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ON MAWHIN'S APPROACH TO MULTIPLE NONABSOLUTELY CONVERGENT INTEGRAL

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J. Mawhin in [1] modified the Riemann-type definition of Perron integral in $\mathbb{R}^n$ by introducing a measure of "irregularity" $\Sigma(\Pi)$ of a partition Π of an n-dimensional interval. The main purpose of this generalization of Perron integral was to obtain the divergence theorem for differentiable vector fields or, in other words, to be able to integrate all derivatives of differentiable functions. Studying the properties of the generalized Perron integral Mawhin pointed out the fact that, unlike the usual Perron integral, the generalized one does not seem to have the additivity property (with respect to the domain of integration): If an n-dimensional interval I is partitioned into intervals I^1, I^2 and if f is generalized Perron integrable over I^i, $i = 1, 2$, then no proof is available of f being generalized Perron integrable over I.

In this paper we first give an example that the generalized Perron integral indeed is not additive in the above sense, and then modify Mawhin's definition, thus obtaining the additivity property mentioned above for functions integrable in our sense (Sec. 2, 3). At the same time, our definition will preserve the good properties of Mawhin's integral, namely, the divergence theorem will hold for all differentiable functions (cf. Sec. 4). In Sec. 5 we give a counterexample to the Fubini theorem for the integrals from Sec. 1 – 3. Sec. 6 contains some general convergence theorems and also the Lebesgue type dominated convergence theorem for the modified integral. Sec. 7 provides a general scheme applicable to all the definitions of integrals introduced in the paper.

1. DEFINITIONS AND A COUNTEREXAMPLE

Let us recall the definitions of Perron and Mawhin's generalized Perron integrals. All intervals $I \subset \mathbb{R}^n$ are assumed to be compact, i.e. $I = [a, b]$, $a, b \in \mathbb{R}^n$, is the Cartesian product of compact intervals $[a_i, b_i] \subset \mathbb{R}$ with $a_i < b_i$, $i = 1, \ldots, n$.

A *P-partition* of the interval I is a finite family

$$\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\} \tag{1}$$

with $x^j \in I^j$, $j = 1, \ldots, m$, where $\{I^1, \ldots, I^m\}$ is a partition of I (consisting of non-overlapping compact intervals).

(Let us note that Mawhin in [1] used right-closed intervals, thus obtaining partitions consisting of disjoint intervals. However, this change does not essentially affect our considerations.)

A function $\delta : I \to \mathbb{R}^+ = (0, \infty)$ is called a *gauge* on I, and a P-partition Π is called *δ-fine* if

$$I^j \subset B(x^j; \delta(x^j)), \quad j = 1, \ldots, m,$$

where $B(c; r) = [c_1 - r, c_1 + r] \times \ldots \times [c_n - r, c_n + r]$.

Definition 1. ([1], Definition 8.) Let X be a Banach space. A function $f: I \to X$ is said to be *P-integrable* if there is $J \in X$ such that for every $\varepsilon > 0$ there is a gauge δ on I such that for every δ-fine P-partition Π of I the inequality

$$\|S(I, f, \Pi) - J\| \leqq \varepsilon \tag{2}$$

holds with $S(I, f, \Pi) = \sum_{j=1}^{m} f(x^j)\, m(I^j)$, where m denotes the n-dimensional Lebesgue measure.

We then write $J = (\mathrm{P}) \int_I f$ and call J the *P-integral of f over I.*

(For detailed accounts of the P-integral see e.g. [2], [3], [4].)

Before proceeding to Mawhin's definition of the generalized Perron integral, let us define the *rate of stretching* of the interval I as

$$\sigma(I) = [\max_i (b_i - a_i)]/[\min_i (b_i - a_i)],$$

$i = 1, \ldots, n$, and the *irregularity* of the partition Π as

$$\Sigma_0(\Pi) = [\max_j \sigma(I^j)]/\sigma(I),$$

$j = 1, \ldots, m$. (Mawhin [1] used Σ instead of Σ_0.)

Definition 2. ([1], Definition 9.) Let X be a Banach space. A function $f: I \to X$ is said to be *GP-integrable* if there is $J \in X$ such that for every $\varepsilon > 0$ and every $C > 0$ there is a gauge δ on I such that for every δ-fine P-partition Π of I with $\Sigma_0(\Pi) \leqq C$ the inequality (2) holds.

We then write $J = (\mathrm{GP}) \int_I f$ and call J the *GP-integral of f over I.*

Remark 1. Notice that δ-fine P-partitions Π with $\Sigma_0(\Pi) \leqq C$ exist for $C \geqq 1$. This can be proved as follows: If there exists such a $t \in I$ that $I \subset B(t, \delta(t))$, then $\Pi = \{(t, I)\}$ is the desired P-partition. Otherwise replace I by intervals I_j, $j = 1, 2, \ldots, 2^n$, which are obtained by cutting I by hyperplanes orthogonal to coordinate axes and passing through the center of I. Let $\mathscr{J}$ be the set of such $j \in \{1, 2, \ldots, 2^n\}$ that there exists a $t \in I_j$ that $I_j \subset B(t, \delta(t))$. For $j \in \mathscr{J}$ choose one of the above points t, denote it by t_j and make (t_j, I_j) an element of Π ; for $j \notin \mathscr{J}$ divide I_j in a similar way etc. As $\delta(t) > 0$ for $t \in I$, after a finite number of steps the desired P-partition Π is obtained. Thus our definition makes good sense (cf. Assumption in Sec.7).

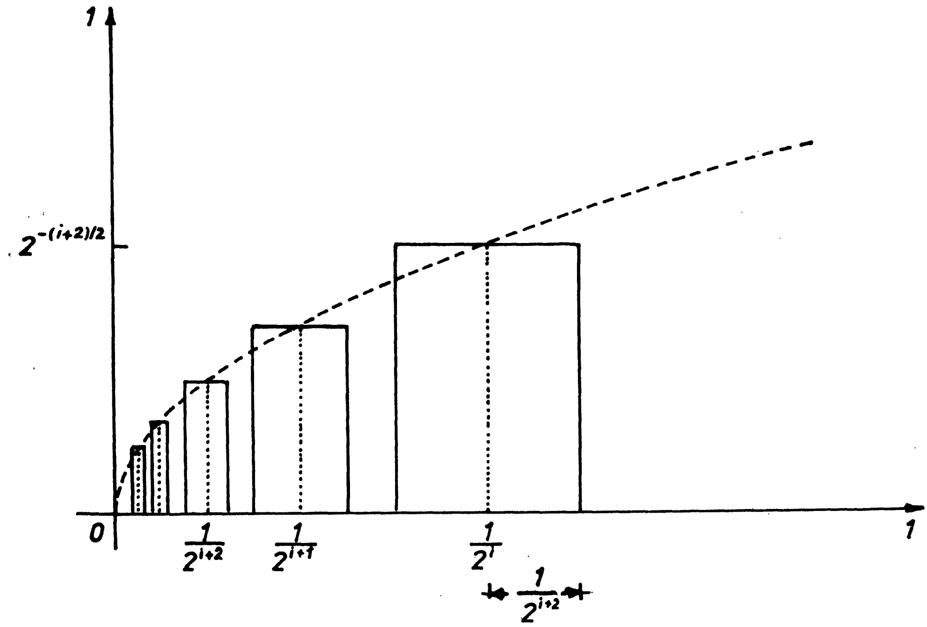

Fig. 1

Example 1. We shall construct a function that is GP-integrable but not P-integrable over a given (twodimensional) interval. (See Fig. 1.)

