Elena Pavlíková

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A REMARK ON THE DIFFERENTIAL EQUATION $y'' + q(x)\,y = r(x)$

Elena Pavlíková, Žilina

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In [4] M. Laitoch introduced the systems of knots of the 1st and 2nd kinds, corresponding to the differential equation

$$(\bar{\mathrm{q}}) \qquad y'' + q(x)\,y = r(x)\,,$$

where $q(x) \in C_2(J)$, $r(x) \in C_0(J)$, $q(x) > 0$ for $x \in J$, J is an open interval, and gave a modification of Sturm's theorem on separating zeros of solutions or zeros of the first derivatives of solutions of the 2nd order linear homogeneous differential equation

$$(\mathrm{q}) \qquad y'' + q(x)\,y = 0\,.$$

In this paper we will extend the above mentioned results from [4] by using the k-th accompanying equation for $(\bar{\mathrm{q}})$ with regard to the basis $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$, where α_j, β_j are real numbers such that $\alpha_j^2 + \beta_j^2 > 0$, $j = 1, \ldots, k$.

1. DEFINITIONS AND NOTATION

In this paper we consider a linear nonhomogeneous differential equation of the 2nd order

$$(1.1) \qquad y'' + q(x)\,y = r(x)\,,$$

where $q(x) > 0$ for $x \in J$, J is an open interval.

We shall suppose the solutions of the corresponding homogeneous equation

$$(1.2) \qquad y'' + q(x)\,y = 0$$

to be oscillatory.

Definition 1.1. Let α_1, β_1 be real numbers such that $\alpha_1^2 + \beta_1^2 > 0$. Denote

$$(1.3) \qquad Q_1(x) = Q_1(x, \alpha_1, \beta_1) = \\ = q + \frac{\alpha_1\beta_1 q'}{\alpha_1^2 + \beta_1^2 q} + \frac{1}{2}\frac{\beta_1^2 q''}{\alpha_1^2 + \beta_1^2 q} - \frac{3}{4}\frac{\beta_1^4 q'^2}{(\alpha_1^2 + \beta_1^2 q)^2}\,,$$

$$(1.4) \qquad R_1(x) = R_1(x, \alpha_1, \beta_1) = \frac{\alpha_1 r + \beta_1 r'}{(\alpha_1^2 + \beta_1^2 q)^{1/2}} - \frac{\beta_1^3 r q}{(\alpha_1^2 + \beta_1^2 q)^{3/2}}\,.$$

Assume $q(x) \in C_2(J)$, $r(x) \in C_1(J)$, $q(x) > 0$ for $x \in J$. The differential equation

$$y'' + Q_1(x)\, y = R_1(x) \tag{1.5}$$

is said to be *the first accompanying equation for the differential equation* (1.1) *with regard to the basis* (α_1, β_1).

It is easy to verify that if $v(x)$ is a solution of (1.1), then the function

$$V_1(x) = \frac{\alpha_1 v + \beta_1 v'}{\sqrt{(\alpha_1^2 + \beta_1^2 q)}} \tag{1.6}$$

is a solution of the differential equation (1.5).

Remark 1.1. If we choose $r(x) \equiv 0$ for $x \in J$ in the above considerations, then we get as a special case the situation studied in [3], concerning the first accompanying equation

$$y'' + Q_1(x)\, y = 0 \tag{1.7}$$

for the differential equation (1.2) with regard to the basis (α_1, β_1).

For $k > 1$ the k-th accompanying equation is defined inductively.

Definition 1.2. Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ be real numbers such that $\alpha_j^2 + \beta_j^2 > 0$, $j = 1, \ldots, k$. Assume $q(x) \in C_{2k}(J)$, $r(x) \in C_k(J)$. The first accompanying equation

$$y'' + Q_k(x)\, y = R_k(x) \tag{1.8}$$

for the $(k-1)$st accompanying equation

$$y'' + Q_{k-1}(x)\, y = R_{k-1}(x) \tag{1.9}$$

with regard to the basis (α_k, β_k) is said to be *the k-th accompanying equation for the differential equation* (1.1) *with regard to the bais* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$. Functions $Q_j(x)$, $R_j(x)$ are defined inductively and we assume $Q_j(x) > 0$, $j = 0, 1, \ldots, k-1$, $Q_0(x) \equiv q(x)$ for $x \in J$.

