ad(H)$-invariant scalar product on the Lie algebra $\mathfrak{k}$ of the group K. Let $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to this product. We get a reductive decomposition $\mathfrak{k} = \mathfrak{m} \dotplus \mathfrak{h}$.

Let us denote by $\tilde{\nabla}$ the canonical connection of the reductive homogeneous space $M = K/H$ (with respect to the decomposition $\mathfrak{k} = \mathfrak{m} \dotplus \mathfrak{h}$). The space $(M, \tilde{\nabla})$ is an affine reductive space. Since the tensor g is K-invariant (the group K consists of isometries), we have $\tilde{\nabla} g = 0$ and $(M, \tilde{\nabla}, g) \in \Omega$.

Definition 3.19. Let (M_1, g_1), (M_2, g_2) be homogeneous Riemannian spaces, $\mathcal{M}_1 = (M_1, \tilde{\nabla}_1, g_1)$, $\mathcal{M}_2 = (M_2, \tilde{\nabla}_2, g_2)$ some Ω-structures on them. Let us suppose that there exists an amalgamation $\mathcal{M} = \mathcal{M}_1 \cup_f \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M} = (M, \tilde{\nabla}, g)$. The homogeneous Riemannian space (M, g) is then called *an amalgamation* of the homogeneous Riemannian spaces (M_1, g_1), (M_2, g_2).

Now our Main Theorem (cf. Introduction) is an immediate consequence of Theorem 3.13.

The simplest homogeneous Riemannian spaces are Lie groups with invariant metrics. We shall show how to amalgamate Lie groups from the class $\mathcal{T}$ (described in Section 2.5) equipped with left invariant metrics.

Let $G \in \mathcal{T}$, denote by $\tilde{\nabla}$ the canonical connection of G. Let $(\mathfrak{g}, g, 0, \tilde{T})$ be the infinitesimal model of the space $(G, \tilde{\nabla}, g)$. Choose $a \notin [\mathfrak{g}, \mathfrak{g}]$ and an orthonormal system $\{w_1, \ldots, w_{n-1}\}$ of vectors in such a way that $\{w_1, \ldots, w_{n-1}, a\}$ is a basis of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] \subset \langle w_1, \ldots, w_{n-1} \rangle$. Let $b = a - \sum_{i=1}^{n-1} g(w_i, a)\, w_i$ and $a' = b/\|b\|$. Then $a' \notin \langle w_1, \ldots, w_{n-1} \rangle$ and $\{w_1, \ldots, w_{n-1}, a'\}$ is an orthonormal basis of $\mathfrak{g}$. The decomposition $\mathfrak{g} = \langle w_1, \ldots, w_{n-1} \rangle \dotplus \langle a' \rangle$ is amalgamating. By virtue of Main Theorem and Remark 3.17 we get

Theorem 3.20. *Let G, H be connected and simply connected Lie groups of dimension >1 with irreducible invariant metrics g, h, respectively, and such that $[\mathfrak{g},\mathfrak{g}] \subsetneq \mathfrak{g}$, $[\mathfrak{h}, \mathfrak{h}] \subsetneq \mathfrak{h}$. Then*

a) *There is at least one amalgamation of the groups (G, g), (H, h). Any such amalgamation is a Lie group with an irreducible invariant metric.*

b) *For every $n \geqq 2$ there is at least one amalgamation power $(G^{(n)}, g^{(n)})$ of the group (G, g). Every amalgamation power of the group (G, g) is a Lie group with an irreducible invariant metric.*

4. EXAMPLES OF AMALGAMATIONS OF GENERALIZED SYMMETRIC RIEMANNIAN SPACES

In this chapter we give examples of amalgamations and selfamalgamations of some generalized symmetric Riemannian spaces from the classification list of [4].

Since the initial spaces are connected, simply connected and irreducible, any amalgamation of them is again an irreducible space. Hence it is possible to construct the infinite series of irreducible generalized symmetric Riemannian spaces in such a way that the dimensions of the spaces form an arithmetic progression.