Let $Q_+ = [0,1] \times [0,1] \subset \mathbb{R}^2$, denote

$$R_i^- = (2^{-i} - 2^{-(i+2)}, 2^{-i}) \times (0, 2^{-(i+2)/2}),$$

$$R_i^+ = (2^{-i}, 2^{-i} + 2^{-(i+2)}) \times (0, 2^{-(i+2)/2})$$

and define a function $f: Q_+ \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} -2^{3(i+2)/2} & \text{for } (x, y) \in R_i^- , \\ 2^{3(i+2)/2} & \text{for } (x, y) \in R_i^+ , \\ 0 & \text{otherwise} . \end{cases}$$

To prove that f is not P-integrable over Q_+ it suffices to recall two facts about the P-integral (cf. e.g. [2]): first, its additivity if the integration domain is partitioned into a finite number of intervals and, secondly, that the P-integral tends to zero if the integration domain contracts into a single point. Thus, if we set

$$I^0 = [0, r] \times [0, s]\,; \quad Q' = [0, r] \times [s, 1]\,;$$
$$Q'' = [r, 1] \times [0, 1]\,,$$

then under the assumption of P-integrability of f over Q_+ we should have

$$\int_{Q_+} f = \int_{I^0} f + \int_{Q'} f + \int_{Q''} f\,. \tag{3}$$

However, choosing $s = 2^{-(i+2)/2}$ and either $r = 2^{-i}$ or $r = 2^{-i} + 2^{-(i+2)}$ we obtain $\int_{Q'} f = 0$, $\int_{Q''} f = 1$ or $\int_{Q''} f = 0$, respectively, which together with the relation $\lim_{i \to \infty} \int_{I^0} f = 0$ contradicts the identity (3).

Let us now proceed to the proof that f is GP-integrable and (GP) $\int_{Q_+} f = 0$. Let $\varepsilon > 0$, $C > 0$. Let us choose a gauge ω: $Q_+ \to \mathbb{R}^+$ so that it satisfies

$$\omega(x, y) \leqq \tfrac{1}{2} \operatorname{dist}\,[(x, y);\ \bigcup_i\,(\partial R_i^- \cup \partial R_i^+)]$$

for $(x, y) \in Q_+ \setminus \bigcup_i\,(\partial R_i^- \cup \partial R_i^+)$, $(x, y) \neq (0, 0)$;

$$\omega(x, y) \leqq \varepsilon\, 2^{-2(i+2)} \quad \text{for} \quad (x, y) \in (\partial R_i^- \cup \partial R_i^+)\,,$$

$i = 1, 2, \ldots,$

$$\omega(0, 0) = \gamma = \text{const} > 0 \quad (\text{to be fixed later})\,.$$

If Π is an ω-fine partition of Q_+, then it obviously includes a pair $((0, 0), I^0)$. Assume $I^0 = [0, r] \times [0, s]$. It is clear that the "worst" case (i.e. the case when $S(Q_+, f, \Pi)$ differs from zero as much as possible) occurs if $r = 2^{-j}$, $s = Cr$. Then the remainder that does not vanish is

$$2^{-(j+2)}C\,.\,2^{-j}\,.\,2^{3(j+2)/2} = 2^{1-j/2}C\,.$$

It is evident that by taking γ sufficiently small (the choice of γ obviously depends on both ε and C) we can make this value smaller than, say, $\frac{1}{2}\varepsilon$.

Now all the other intervals of the partition Π split into three groups: those lying inside of either R_i^+ or R_i^-; those lying outside of all the rectangles R_i^+, R_i^-; and those intersecting the boundary of some of the rectangles. The contribution to the sum $S(Q_+, f, \Pi)$ corresponding to the first group of intervals is small because the individual terms for R_i^+ and R_i^- "almost" cancel each other; the sum corresponding to the second group vanishes since $f(x, y) = 0$ outside the rectangles; and the third group of intervals again gives a very small contribution because of the properties of the gauge ω. This shows that

$$|S(Q_+, f, \Pi)| \leqq \varepsilon$$

for every ω-fine partition Π with $\Sigma_0(\Pi) \leqq C$. (A rigorous proof requires merely a greater amount of elementary calculations.) Hence

$$(\mathrm{GP})\int_{Q_+} f = 0\,.$$

Example 2. Let $Q_- = [-1, 0] \times [0, 1]$ and let us extend the function f from Example 1 to Q_- by defining

$$f(x, y) = 0 \quad \text{for} \quad (x, y) \in Q_- .$$

Then evidently $(\mathrm{GP}) \int_{Q_-} f = (\mathrm{P}) \int_{Q_-} f = 0$, $(\mathrm{GP}) \int_{Q_+} f = 0$ by Example 1, but $(\mathrm{GP}) \int_{Q_- \cup Q_+} f$ does not exist.

Indeed, the existence of $(\mathrm{GP}) \int_{Q_+} f$ followed from the fact that the rate of stretching of the intervals I^0 which would "spoil" the sum $S(Q_+, f, \Pi)$ was too big, so that the irregularity of the corresponding partition was greater than C. This fact excluded such "bad" partitions, thus guaranteeing the GP-integrability (over Q_+) of f.

However, now, when partitioning the whole interval $Q_- \cup Q_+$, we can modify the interval I^0 by extending it into Q_- in such a way that it becomes a square (which means $\sigma(I^0) = 1$) and at the same time remains ω-fine. Partitions including such intervals then give sums that are not near to zero, as was shown in Example 1 when P-integrability was considered. This shows that f is not GP-integrable over $Q_- \cup Q_+$.

2. MODIFIED DEFINITION: M_1-INTEGRAL

For a P-partition Π of an interval $I \subset \mathbb{R}^n$ let us introduce the modified irregularity as

$$\Sigma_1(\Pi) = \sum_{j=1}^{m} m_{n-1}(\partial I^j) \operatorname{diam}(I^j),$$

where ∂ denotes the boundary, m_{n-1} is the $(n-1)$-dimensional Lebesgue measure and diam stands for the diameter of a set.

Definition 3. A function $f: I \to X$ (X a Banach space) is said to be *M_1-integrable* if there is $J \in X$ such that for every $\varepsilon > 0$ and every constant $C > 0$ there is a gauge δ on I such that for every δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq C$ the inequality (2) holds.

We then write $J = (\mathrm{M}_1) \int_I f$ and call J the *M_1-integral* of f over I.

Lemma 1. *For every constant C there is a constant K such that any P-partition Π with $\Sigma_0(\Pi) \leqq C$ satisfies $\Sigma_1(\Pi) \leqq K$.*

Proof requires only elementary calculations.

Corollary. *If a function $f: I \to X$ is M_1-integrable, then it is GP-integrable and both integrals coincide.*

Lemma 2. *Let $C \geqq m_{n-1}(\partial I) \operatorname{diam}(I)$. Then for every gauge δ on I there exists a δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq C$.*

Such a P-partition Π can be obtained in the same way as in Remark 1.

The following theorem is a modification of a theorem holding for the GP-integral (cf. [1]) to the M_1-integral.

Theorem 1. *Let I, K, L be compact non-overlapping intervals in $\mathbb{R}^n$, $I = K \cup L$. Let f be M_1-integrable over I. Then f is M_1-integrable over both K and L and*

$$(M_1)\int_I f = (M_1)\int_K f + (M_1)\int_L f. \tag{4}$$

Moreover, if $C \geqq m_{n-1}(\partial I)\,\mathrm{diam}\,(I)$, $\varepsilon > 0$ and if δ is such a gauge on I that

$$\left\| S(I, f, \Pi) - (M_1)\int_I f \right\| \leqq \varepsilon$$

for every δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq 2C$, then

$$\left\| S(K, f, \Pi_1) - (M_1)\int_K f \right\| \leqq \varepsilon \tag{5}$$

for every δ-fine P-partition Π_1 of K with $\Sigma_1(\Pi_1) \leqq 2C$.

Proof. Let C, ε, δ be the same as in Theorem 1. Let Π_1, Π_2 be δ-fine P-partitions of K with $\Sigma_1(\Pi_1) \leqq C$, $\Sigma_1(\Pi_2) \leqq C$ and let Π_3 be a δ-fine P-partition of L with $\Sigma_1(\Pi_3) \leqq C$ (cf. Lemma 2). Then $\Pi_4 = \Pi_1 \cup \Pi_3$ and $\Pi_5 = \Pi_2 \cup \Pi_3$ are δ-fine P-partitions of I with $\Sigma_1(\Pi_4) \leqq 2C$, $\Sigma_1(\Pi_5) \leqq 2C$. We have

$$S(K, f, \Pi_1) - S(K, f, \Pi_2) = S(I, f, \Pi_4) - S(I, f, \Pi_5),$$

$$\|S(I, f, \Pi_4) - S(I, f, \Pi_5)\| \leqq 2\varepsilon,$$

so that

$$\|S(K, f, \Pi_1) - S(K, f, \Pi_2)\| \leqq 2\varepsilon,$$

which proves the existence of the integral $(M_1)\int_K f$. Analogously, $(M_1)\int_L f$ exists. The validity of (4) follows directly from Definition 3.