A straightforward calculation shows that if $v(x)$ is a solution of (1.1), then the function

$$(1.10_v) \qquad V_k(x) = V_k(x, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k) =$$

$$= \left[\alpha_k \left[\ldots \alpha_3 \left[\alpha_2 \frac{\alpha_1 v + \beta_1 v'}{\sqrt{(\alpha_1^2 + \beta_1^2 q)}} + \beta_2 \left[\frac{\alpha_1 v + \beta_1 v'}{\sqrt{(\alpha_1^2 + \beta_1^2 q)}}\right]'\right] (\alpha_2^2 + \beta_2^2 Q_1)^{-1/2} + \ldots\right] \cdot\right.$$

$$\cdot (\alpha_{k-1}^2 + \beta_{k-1}^2 Q_{k-2})^{-1/2} + \beta_k \left[\left[\ldots \alpha_3 \left[\alpha_2 \frac{\alpha_1 v + \beta_1 v'}{\sqrt{(\alpha_1^2 + \beta_1^2 q)}} + \beta_2 \left[\frac{\alpha_1 v + \beta_1 v'}{\sqrt{(\alpha_1^2 + \beta_1^2 q)}}\right]'\right] \cdot\right.\right.$$

$$\left.\left.\left.\cdot (\alpha_2^2 + \beta_2^2 Q_1)^{-1/2} + \ldots\right] (\alpha_{k-1}^2 + \beta_{k-1}^2 Q_{k-2})^{-1/2}\right]'\right] (\alpha_k^2 + \beta_k^2 Q_{k-1})^{-1/2}$$

is a solution of the differential equation (1.8).

Remark 1.2. If $r(x) \equiv 0$, $x \in J$, then $R_k(x) \equiv 0$ and the differential equation

$$y'' + Q_k(x)\,y = 0 \tag{1.11}$$

is the k-th accompanying equation for the differential equation (1.2) with regard to the basis $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$.

2. A MODIFICATION OF STURM'S THEOREM ON SEPARATING ZEROS OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF THE 2ND ORDER

O. Borůvka in [1] introduced the n-th central dispersions corresponding to the differential equation (1.2).

Throughout this section we suppose that for $n = 0, \pm 1, \ldots$; $k = 1, 2, \ldots$,

$$\varphi_{k,n}\,,\ \psi_{k,n}$$

are the n-th central dispersions of the first and second kinds corresponding to the k-th accompanying equation for the diffrential equation (1.2) with regard to the basis $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$.

Lemma 2.1. *Let* (1.11) *be the k-th accompanying equation for the differential equation* (1.2) *with regard to the basis* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$. *Then the differential equation* (1.11) *is oscillatory if and only if the differential equation* (1.2) *is oscillatory.*

The proof is quite similar to the proof of Theorem 1.2 in [2].

Theorem 2.1. *Let* $x \in J$ *and let* v_0 *be a real number. Let* $k \geqq 1$ *be an integer and let* $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ *be real numbers such that* $\alpha_j^2 + \beta_j^2 > 0$, $j = 1, 2, \ldots, k$. *Let*

$$q(x) \in C_{2k}(J)\,, \quad r(x) \in C_k(J)\,,$$

$$Q_j(x) > 0\,, \quad j = 0, 1, \ldots, k-1\,, \quad for \quad x \in J\,,$$

$$Q_0(x) \equiv q(x) \quad for \quad x \in J\,.$$

Let $v_1(x), v_2(x)$ *be arbitrary particular solutions of* (1.1), *let* $V_{k,1}(x)$, $V_{k,2}(x)$ *be functions defined by* (1.10_{v_1}), (1.10_{v_2}), *respectively. If*

$$V_{k,1}(x_0) = V_{k,2}(x_0) = v_0\,,$$

then we have

$$V_{k,1}[\varphi_{k,n}(x_0)] = V_{k,2}[\varphi_{k,n}(x_0)]$$

and $V_{k,1}(x) \neq V_{k,2}(x)$ *for* $x \in J$, $x \neq \varphi_{k,n}(x_0)$ *for every* $n = 0, \pm 1, \ldots$.