Example 1. A space of dimension 3 and of order 4. ([4], pp. 16–17.) Let us consider n copies of this space with the invariants $\lambda_1, \ldots, \lambda_n$. By an amalgamation of these g.s. Riemannian spaces we obtain a g.s. Riemannian space $(M' = \mathbf{R}^{2n+1}(x_1, y_1, \ldots, x_n, y_n, z), g')$, where

$$g' = e^{2z}(dx_1)^2 + e^{-2z}(dy_1)^2 + \sum_{i=2}^{n} [e^{2\mu_i z}(dx_i)^2 + e^{-2\mu_i z}(dy_i)^2] + \mu_1 dz^2 ,$$

$$\mu_1 = \lambda_1 , \quad \mu_i = \lambda_1/\lambda_i , \quad i = 2, \ldots, n .$$

As a homogeneous space, M' is a matrix group

$$\left\| \begin{matrix} e^{-z} & & & & & & & & x_1 \\ & e^{z} & & & & & & & y_1 \\ & & e^{-\mu_2 z} & & & 0 & & & x_2 \\ & & & e^{\mu_2 z} & & & & & y_2 \\ & & & & \ddots & & & & \vdots \\ & & 0 & & & e^{-\mu_n z} & & & x_n \\ & & & & & & e^{\mu_n z} & & y_n \\ & & & & & & & & 1 \end{matrix} \right\| ,$$

$x_1, y_1, \ldots, x_n, y_n, z \in \mathbf{R}$. The space (M', g') is of order 4. A typical symmetry at the origin of $\mathbf{R}^{2n+1}$ is $x_i' = -y_i$, $y_i' = x_i$, $z' = -z$, $i = 1, 2, \ldots, n$.

Example 2. A space of dimension 4 and of order 3. ([4], pp. 18–21.) An n-th amalgamation power of this space is a g.s. Riemannian space $(M' = \mathbf{R}^{2n+2}(u_1, v_1, \ldots \ldots, u_n, v_n, x, y), g')$, where

$$g' = (-x + \sqrt{(x^2 + y^2 + 1)}) \sum_{i=1}^{n} (du_i)^2 + (x + \sqrt{(x^2 + y^2 + 1)})$$
$$\sum_{i=1}^{n} (dv_i)^2 - 2y \sum_{i=1}^{n} du_i \, dv_i +$$
$$+ [(1 + y^2) dx^2 + (1 + x^2) dy^2 - 2xy \, dx \, dy]/[2\varrho^2(1 + x^2 + y^2)] ,$$

$\varrho > 0$ being a real invariant. The space (M', g') is of order 3. A typical symmetry at the origin of $\mathbf{R}^{2n+2}$ is given by

$$x' = \cos\tfrac{2}{3}\pi x - \sin\tfrac{2}{3}\pi y\,, \qquad y' = \sin\tfrac{2}{3}\pi x + \cos\tfrac{2}{3}\pi y\,,$$
$$u'_i = \cos\tfrac{2}{3}\pi u_i - \sin\tfrac{2}{3}\pi v_i\,, \qquad v'_i = \sin\tfrac{2}{3}\pi\, u_i + \cos\tfrac{2}{3}\pi v_i\,,$$
$$i = 1, 2, \ldots, n\,.$$

Example 3. A space of dimension 5 and of order 4, type 1. ([4], pp. 27–28.) An amalgamation of n copies of this space with the invariants $\varrho_1, \ldots, \varrho_n$ is a g.s. Riemannian space $(M' = \mathbf{R}^{3n+2}(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, u, v), g')$, where

$$g' = \mathrm{d}u^2 + \mathrm{d}v^2 + \sum_{i=1}^{n} [(\mathrm{d}x_i)^2 + (\mathrm{d}y_i)^2 + \varrho_i^2(x_i\,\mathrm{d}u - y_i\,\mathrm{d}v + \mathrm{d}z_i)^2]\,.$$

The space (M', g') is of order 4. A typical symmetry at the origin of $\mathbf{R}^{3n+2}$ is given by $x'_i = -y_i$, $y'_i = x_i$, $z'_i = -z_i$, $u' = -v$, $v' = u$, $i = 1, \ldots, n$.