Let $\eta > 0$. Now we can assume in addition that

$$\left\| S(L, f, \Pi_3) - (M_1)\int_L f \right\| \leqq \eta$$

and we obtain by (4) that

$$\left\| S(K, f, \Pi_1) - (M_1)\int_K f \right\| =$$

$$= \left\| S(I, f, \Pi_4) - (M_1)\int_I f - S(L, f, \Pi_3) + (M_1)\int_L f \right\| \leqq$$

$$\leqq \left\| S(I, f, \Pi_4) - (\mathrm{M}_1)\int_I f \right\| + \left\| S(L, f, \Pi_3) - (\mathrm{M}_1)\int_L f \right\| \leqq$$

$$\leqq \varepsilon + \eta\,,$$

which proves (5).

Corollary. If I, H are compact intervals in $\mathbb{R}^n$, $H \subset I$, and if f is M_1-integrable over I, then f is M_1-integrable over H as well.

Theorem 2. *Let I, K, L be compact non-overlapping intervals in $\mathbb{R}^n$, $I = K \cup L$. Let a function $f: I \to X$ be M_1-integrable both over K and over L. Then f is M_1-integrable over I and (4) holds.*

Proof. Let $\varepsilon > 0$, $C > 0$. Find gauges δ_K, δ_L on K, L, respectively, "associated" with the constant $\frac{1}{2}\varepsilon$, C. Put

$$\delta(x) = \begin{cases} \min\,[\delta_K(x),\ \operatorname{dist}(x, L)] & \text{for } x \in K \setminus L\,, \\ \min\,[\delta_L(x),\ \operatorname{dist}(x, K)] & \text{for } x \in L \setminus K\,, \\ \min\,[\delta_K(x),\ \delta_L(x)] & \text{for } x \in K \cap L\,. \end{cases}$$

Let a P-partition Π of I be δ-fine and $\Sigma_1(\Pi) \leqq C$. Then

$$\Pi_K = \{(x^*, K^*);\ \emptyset \neq K^* = J^* \cap K, \text{ where } (x^*, J^*) \in \Pi\}$$

and analogously

$$\Pi_L = \{(x^*, L^*);\ \emptyset \neq L^* = J^* \cap L, \text{ where } (x^*, J^*) \in \Pi\}$$

are P-partitions of the intervals K, L, which are δ_K- and δ_L-fine, respectively. Moreover, $\Sigma_1(\Pi) \leqq C$ implies that $\Sigma_1(\Pi_K) \leqq C$, $\Sigma_1(\Pi_L) \leqq C$ since some of the summands of the sum defining $\Sigma_1(\Pi)$ vanish and some other may decrease when we pass to $\Sigma_1(\Pi_K)$, $\Sigma_1(\Pi_L)$, but none of them increase. Hence

$$\left| S(K, f, \Pi_K) - (\mathrm{M}_1)\int_K f \right| \leqq \tfrac{1}{2}\varepsilon\,,$$

$$\left| S(L, f, \Pi_L) - (\mathrm{M}_1)\int_L f \right| \leqq \tfrac{1}{2}\varepsilon\,,$$

which by the obvious identity

$$S(I, f, \Pi) = S(K, f, \Pi_K) + S(L, f, \Pi_L)$$

yields

$$\left| S(I, f, \Pi) - \left[(\mathrm{M}_1)\int_K f + (\mathrm{M}_1)\int_L f\right] \right| \leqq \varepsilon\,.$$

This completes the proof of (4) and hence of Theorem 2.

Remark 2. While Example 1 shows that the GP-integral generally fails to depend continuously on the domain of integration, the following result can be proved for the M_1-integral: Let

$$I = [a, b] = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n],$$
$$K_k = [a^{(k)}, b] = [a_1^{(k)}, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n],$$

where $a_1^{(k)} \to a_1$, $a_1^{(k)} \geqq a_1$. Let f be M_1-integrable over I. Then

$$\lim_{k\to\infty} (M_1) \int_{K_k} f = (M_1) \int_I f . \tag{6}$$

Proof. Denote $L_k = \mathrm{cl}(I \setminus K_k)$. Given $\varepsilon > 0$, $C \geqq m_{n-1}(\partial I) \operatorname{diam}(I)$, let δ be such a gauge on I that

$$\left\| S(I, f, \Pi) - (M_1) \int_I f \right\| \leqq \varepsilon$$

for every δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq 2C$. Put

$$G = [a_2, b_2] \times [a_3, b_3] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^{n-1}$$

and let $\Pi^* = \{(g_j, G_j);\ j = 1, 2, \ldots, m\}$ be such a P-partition of G that $G_j \subset B(g_j, \frac{1}{2}\delta((a_1, g_j))) \subset \mathbb{R}^{n-1}$, $j = 1, 2, \ldots, m$. There exists such an r that

$$a_1^{(k)} - a_1 < \min \{\tfrac{1}{2}\delta((a_1, g_j));\ j = 1, 2, \ldots, m\}$$

for $k \geqq r$, so that

$$H_j^{(k)} = [a_1, a_1^{(k)}] \times G_j \subset B((a_1, g_j), \delta((a_1, g_j))) \subset \mathbb{R}^n$$

for $j = 1, 2, \ldots, m$. Further,

$$\Pi_k = \{((a_1, g_j), H_j^{(k)});\ j = 1, 2, \ldots, m\}$$

is a δ-fine P-partition of L_k for $k \geqq r$. Evidently, since Π^* is independent of k, we have $\Sigma_1(\Pi_k) \leqq C$ for k sufficiently large and thus (4) and (5) from Theorem 1 yield

$$\left\| (M_1) \int_I f - (M_1) \int_{K_k} f \right\| = \left\| (M_1) \int_{L_k} f \right\| \leqq$$
$$\leqq \left\| (M_1) \int_{L_k} f - S(L_k, f, \Pi_k) \right\| + \| S(L_k, f, \Pi k) \| \leqq$$
$$\leqq \varepsilon + (a_1^{(k)} - a_1) \sum_{j=1}^{m} f((a_1, g_j))\, m_{n-1}(G_j) .$$

which implies that (6) holds.

3. ANOTHER MODIFICATION: M_2-INTEGRAL

A finite family (1), where $\{I^1, \ldots, I^m\}$ is a partition of I and $x^j \in I$, $j = 1, \ldots, m$, is called an *L-partition* of I.

(Notice that a P-partition is an L-partition satisfying the additional condition $x^j \in I^j$, $j = 1, \dots, m$.) For an L-partition Π of an interval $I \subset \mathbb{R}^n$ let us introduce another measure of irregularity as

$$\Sigma_2(\Pi) = \sum_{j=1}^{m} m_{n-1}(\partial I^j)\, q_j ,$$

where $q_j = \max\{\operatorname{dist}(x^j, x);\ x \in I^j\}$.

Definition 4. A function $f: I \to X$ (X a Banach space) is said to be *M_2-integrable* if there is $J \in X$ such that for every $\varepsilon > 0$ and every constant $C > 0$ there is a gauge δ on I such that for every δ-fine L-partition Π of I with $\Sigma_2(\Pi) \leqq C$ the inequality (2) holds.

We then write $J = (\mathrm{M}_2) \int_I f$ and call J the M_2-integral of f over I.

Remark 3. It is almost evident that every M_2-integrable function is M_1-integrable (over the same interval). Moreover, for $n = 1$ the sets of P-, GP- and M_1-integrable functions coincide, while the set of M_2-integrable functions is contained (as a proper subset) in each of them. (Cf. [2]: for $n = 1$ a function is M_2-integrable if and only if it is Lebesgue integrable.)

Example 3. Let again $Q_+ = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and denote

$$R_i^- = (2^{-i} - 2^{-(i+2)}, 2^{-i}) \times (0, 2^{-(i+2)}),$$
$$R_i^+ = (2^{-i}, 2^{-i} + 2^{-(i+2)}) \times (0, 2^{-(i+2)}).$$

Define

$$f(x, y) = \begin{cases} -\alpha_i & \text{for } (x, y) \in R_i^- , \\ \alpha_i & \text{for } (x, y) \in R_i^+ , \\ 0 & \text{otherwise} , \end{cases}$$

where $\alpha_i > 0$, $2^{-4i}\alpha_i \to 0$, $\sum_{i=1}^{\infty} 2^{-4i}\alpha_i = \infty$.