Proof. Let $v_1(x)$, $v_2(x)$ be solutions of the differential equation (1.1). It follows from (1.10_{v_1}), (1.10_{v_2}) that the functions $V_{k,1}(x)$, $V_{k,2}(x)$ are solutions of the equation (1.8).

Put $U_k(x) = V_{k,2}(x) - V_{k,1}(x)$, $x \in J$. Then by Lemma 3 in [4], $U_k(x)$ is a solution of (1.11) and by Lemma 2.1, the differential equation (1.11) is oscillatory.

At the point x_0, by hypotheses, we have

$$U_k(x_0) = V_{k,2}(x_0) - V_{k,1}(x_0) = 0\,.$$

According to Lemma 1 in [4] we have

$$U_k[\varphi_{k,n}(x_0)] = 0$$

and $U_k(x) \neq 0$ for $x \in J$, $x \neq \varphi_{k,n}(x_0)$ for every $n = 0, \pm 1, \ldots$. This implies

$$0 = U_k[\varphi_{k,n}(x_0)] = V_{k,2}[\varphi_{k,n}(x_0)] - V_{k,1}[\varphi_{k,n}(x_0)]$$

and

$$0 \neq U_k(x) = V_{k,2}(x) - V_{k,1}(x)$$

for $x \in J$, $x \neq \varphi_{k,n}(x_0)$. The theorem is proved.

Theorem 2.2. *Let $x_0 \in J$ and let v_0' be a real number. Let $k \geqq 1$ be an integer and let $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ be real numbers such that $\alpha_j^2 + \beta_j^2 > 0$, $j = 1, 2, \ldots, k$. Let*

$$q(x) \in C_{2k}(J)\,, \quad r(x) \in C_k(J)\,,$$

$$Q_j(x) > 0\,, \quad j = 0, 1, \ldots, k-1\,, \quad \text{for} \quad x \in J\,,$$

$$Q_0(x) \equiv q(x) \quad \text{for} \quad x \in J\,.$$

Let $v_1(x)$, $v_2(x)$ be arbitrary particular solutions of (1.1), let $V_{k,1}(x)$, $V_{k,2}(x)$ be functions defined by (1.10_{v_1}), (1.10_{v_2}), respectively. If

$$V_{k,1}'(x_0) = V_{k,2}'(x_0) = v_0'\,,$$

then we have

$$V_{k,1}'[\psi_{k,n}(x_0)] = V_{k,2}'[\psi_{k,n}(x_0)]$$

and $V_{k,1}'(x) \neq V_{k,2}'(x)$ for $x \in J$, $x \neq \psi_{k,n}(x_0)$ for every $n = 0, \pm 1, \ldots$.

Proof. By hypotheses, the function $U_k(x) = V_{k,2}(x) - V_{k,1}(x)$ for $x \in J$ is a solution of (1.11) and by Lemma 2.1, the differential equation (1.11) is oscillatory.

At the point x_0 we have

$$U_k'(x_0) = V_{k,2}'(x_0) - V_{k,1}'(x_0) = 0\,.$$

According to Lemma 1 in [4] we have

$$U_k'[\psi_{k,n}(x_0)] = 0$$

and $U_k'(x) \neq 0$ for $x \in J$, $x \neq \psi_{k,n}(x_0)$ for every $n = 0, \pm 1, \ldots$. Therefore

$$0 = U_k'[\psi_{k,n}(x_0)] = V_{k,2}'[\psi_{k,n}(x_0)] - V_{k,1}'[\psi_{k,n}(x_0)]$$

and

$$0 \neq U_k'(x) = V_{k,2}'(x) - V_{k,1}'(x)$$

for every $x \in J$, $x = \psi_{k,n}(x_0)$. This completes the proof.