Example 4. A space of dimension 5 and of order 4, type 7. ([4], pp. 45–47.)

a) An amalgamation of n copies of this space with the invariants a_i, γ_i, λ_i, $i = 1, \ldots, n$, is a g.s. Riemannian space $(M' = \mathbf{R}^{4n+1}(x_1, y_1, u_1, v_1, \ldots, x_n, y_n, u_n, v_n, t), g')$, where

$$g' = \mathrm{d}t^2 + \sum_{i=1}^{n} [\mathrm{e}^{-2\lambda_i t}(t\,\mathrm{d}x_i - \mathrm{d}u_i)^2 + \mathrm{e}^{2\lambda_i t}(t\,\mathrm{d}y_i + \mathrm{d}v_i)^2 +$$
$$+ a_i^2(\mathrm{e}^{-2\lambda_i t}(\mathrm{d}x_i)^2 + \mathrm{e}^{2\lambda_i t}(\mathrm{d}y_i)^2) + 2\gamma_i(\mathrm{d}y_i\,\mathrm{d}u_i - \mathrm{d}x_i\,\mathrm{d}v_i)]\,.$$

The g.s. Riemannian space (M', g') is of order 4. A typical symmetry at the origin of $\mathbf{R}^{4n+1}$ is $x'_i = -y_i$, $y'_i = x_i$, $u'_i = -v_i$, $v'_i = u_i$, $t' = -t$, $i = 1, \ldots, n$.

b) In the case of $\lambda = \gamma = 0$, and n-th amalgamation power of this space is a g.s. Riemannian space $(M'' = \mathbf{R}^{3n+2}(x, y, u_1, v_1, t_1, \ldots, u_n, v_n, t_n), g'')$, where

$$g'' = a^2(\mathrm{d}x^2 + \mathrm{d}y^2) + \sum_{i=1}^{n} [(\mathrm{d}t_i)^2 + (t_i\,\mathrm{d}x - \mathrm{d}u_i)^2 + (t_i\,\mathrm{d}y - \mathrm{d}v_i)^2]\,.$$

For n odd the g.s. Riemannian space (M'', g'') is of order 4, for n even it is 4-symmetric. A typical symmetry of order 4 at the origin of $\mathbf{R}^{3n+2}$ is $x' = -y$, $y' = x$, $u'_i = -v_i$, $v'_i = u_i$, $t'_i = -t_i$, $i = 1, \ldots, n$.

Example 5. A space of dimension 5 and of order 6, type 9. ([4], pp. 51–53.) An amalgamation of n copies of this space with the invariants $a_1, b_1, \ldots, a_n, b_n$, $a_1 = a_2 = \ldots = a_n = a$, is a g.s. Riemannian space $(M' = \mathbf{R}^{3n+2}(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, u, v), g')$, where

$$g' = \tfrac{4}{3}a^2(\mathrm{d}u^2 + \mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2) + \sum_{i=1}^{n} [(2b_i^2 + 1)(\mathrm{e}^{2(u+v)}(\mathrm{d}x_i)^2 +$$
$$+ \mathrm{e}^{-2u}(\mathrm{d}y_i)^2 + \mathrm{e}^{-2v}(\mathrm{d}z_i)^2 + (2b_i^2 - 1)(\mathrm{e}^{v}\,\mathrm{d}x_i\,\mathrm{d}y_i +$$
$$+ \mathrm{e}^{u}\,\mathrm{d}x_i\,\mathrm{d}z_i - \mathrm{e}^{(u+v)}\,\mathrm{d}y_i\,\mathrm{d}z_i)]\,.$$

A typical symmetry of order 6 at the origin of $\mathbf{R}^{3n+2}$ is $u' = v$, $v' = -(u + v)$, $x_i' = y_i$, $y_i' = -z_i$, $z_i' = x_i$, $i = 1, \ldots, n$.

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