Then arguments similar to those in Example 1 show that f is M_1-integrable but not M_2-integrable. Indeed, for every gauge δ we can find an L-partition Π_1 such that $|S(Q_+, f, \Pi_1)| < 1$ and another L-partition Π_2 with $S(Q_+, f, \Pi_2) > 2$. (The partition Π_2 is obtained by putting $(0, R_i^-) \in \Pi_2$ for $i = p + 1, \dots, p + q$, where p is such that $R_i^- \subset B(0, \delta(0))$ for $i > p$ and q is such that $\sum_{i=p+1}^{p+q} 2^{-4i}\alpha_i > 2$.) Moreover, we can at the same time satisfy the conditions $\Sigma_2(\Pi_1) \leqq C$, $\Sigma_2(\Pi_2) \leqq C$ with C independent of δ, Π_1, Π_2.

It also can be proved that f is P-integrable over Q_+.

Remarks. 4. Denoting by **Int (P)**, **Int (GP)**, **Int (M$_1$)** and **Int (M$_2$)** the families of functions integrable in the respective sense, we thus have the following inclusions (for $n > 1$):

$$\mathbf{Int}\,(\mathrm{GP}) \supsetneqq \mathbf{Int}\,(\mathrm{M}_1) \supsetneqq \mathbf{Int}\,(\mathrm{M}_2), \quad \mathbf{Int}\,(\mathrm{M}_1) \supset \mathbf{Int}\,(\mathrm{P}),$$

$$\mathbf{Int}(M_2) \not\supset \mathbf{Int}(P).$$

5. Theorem 1 holds with M_1 replaced by M_2 since, when splitting a partition Π with $\Sigma_2(\Pi) \leqq C$ into partitions Π_K, Π_L as in the proof of Theorem 1, we conclude by the same argument that $\Sigma_2(\Pi_K) \leqq C$, $\Sigma_2(\Pi_L) \leqq C$ and the whole proof works in the case of the M_2-integral.

We conclude the present section by mentioning some elementary facts on the M_1- and M_2-integral that will be used without special reference in the sequel, especially in Sec. 6. We formulate them for the M_1-integral only.

Remarks. 6. If $N \subset I$ with $m(N) = 0$ and $f: I \to X$ satisfies $f(x) = 0$ for $x \in I \setminus N$ then f is M_1-integrable over I and $(M_1) \int_I f = 0$. This follows from the fact that such a function f is Lebesgue integrable and $0 = (L) \int_I f = (M_1) \int_I f$.

7. If $h: I \to R$ is M_1-integrable and satisfies $h(x) \geqq 0$ for all $x \in I$ then $(M_1) \int_I h \geqq \geqq 0$. Indeed, the converse inequality would contradict the fact that $S(I, f, \Pi) \geqq 0$ for every P-partition Π of I. Consequently, if $f, g: I \to R$ are M_1-integrable over I and $f(x) \leqq g(x)$ for all $x \in I$, then $(M_1) \int_I f \leqq (M_1) \int_I g$.

4. EVERY DERIVATIVE IS BOTH M_1- AND M_2-INTEGRABLE

Mawhin's Theorem 1 [1] (the divergence theorem for differentiable functions) holds for the M_1- and M_2-integral as well, the proof being a mere verbatim transcription of Mawhin's proof. Let us therefore present a closely related theorem, the contents of which is expressed by the headline of the present section.

Theorem 3. *Let $I = [a, b] \subset R^n$ be an interval, Ω a domain such that $I \subset \Omega \subset \subset R^n$. Let a function $f: R^n \to R$ be differentiable on Ω. Then $\partial f / \partial x_1$ is both M_1- and M_2-integrable over I and*

$$\text{(7)} \quad (M) \int_I \frac{\partial f}{\partial x_1} = \int_{a_2}^{b_2} \dots \int_{a_n}^{b_n} [f(b_1, \xi_2, \dots, \xi_n) - f(a_1, \xi_2, \dots, \xi_n)] \, d\xi_2 \dots d\xi_n ,$$

where (M) stands either for (M_1) or (M_2).

Proof. For any interval $L = [c_1, d_1] \times \dots \times [c_n, d_n]$ denote

$$\Phi(L, f) = \int_{c_2}^{d_2} \dots \int_{c_n}^{d_n} [f(d_1, \xi_2, \dots, \xi_n) - f(c_1, \xi_2, \dots, \xi_n) \, d\xi_2 \dots d\xi_n .$$

(Thus the right-hand side of (7) is denoted by $\Phi(I, f)$.)

We shall need the following auxiliary result: If $\{I^1, \dots, I^m\}$ is a partition of I, then

$$\text{(8)} \qquad \Phi(I, f) = \sum_{j=1}^{m} \Phi(I^j, f) .$$

An elementary rigorous proof of this identity is rather lengthy; nonetheless, let us present at least its main points. First of all, the identity (8) holds if the partition is "net-like", that is, if there are finite sequences

$$a_i = c_i^1 < c_i^2 < \ldots < c_i^{m_i} = b_i\,, \quad i = 1, \ldots, n\,,$$

such that the partition consists of all intervals

$$[c_1^{j_1}, c_1^{j_1+1}] \times \ldots \times [c_n^{j_n}, c_n^{j_n+1}]\,, \quad j_i = 1, \ldots, m_i\,.$$

Let us show this at least for $n = 2$ to avoid too complicated indices. Let $I = \bigcup_{i,j} K^{ij}$, where

$$K^{ij} = [c_i, c_{i+1}] \times [d_i, d_{i+1}]\,,$$

$a_1 = c_1 < c_2 < \ldots < c_p = b_1$, $a_2 = d_1 < d_2 < \ldots < d_q = b_2$. Then

$$\sum_{i,j} \Phi(K^{ij}, f) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \int_{d_j}^{d_{j+1}} [f(c_{i+1}, \xi) - f(c_i, \xi)]\, \mathrm{d}\xi =$$

$$= \sum_{i=1}^{p-1} \int_{d_1}^{d_q} [f(c_{i+1}, \xi) - f(c_i, \xi)]\, \mathrm{d}\xi = \sum_{i=2}^{p} \int_{a_2}^{b_2} f(c_i, \xi)\, \mathrm{d}\xi - \sum_{i=1}^{p-1} \int_{a_2}^{b_2} f(c_i, \xi)\, \mathrm{d}\xi =$$

$$= \int_{a_2}^{b_2} [f(c_p, \xi) - f(c_1, \xi)\, \mathrm{d}\xi = \int_{a_2}^{b_2} [f(b_1, \xi) - f(a_1, \xi)]\, \mathrm{d}\xi = \Phi(I, f)\,.$$

For $n > 2$, the proof is analogous.

Now, if $\{I^1, \ldots, I^m\}$ is an arbitrary partition of I, it is easy to construct a "net-like" partition Λ of I such that its "restriction" to any I^j, $j = 1, \ldots, m$, again represents a "net-like" partition of I^j (this is achieved by arranging the i-th coordinates ($i = 1, \ldots, n$) of all intervals $I^1, \ldots, I^m$ in increasing sequences and taking all intervals whose end-points have these coordinates). Thus, if we write

$$I^j = \bigcup_{k=1}^{k_j} L_k^j\,, \quad L_k^j \in \Lambda\,,$$

then

$$I = \bigcup_{j=1}^{m} \bigcup_{k=1}^{k_j} L_k^j$$

and, using (8) (for "net-like" partitions!) once for I and once for I^j, we immediately obtain

$$\Phi(I, f) = \sum_{L \in \Lambda} \Phi(L, f) = \sum_{j=1}^{m} \sum_{k=1}^{k_j} \Phi(L_k^j, f) = \sum_{j=1}^{m} \Phi(I^j, f)\,,$$

that is, (8) holds for any partition.