Let $x_0 \in J$ and let v_0, v_0' be real numbers. Let $k \geqq 1$ be an integer and let $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ be real numbers such that $\alpha_j^2 + \beta_j^2 > 0$, $j = 1, 2, \ldots, k$. Let

$$q(x) \in C_{2k}(J)\,, \quad r(x) \in C_k(J)\,,$$
$$Q_j(x) > 0\,, \quad j = 0, 1, \ldots, k-1\,, \quad \text{for} \quad x \in J\,,$$
$$Q_0(x) \equiv q(x) \quad \text{for} \quad x \in J\,.$$

Let $v(x)$ be an arbitrary particular solution of (1.1), let $V_k(x)$ be the function defined by (1.10_v). Let $V_k(x_0) = v_0$ or $V_k'(x_0) = v_0'$.

Definition 2.1. The set of all points $\{\varphi_{k,n}(x_0), V_k[\varphi_{k,n}(x_0)]\}$ for $n = 0, \pm 1, \ldots$ will be called *the system of knots of the* $(2k+1)$*st kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v_0) *and to the basis* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots$ $\ldots, \beta_k)$. It will be denoted by

$$S_{2k+1}(x_0, v_0, r) = S_{2k+1}(x_0, v_0, r, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)\,.$$

Remark 2.1. Let $S_1(x_0, v_0, R_k) = S_1(x_0, v_0, R_k(\alpha_1, \ldots, \alpha_k, \ \beta_1, \ldots, \beta_k))$ be the system of knots of the 1st kind corresponding to the differential equation (1.8) and to the condition (x_0, v_0). Then

$$S_1(x_0, v_0, R_k) = S_{2k+1}(x_0, v_0, r)\,.$$

Definition 2.2. By *the bundle of solutions of the* $(2k+1)$*st kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v_0) *and to the basis* $(\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k)$ we mean all solutions $v(x)$ of (1.1) satisfying the condition

$$V_k(x_0) = v_0\,.$$

It will be denoted by

$$T_{2k+1}(x_0, v_0, r) = T_{2k+1}(x_0, v_0, r, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)\,.$$

Definition 2.3. The set of all points $\{\psi_{k,n}(x_0), V_k'[\psi_{k,n}(x_0)]\}$ for $n = 0, \pm 1, \ldots$ will be called *the system of knots of the* $(2k+2)$*nd kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v_0') *and to the basis* $(\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k)$. It will be denoted by

$$S_{2k+2}(x_0, v_0', r) = S_{2k+2}(x_0, v_0', r, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)\,.$$

Remark 2.2. Let $S_2(x_0, v_0', R_k) = S_2(x_0, v_0', R_k(\alpha_1, \ldots, \alpha_k, \ \beta_1, \ldots, \beta_k))$, be the system of knots of the 2nd kind corresponding to the differential equation (1.8) and to the condition (x_0, v'_0). Then

$$S_2(x_0, v'_0, R_k) = S_{2k+2}(x_0, v'_0, r).$$

Definition 2.4. By *the bundle of solutions of the* $(2k + 2)$*nd kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v'_0) *and to the basis* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$ we mean all solutions $v(x)$ of (1.1) satisfying the condition

$$V'_k(x_0) = v'_0.$$

It will be denoted by

$$T_{2k+2}(x_0, v'_0, r) = T_{2k+2}(x_0, v'_0, r, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k).$$

Let $S_{2k+1}(x_0, v_0, r)$ and $S_{2k+2}(x_0, v'_0, r)$ be the systems of knots of the $(2k + 1)$st and $(2k + 2)$nd kinds corresponding to the differential equation (1.1), to the initial conditions (x_0, v_0), (x_0, v'_0), respectively, and to the basis $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$.