Now it is not difficult to complete the proof of Theorem 3. Let $\varepsilon > 0$, $C > 0$, $x \in I$ and denote $\partial f / \partial x_k = f_k'$. Then there exists $\delta = \delta(x) > 0$ such that for every $y \in B(x; \delta(x))$ we have

$$\left|f(y) - f(x) - \sum_{i=1}^{n} f_i'(x)(y_i - x_i)\right| \leqq \varepsilon_1 \|y - x\| .$$

Thus $\delta: I \to \mathbb{R}^+$ can be viewed as a gauge on I. Let a δ-fine P-partition Π be defined by (1) and set

$$g^j(y) = f(x^j) + \sum_{i=1}^{n} f_i'(x^j)(y_i - x_i^j), \quad h^j(y) = f(y) - g^j(y).$$

Then we easily find that $\Phi(I^j, g^j) = f_1'(x^j)\, m(I^j)$ and we can estimate

$$\left|S(I, f_1', \Pi) - \Phi(I, f)\right| =$$

$$= \left|\sum_{j=1}^{m} [f_1'(x^j)\, m(I^j) - \Phi(I^j, g^j) - \Phi(I^j, h^j)]\right| = \left|\sum_{j=1}^{m} \Phi(I^j, h^j)\right| =$$

$$= \left|\sum_{j=1}^{m} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} [h^j(b_1, \xi_2, \ldots, \xi_n) - h^j(a_1, \xi_2, \ldots, \xi_n)]\, d\xi_2 \ldots d\xi_n\right| \leqq$$

$$\leqq 2\varepsilon_1 \sum_{j=1}^{m} \operatorname{diam}(I^j) \frac{m(I^j)}{b_1^j - a_1^j} \leqq 2\varepsilon_1\, \Sigma_1(\Pi)$$

provided the sum $S(I, f_1', \Pi)$ corresponds to the M_1-integral; hence choosing $\varepsilon_1 = \frac{1}{2}\varepsilon C^{-1}$ and assuming $\Sigma_1(\Pi) \leqq C$ we obtain $2\varepsilon_1\, \Sigma_1(\Pi) \leqq \varepsilon$. Similarly, considering the M_2-integral we obtain

$$\left|S(I, f_1', \Pi) - \Phi(I, f)\right| \leqq 2\varepsilon_1 \sum_{j=1}^{m} q_j \frac{m(I^j)}{b_1^j - a_1^j} \leqq 2\varepsilon_1\, \Sigma_2(\Pi),$$

which yields the same estimate as above for δ-fine L-partitions with $\Sigma_2(\Pi) \leqq C$.

5. A COUNTEREXAMPLE TO THE FUBINI THEOREM

In the next example we shall construct a differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$ such that its partial derivative $g \equiv \partial f/\partial y$ is not P-integrable in x, i.e., (P) $\int_0^1 g(\xi, y)\, d\xi$ does not exist for almost all y (cf. Remark 2). This fact disproves the Fubini theorem for the GP-, M_1- and M_2-integral, since by the results of Sec. 4 the function g, being a derivative of a differentiable function, is integrable in each of the above senses (but not P-integrable).

Example 4. We shall construct the function g on the square $Q_+ = [0, 1] \times [0, 1]$ and put

$$f(x, y) = \begin{cases} \displaystyle\int_0^y g(x, \eta)\, d\eta & \text{for } (x, y) \in Q_+, \\ 0 & \text{otherwise}. \end{cases} \tag{9}$$

Let us first construct an auxiliary function $\varphi: [0, 1] \to \mathbb{R}$, $\varphi \in C^\infty(0, 1)$. (See Fig. 2.)

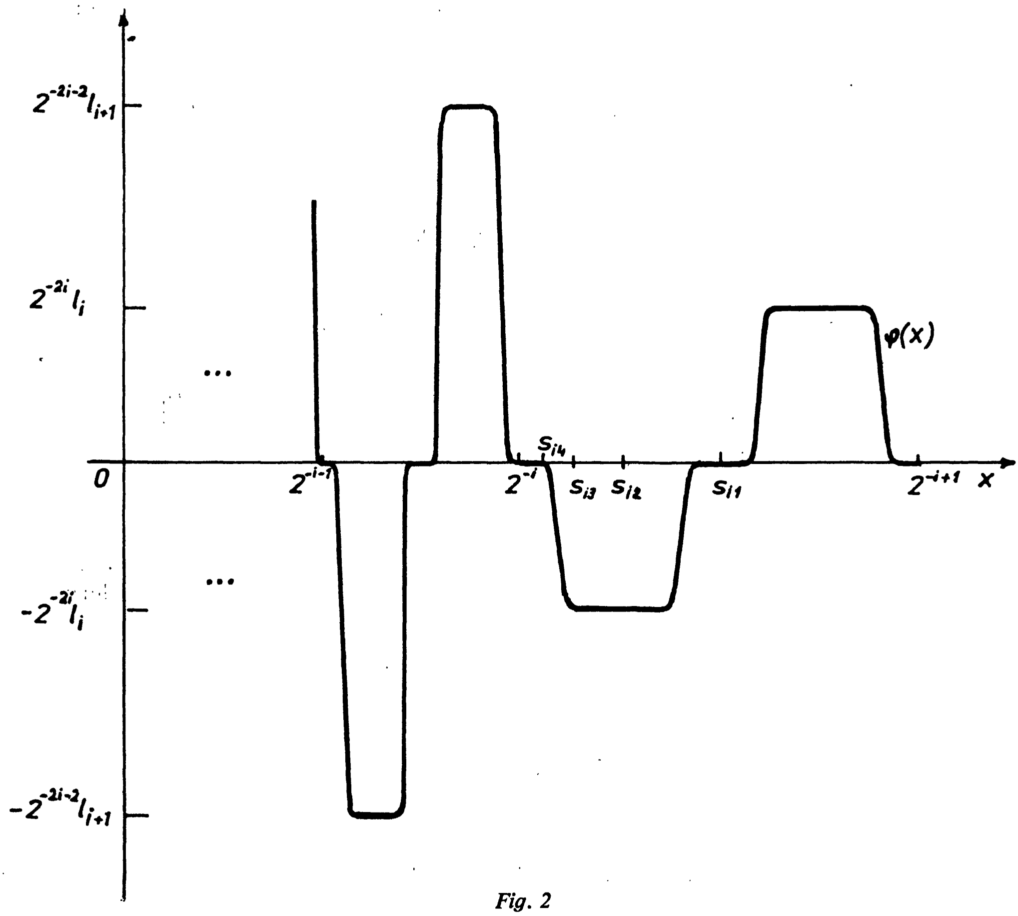

Fig. 2

Denote $V_i = (2^{-i}, 2^{-i+1}]$, $i = 1, 2, \ldots$ and let $s_{i1} = \frac{1}{2}(2^{-i} + 2^{-i+1})$ be the center of the segment V_i, $s_{i,k+1} = \frac{1}{2}(2^{-i} + s_{ik})$ for $k = 1, 2, 3$. We set

$\varphi(x) = 0$ for $x \in (2^{-i}, s_{i4}]$ and for $x = 0$,

φ decreasing in $[s_{i4}, s_{i3}]$;

$\varphi(x) = -2^{-2i}l_i$ for $x \in [s_{i3}, s_{i2}]$;

$\varphi(s_{i2} + \xi) = \varphi(s_{i2} - \xi)$ for $\xi \in [0, 2^{-i-2}]$;

$\varphi(s_{i1} + \xi) = -\varphi(s_{i1} - \xi)$ for $\xi \in [0, 2^{-i-1}]$, $i = 1, 2, \ldots$.

It is easy to establish the estimates

$$\int_{s_{i1}}^{1} \varphi(x)\,dx \geq 2^{-i-2} \cdot 2^{-2i}l_i = 2^{-3i-2}l_i,$$

$$\int_{2^{-i}}^{1} \varphi(x)\,dx = 0\,.$$

Now put

$$g(x, y) = \varphi(x)\,[\sin \pi l_i y]^{1/i} \quad \text{for} \quad (x, y) \in V_i \times [0, 1]\,,$$

defining $[\alpha]^\beta = |\alpha|^\beta \operatorname{sign} \alpha$. Then

$$\text{(10)} \qquad \int_{2^{-i}}^{1} g(\xi, y)\,d\xi = 0\,,$$

$$\left|\int_{s_{i1}}^{1} g(\xi, y)\,d\xi\right| \geqq 2^{-3i-2} l_i |\sin \pi l_i y|^{1/i}\,.$$

Denote $A_i = \{y \in (0, 1];\ |\sin \pi l_i y| \geqq 2^{-i}\}$, $i = 1, 2, \ldots$. Then there is a constant c such that

$$m(A_i) \geqq 1 - c\,2^{-i}\,.$$

Put $A = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$. By a standard argument we obviously have $m(A) \geqq 1 - c\,2^{-j}$ for every $j = 1, 2, \ldots$, hence $m(A) = 1$. Thus for a.e. $y \in (0, 1]$ we have

$$\left|\int_{s_{i1}}^{1} g(\xi, y)\,d\xi\right| \geqq 2^{-3i-2} l_i [2^{-i}]^{1/i} = 2^{-3(i+1)} l_i\,.$$

Choosing l_i suitably and combining this estimate with the identity (10), we immediately conclude that the P-integral $\int_0^1 g(\xi, y)\,d\xi$ does not exist for a.e. $y \in [0, 1]$. (By Remark 3, this means that the GP-, M_1- and M_2-integrals do not exist, either.)

It remains to prove that the function f defined by (9) is differentiable in $\mathbb{R}^2$. It is evident that it is only necessary to prove differentiability at the points $(0, y)$, $y \in [0, 1]$. However, we easily obtain the estimate

$$|f(x, y)| \leqq |\varphi(x)| \cdot \left|\int_0^y [\sin \pi l_i \eta]^{1/i}\,d\eta\right| \leqq$$

$$\leqq 2^{-2i} l_i \,.\, l_i^{-1} = 2^{-2i} \quad \text{for} \quad x \in V_i\,, \quad i = 1, 2, \ldots\,.$$

This estimate implies that $f(x, y) = o(x)$, which immediately yields differentiability of f at the points $(0, y)$.

Thus, Theorem 2 implies that g is GP-, M_1- and M_2-integrable over Q_+ and

$$\int_{Q_+} g = \int_0^1 \int_0^1 g(\xi, \eta)\,d\eta\,d\xi$$

(the left-hand side integral being one of the three just mentioned).

Remark 8. Since the Fubini theorem holds for the P-integral (see again [2]), our example enables us to amend Remark 4:

$$\mathrm{Int}(\mathrm{M}_1) \supsetneqq \mathrm{Int}(\mathrm{P}), \quad \mathrm{Int}(\mathrm{M}_2) \not\subset \mathrm{Int}(\mathrm{P})$$

(again for $n > 1$).

Example 5. We know (cf. Remark 2) that the function $H(y) = (\mathrm{M}_1) \int_{Q_y} h$, where $Q_y = [0, 1] \times [0, y]$, is continuous on $[0, 1]$ provided the integral $(\mathrm{M}_1) \int_{Q_1} h$ exists. We will show that H is generally not differentiable. Put $h(x, y) = |\varphi(x)| \, . \, [\sin \pi l_i y]^{1/i}$ similarly as in Example 4 for $(x, y) \in V_i \times [0, 1]$. Fix positive integers i, k and evaluate

$$\left| H\left(\frac{k+1}{l_i}\right) - H\left(\frac{k}{l_i}\right) \right| = \left| \int_0^1 \int_{k/l_i}^{(k+1)/l_i} h(x, y) \, \mathrm{d}y \, \mathrm{d}x \right| =$$

$$= \left| \sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} |\varphi(x)| \int_{k/l_i}^{(k+1)/l_i} [\sin \pi l_j y]^{1/j} \, \mathrm{d}y \, \mathrm{d}x \right| .$$

However, for $j > i$ the inner integral vanishes because of the oscillations of sine, so that we may write

$$\left| H\left(\frac{k+1}{l_i}\right) - H\left(\frac{k}{l_i}\right) \right| \geqq \left| \int_{2^{-i}}^{2^{-i+1}} |\varphi(x)| \int_{k/l_i}^{(k+1)/l_i} [\sin \pi l_i y]^{1/i} \, \mathrm{d}y \, \mathrm{d}x \right| -$$

$$- \sum_{j=1}^{i-1} \int_{2^{-j}}^{2^{-j+1}} |\varphi(x)| \, \mathrm{d}x \int_{k/l_i}^{(k+1)/l_i} |\sin \pi l_j y|^{1/j} \, \mathrm{d}y .$$

Routine calculation yields

$$2^{-3j-1} l_j = 2^{-(j+1)} \, . \, 2^{-2j} l_j \leqq \int_{2^{-j}}^{2^{-j+1}} |\varphi(x)| \, \mathrm{d}x \leqq 2^{-j} \, . \, 2^{-2j} l_j = 2^{-3j} l_j .$$

Hence

$$\left| H\left(\frac{k+1}{l_i}\right) - H\left(\frac{k}{l_i}\right) \right| \geqq 2^{-3i-1} l_i \, . \, \frac{2}{\pi l_i} - \sum_{j=1}^{i-1} 2^{-3j} l_j l_i^{-1} = \frac{1}{\pi} 2^{-3i} - \sum_{j=1}^{i-1} 2^{-3j} l_j / l_i .$$

Consequently,

$$l_i \left| H\left(\frac{k+1}{l_i}\right) - H\left(\frac{k}{l_i}\right) \right| \geqq \frac{1}{\pi} 2^{-3i} l_i - \sum_{j=1}^{j-1} 2^{-3j} l_j .$$

It is clear that by a suitable choice of l_i's we can make the right-hand side tending to infinity as quickly as required (with $i \to \infty$).

Thus we may infer: (i) H has a finite derivative for no $z \in [0, 1]$; (ii) no a-priori modulus of continuity for H exists.

6. CONVERGENCE THEOREMS

In [1] J. Mawhin proved the Levi-type monotone convergence theorem for the GP-integral. We follow here the idea of the proof of convergence theorems for the

P-integral as presented in [2] and give the corresponding results for the case of the M_1- and M_2-integrals. Since the results as well as their proofs are completely analogous in both cases, we formulate them for the M_1-integral only. The results for the M_2-integral are obtained by replacing M_1 by M_2 and, in the proofs, the P-partitions by the L-partitions.

First, we prove a general convergence theorem.

Theorem 4. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions $f_k : I \to X$ (X a Banach space, $I \subset \mathbb{R}^n$ a compact interval) satisfying the following conditions:*

(i) *For each $k \in \mathbb{N}$, f_k is M_1-integrable over I.*

(ii) *The sequence $(f_k)_{k\in\mathbb{N}}$ converges pointwise on I to a function $f : I \to X$.*

(iii) *For every $\varepsilon > 0$ and every constant $C > 0$ there is a gauge δ on I such that for every δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq C$ the inequality*

$$\left\| S(I, f_k, \Pi) - (\mathrm{M}_1)\int_I f_k \right\| \leqq \varepsilon \tag{11}$$

holds for every $k \in \mathbb{N}$.

Then f is M_1-integrable over I and

$$\lim_{k\to\infty} (\mathrm{M}_1)\int_I f_k = (\mathrm{M}_1)\int_I f\,. \tag{12}$$

Proof. Given $\varepsilon > 0$, $C > 0$, assume that δ is the gauge corresponding to $\frac{1}{2}\varepsilon$, C by the assumption (iii), i.e. we have

$$\left\| S(I, f_k, \Pi) - (\mathrm{M}_1)\int_I f_k \right\| \leqq \tfrac{1}{2}\varepsilon \tag{13}$$

for every δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq C$ and for every $k \in \mathbb{N}$.

By (ii), for every fixed δ-fine P-partition Π with $\Sigma_1(\Pi) \leqq C$ there is $k_0 \in \mathbb{N}$ such that

$$\| S(I, f_k, \Pi) - S(I, f, \Pi) \| \leqq \tfrac{1}{2}\varepsilon \tag{14}$$

for $k \in \mathbb{N}$, $k \geqq k_0$.