Let $x_1, x_2 \in J$. Let v_1, v_2, v'_1, v'_2 be real numbers such that

$$[x_1, v_1], [x_2, v_2] \in S_{2k+1}(x_0, v_0, r)$$

and

$$[x_1, v'_1], [x_2, v'_2] \in S_{2k+2}(x_0, v'_0, r).$$

Definition 2.5. The points $[x_1, v_1]$, $[x_2, v_2]$ will be called *the neighbouring knots of the* $(2k + 1)$*st kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v_0) *and to the basis* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$ if the numbers x_1 and x_2 are the neighbouring numbers of the 1st kind corresponding to the differential equation (1.2).

Definition 2.6. The points $[x_1, v'_1]$, $[x_2, v'_2]$ will be called *the neighbouring knots of the* $(2k + 2)$*nd kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v'_0) *and to the basis* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$ if the numbers x_1 and x_2 are the neighbouring numbers of the 2nd kind corresponding to the differential equation (1.2).

Theorem 2.3. *Let* $S_{2k+1}(x_0, v_0, r)$ *be the system of knots of the* $(2k + 1)$*st kind corresponding to the differential equation* (1.1), *to the condition* (x_0, v_0) *and to the basis* $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$. *Let* $x_1, x_2 \in J$, $x_1 < x_2$. *Let* v_1, v_2 *be real numbers such that the points* $[x_1, v_1]$, $[x_2, v_2]$ *are two neighbouring knots of the* $(2k + 1)$*st kind from the system* $S_{2k+1}(x_0, v_0, r)$. *Let* $v(x)$ *be a solution of* (1.1) *such that*

$$V_k(x_0) = v_0,$$

where $V_k(x)$ *is defined by* (1.10_v). *If* $\bar{v}(x)$ *is a solution of* (1.1) *such that the function* $\bar{V}_k(x)$ *defined by* $(1.10_{\bar{v}})$ *is not passing through these knots, then there exists precisely one number* τ *in the interval* (x_1, x_2) *such that*

$$[\tau, V_k(\tau)] = [\tau, \bar{V}_k(\tau)].$$

Proof. By Remark 1, we have

$$S_{2k+1}(x_0, v_0, r) = S_1(x_0, v_0, R_k)$$

and $[x_1, v_1]$, $[x_2, v_2]$ are two neighbouring knots of the 1st kind from the system $S_1(x_0, v_0, R_k)$.

By hypotheses, the function $V_k(x)$ is the solution of the equation (1.8) for which $V_k(x_0) = v_0$ and the function $\bar{V}_k(x)$ is the solution of the equation (1.8) not passing through these knots.

It is obvious that the conditions of Theorem 3 in [4] are fulfilled. Consequently, there exists exactly one number τ in the interval (x_1, x_2) such that

$$[\tau, V_k(\tau)] = [\tau, \bar{V}_k(\tau)]$$

and the theorem is proved.

Theorem 2.4. *Let $S_{2k+2}(x_0, v_0', r)$ be the system of knots of the $(2k + 2)$nd kind corresponding to the differential equation* (1.1), *to the condition (x_0, v_0') and to the basis $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$. Let $x_1, x_2 \in J$, $x_1 < x_2$. Let v_1', v_2' be real numbers such that the points $[x_1, v_1']$, $[x_2, v_2']$ are two neighbouring knots of the $(2k + 2)$nd kind from the system $S_{2k+2}(x_0, v_0', r)$. Let $v(x)$ be a solution of* (1.1) *for which*

$$V_k'(x_0) = v_0',$$

where $V_k(x)$ is defined by (1.10_v). If $\bar{v}(x)$ is a solution of (1.1) *such that the function $\bar{V}_k'(x)$, where $\bar{V}_k(x)$ is defined by $(1.10_{\bar{v}})$, does not pass through these knots, then there exists exactly one number τ in the interval (x_1, x_2) such that*

$$[\tau, V_k'(\tau)] = [\tau, \bar{V}_k'(\tau)].$$

The proof is quite similar to the proof of Theorem 2.3.

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Author's address: 010 88 Žilina, Marxa-Engelsa 25 (Katedra matematiky VŠDS).