Combining (13) and (14) we infer that for every $\varepsilon > 0$ and $C > 0$ there is a gauge δ such that for every δ-fine P-partition Π of I with $\Sigma_1(\Pi) \leqq C$ there is $k_0 \in \mathbb{N}$ such that the inequality

$$\left\| S(I, f, \Pi) - (\mathrm{M}_1)\int_I f_k \right\| \leqq \varepsilon \tag{15}$$

holds for $k \in \mathbb{N}$, $k \geqq k_0$. Hence for $k, l \in \mathbb{N}$, $k \geqq k_0$, $l \geqq k_0$ we have

$$\left\|(\mathrm{M}_1)\int_I f_k - (\mathrm{M}_1)\int_I f_l\right\| \leqq 2\varepsilon ,$$

which implies that the sequence $((\mathrm{M}_1)\int_I f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Thus it possesses a limit in the Banach space X, i.e.

$$\lim_{k\to\infty}(\mathrm{M}_1)\int_I f_k = \alpha .$$

Finally, by (15) we have

$$\|S(I, f, \Pi) - \alpha\| = \lim_{k\to\infty}\left\|S(I, f, \Pi) - (\mathrm{M}_1)\int_I f_k\right\| \leqq \varepsilon$$

for every δ-fine P-partition Π with $\Sigma_1(\Pi) \leqq C$, which implies the M_1-integrability of f as well as the equality

$$(\mathrm{M}_1)\int_I f = \alpha = \lim_{k\to\infty}(\mathrm{M}_1)\int_I f_k .$$

Remark 9. Notice that Theorem 3 is valid even for the GP-integral. The assumption (iii) may be called the M_1-equiintegrability of the sequence $(f_k)_{k\in\mathbb{N}}$. It is this assumption that makes the proof of Theorem 3 so easy. On the other hand, (iii) is a very strong assumption and not easy to verify. In the sequel, restricting ourselves to real functions, we replace (iii) by another condition which together with Theorem 3 easily yields both the Levi-type monotone convergence and the Lebesgue-type dominated convergence theorems.

Let us now prove the following Saks-Henstock lemma (see also [1] for a slightly different version).

Lemma (Saks-Henstock). *Let $f : I \to X$ (X a Banach space, $I \subset \mathbb{R}^n$ a compact interval) be M_1-integrable over I. Let δ be a gauge on I corresponding to $\varepsilon > 0$, $C < 0$ in the definition of the M_1-integral (cf. Definitions 3, 4). Assume $\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$ is a δ-fine P-partition of I with $\Sigma_1(\Pi) \leqq C$.*

Then for any finite sequence of integers m_j, $j = 1, \ldots, l$, such that $1 < m_1 < \ldots < m_l < m$ the inequality

$$\left\|\sum_{j=1}^{l}\left[f(x^{m_j})\, m(I^{m_j}) - (\mathrm{M}_1)\int_{I^{m_j}} f\right]\right\| \leqq \varepsilon \tag{16}$$

holds.

Proof. For $m = l$ the lemma is a trivial consequence of the definition. Thus assume $m - l = k > 0$. Denote the intervals I^j not appearing in the sum (16) by K^i, $i = 1, \ldots, k$.

Let $\eta > 0$. Since f is M_1-integrable over every interval K^i, $i = 1, \ldots, k$, there exists a gauge $\delta_i : K^i \to \mathbb{R}^+$ on K^i such that $\delta_i(x) \leqq \delta(x)$ for all $x \in K^i$ and such that for every δ_i-fine partition Π_i of K^i with $\Sigma_1(\Pi_i) \leqq \mathscr{m}_{n-1}(\partial K^i)\operatorname{diam}(K^i)$ the inequality

$$\left\| S(K^i, f, \Pi_i) - (\mathrm{M}_1) \int_{K^i} f \right\| < \eta/(k+1)$$

holds. Let us now define the P-partition

$$\tilde{\Pi} = \{(x^{m_1}, I^{m_1}), \ldots, (x^{m_l}, I^{m_l})\} \cup \bigcup_{i=1}^{k} \Pi_i .$$

Then $\tilde{\Pi}$ evidently is δ-fine and

$$\Sigma_1(\tilde{\Pi}) = \sum_{j=1}^{l} m_{n-1}(\partial I^{m_j}) \operatorname{diam}(I^{m_j}) + \sum_{i=1}^{k} \Sigma_1(\Pi_i) \leqq$$

$$\leqq \sum_{j=1}^{l} m_{n-1}(\partial I^{m_j}) \operatorname{diam}(I^{m_j}) + \sum_{i=1}^{k} m_{n-1}(\partial K^i) \operatorname{diam}(K^i) = \Sigma_1(\Pi) \leqq C .$$

Hence we have by the assumption

$$\left\| S(I, f, \tilde{\Pi}) - (\mathrm{M}_1) \int_I f \right\| =$$

$$= \left\| \sum_{j=1}^{l} f(x^{m_j})\, m(I^{m_j}) + \sum_{i=1}^{k} S(K^i, f, \Pi_i) - \sum_{j=1}^{l} (\mathrm{M}_1) \int_{I^{m_j}} f - \sum_{i=1}^{k} (\mathrm{M}_1) \int_{K^i} f \right\| \leqq \varepsilon$$

and, consequently,

$$\left\| \sum_{j=1}^{l} \left[f(x^{m_j})\, m(I^{m_j}) - (\mathrm{M}_1) \int_{I^{m_j}} f \right] \right\| \leqq$$

$$\leqq \varepsilon + \left\| \sum_{i=1}^{k} \left[S(K^i, f, \Pi_i) - (\mathrm{M}_1) \int_{K^i} f \right] \right\| \leqq \varepsilon + \sum_{i=1}^{k} \frac{\eta}{k+1} = \varepsilon + \eta$$

as well. Since η has been arbitrary, (16) immediately follows.

Theorem 5. *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of functions $f_k : I \to \mathbb{R}$ ($I \subset \mathbb{R}^n$ a compact interval) satisfying* (i), (ii) *from Theorem* 3 *and*

(iv) *there is a constant $K > 0$ such that for every finite partition $\{I^1, \ldots, I^m\}$ of I and every finite sequence of positive integers $k_1, \ldots, k_m$ the inequality*

$$\left| \sum_{j=1}^{m} (\mathrm{M}_1) \int_{I^j} f_{k_j} \right| \leqq K$$

holds.

Then the function f is M_1-integrable over I and (12) *holds.*

Proof. We shall prove that the assumptions of Theorem 5 imply those of Theorem 4, i.e., (iv) implies (iii).

For $p \in \mathbb{N}$, denote by $\mathscr{F}_p$ the collection of all functions $h : I \to \mathbb{R}$ for which there exists a partition $\{J^1, \ldots, J^m\}$ of the interval I (i.e., $J^1, \ldots, J^m$ are non-overlapping compact intervals, $\bigcup_{j=1}^{m} J^j = I$) and $k_1, \ldots, k_m \in \mathbb{N}$, $k_i > p$ for $i = 1, \ldots, m$, such that

$$h(x) = f_{k_i}(x) \quad \text{for} \quad x \in J^i, \quad i = 1, \ldots, m.$$

(If $x \in \partial I^i \cap \partial I^j$, $i \neq j$, we choose one of the values $f_{k_i}(x), f_{k_j}(x)$ arbitrarily.)

The collections $\mathscr{F}_p$ have the following properties:

(α) If $p_1, p_2 \in \mathbb{N}$, $p_1 > p_2$, then $\mathscr{F}_{p_1} \subset \mathscr{F}_{p_2}$.

(β) Every function $h \in \mathscr{F}_1$ (and hence, by (α), every function $h \in \mathscr{F}_p$ for any $p \in \mathbb{N}$) is M_1-integrable. (This immediately follows from Theorems 1, 2 and Remark 6.) Moreover, by (iv) we have

$$\left|(\mathrm{M}_1)\int_I h\right| \leqq K$$

provided $h \in \mathscr{F}_1$.

(γ) We have

$$\lim_{p\to\infty} h_p(x) = f(x)$$

for $x \in I$ provided $h_p \in \mathscr{F}_p$, $p \in \mathbb{N}$. (Cf. (ii).)

(δ) Given $\varepsilon > 0$, then for every $p \in \mathbb{N}$ there exist $g_p, G_p \in \mathscr{F}_p$ such that

$$\inf\left\{(\mathrm{M}_1)\int_I h;\ h \in \mathscr{F}_p\right\} + \frac{\varepsilon}{2^{p+2}} > (\mathrm{M}_1)\int_I g_p$$

and

$$\sup\left\{(\mathrm{M}_1)\int_I h;\ h \in \mathscr{F}_p\right\} - \frac{\varepsilon}{2^{p+2}} < (\mathrm{M}_1)\int_I G_p.$$

(ε) Given $\varepsilon > 0$, $p \in \mathbb{N}$, $h \in \mathscr{F}_p$ and a finite system of non-overlapping (compact) intervals $J^1, J^2, \ldots, J^q \subset I$, then

$$\sum_{j=1}^{q}(\mathrm{M}_1)\int_{J^j} g_p - \frac{\varepsilon}{2^{p+2}} \leqq \sum_{j=1}^{q}(\mathrm{M}_1)\int_{J^j} h \leqq \sum_{j=1}^{q}(\mathrm{M}_1)\int_{J^j} G_p + \frac{\varepsilon}{2^{p+2}}.$$

Indeed, if e.g. the second inequality did not hold, then there would exist a function $\hat{h} \in \mathscr{F}_p$ such that

$$\sum_{j=1}^{q}(\mathrm{M}_1)\int_{J^j} G_p + \frac{\varepsilon}{2^{p+2}} < \sum_{j=1}^{q}(\mathrm{M}_1)\int_{J^j} \hat{h}.$$

Defining now a function h^0 on I by

$$h^0(x) = \begin{cases} \hat{h}(x) & \text{for} \quad x \in \bigcup_{j=1}^{q} J^j \\ G_p(x) & \text{otherwise}, \end{cases}$$

then $h^0 \in \mathscr{F}_p$ and by (δ) we have (using Theorem 1)

$$(\mathrm{M}_1)\int_I h^0 > (\mathrm{M}_1)\int_I G_p + \frac{\varepsilon}{2^{p+2}} > \sup\left\{(\mathrm{M}_1)\int_I h;\ h \in \mathscr{F}_p\right\},$$

a contradiction. The other inequality can be proved similarly.

Let us now proceed to the proof of (iii). Let $\varepsilon > 0$ and $C > 0$. By (γ) we have $g_p(x) \to f(x)$, $G_p(x) \to f(x)$ with $p \to \infty$ for every $x \in I$. Hence for every $x \in I$ there is $p \coloneqq p(x) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k > p(x)$ we have

$$\left|f_k(x) - g_{p(x)}(x)\right| < \tfrac{1}{2}\varepsilon(m(I) + 1)^{-1}, \tag{17}$$

$$\left|f_k(x) - G_{p(x)}(x)\right| < \tfrac{1}{2}\varepsilon(m(I) + 1)^{-1}.$$

Further, for $p \in \mathbb{N}$ there exists a gauge δ_p on I such that for every δ_p-fine P-partition Π of I with $\Sigma_1(\Pi) < C$ the inequality

$$\left|S(I, h, \Pi) - (\mathrm{M}_1)\int_I h\right| < \frac{\varepsilon}{2^{p+2}} \tag{18}$$

holds with $h = g_p$ and $h = G_p$ and $h = f_p$.

Choose a gauge δ on I so that

$$\delta(x) \leqq \min\left(\delta_1(x), \delta_2(x), \ldots, \delta_{p(x)}(x)\right)$$

for every $x \in I$ and assume that

$$\Pi = \{(x^1, I^1), (x^2, I^2), \ldots, (x^m, I^m)\}$$

is a δ-fine P-partition of I with $\Sigma_1(\Pi) \leqq C$ and $k \in \mathbb{N}$. Then

$$S(I, f_k, \Pi) = \sum_{j=1}^{m} f_k(x^j)\, m(I^j) =$$

$$= \sum_{\substack{j=1 \\ p(x^j) \geqq k}}^{m} f_k(x^j)\, m(I^j) + \sum_{\substack{j=1 \\ p(x^j) < k}}^{m} f_k(x^j)\, m(I^j).$$

In the second sum we have by (17)

$$f_k(x^j) > G_{p(x^j)}(x^j) - \tfrac{1}{2}\varepsilon(m(I) + 1)^{-1},$$

hence

$$\sum_{\substack{j=1 \\ p(x^j) < k}}^{m} f_k(x^j)\, m(I^j) > \sum_{\substack{j=1 \\ p(x^j) < k}}^{m} G_{p(x^j)}(x^j)\, m(I^j) - \tfrac{1}{2}\varepsilon(m(I) + 1)^{-1} \sum_{\substack{j=1 \\ p(x^j) < k}}^{m} m(I^j) \geqq$$

$$\geqq \sum_{\substack{j=1 \\ p(x^j) < k}}^{m} G_{p(x^j)}(x^j)\, m(I^j) - \tfrac{1}{2}\varepsilon$$

and

$$S(I,f_k,\Pi) > \sum_{\substack{j=1\\ p(x^j)\geq k}}^{m} f_k(x^j)\, m(I^j) + \sum_{\substack{j=1\\ p(x^j)<k}}^{m} G_{p(x^j)}(x^j)\, m(I^j) - \tfrac{1}{2}\varepsilon =$$

$$= \sum_{\substack{j=1\\ p(x^j)\geq k}}^{m} f_k(x^j)\, m(I^j) + \sum_{r=1}^{k-1} \sum_{\substack{j=1\\ p(x^j)=r}}^{m} G_r(x^j)\, m(I^j) - \tfrac{1}{2}\varepsilon\,.$$

Applying (18) with $h = f_k$, $h = G_r$ and the Saks-Henstock Lemma to the sums on the right hand side we obtain

(19)

$$S(I,f_k,\Pi) > \sum_{\substack{j=1\\ p(x^j)\geq k}}^{m} (\mathrm{M}_1)\int_{I^j} f_k - \frac{\varepsilon}{2^{k+2}} + \sum_{r=1}^{k-1}\left[\sum_{\substack{j=1\\ p(x^j)=r}}^{m} (\mathrm{M}_1)\int_{I^j} G_r - \frac{\varepsilon}{2^{r+2}}\right] - \tfrac{1}{2}\varepsilon\,.$$

Since the function f_k by definition belongs to all systems $\mathscr{F}_r$ with $r = 1, 2, \ldots, k-1$, we have by (ε)

$$\sum_{r=1}^{k-1} \sum_{\substack{j=1\\ p(x^j)=r}}^{m} (\mathrm{M}_1)\int_{I^j} G_r \geq \sum_{r=1}^{k-1}\left[\sum_{\substack{j=1\\ p(x^j)=r}}^{m} (\mathrm{M}_1)\int_{I^j} f_k - \frac{\varepsilon}{2^{r+2}}\right] = \sum_{\substack{j=1\\ p(x^j)<k}}^{m} (\mathrm{M}_1)\int_{I^j} f_k - \varepsilon\sum_{r=1}^{k-1} 2^{-r-2}\,.$$

This together with (19) yields the inequality

$$S(I,f_k,\Pi) > \sum_{j=1}^{m} (\mathrm{M}_1)\int_{I^j} f_k - 2\varepsilon\sum_{r=1}^{k-1} 2^{-r-2} - \tfrac{1}{2}\varepsilon = (\mathrm{M}_1)\int_I f_k - \varepsilon\,.$$

In a completely analogous way (making use of g_p instead of G_p) it can be shown that

$$S(I,f_k,\Pi) < (\mathrm{M}_1)\int_I f_k + \varepsilon\,,$$

which together yields (11) from (iii) (Theorem 4). Hence the assertion of Theorem 5 follows by virtue of Theorem 4.

Theorem 5 makes it immediately possible to derive the Levi-type monotone convergence theorem and the Lebesgue-type dominated convergence theorem. We shall present the proofs of both of them.

Theorem 6. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of real functions, $f_k : I \to \mathbb{R}$, $I \subset \mathbb{R}^n$ an interval, such that (i), (ii) from Theorem 4 holds,*

(v) *there is $K > 0$ such that $\left|(\mathrm{M}_1)\int_I f_k\right| \leq K$ for all $k \in \mathbb{N}$, and*

(vi) *for all $k \in \mathbb{N}$ and $x \in I$ the inequality $f_{k+1}(x) \geq f_k(x)$ (or $f_{k+1}(x) \leq f_k(x)$) holds.*

Then $f = \lim_{k\to\infty} f_k$ is M_1-integrable over I and (12) holds.

Proof. Given a partition $\{I^1, \ldots, I^m\}$ of I and a finite sequence $k_1, \ldots, k_m$ of positive integers, then using the monotonicity property of the M_1-integral (Remark 7) and (vi) we obtain

$$(M_1) \int_{I^j} f_\mu \leqq (M_1) \int_{I^j} f_{k_j} \leqq (M_1) \int_{I^j} f_\nu ,$$

where $\mu = \min(k_1, \ldots, k_m)$ and $\nu = \max(k_1, \ldots, k_m)$. Consequently, by (v) we have

$$-K \leqq (M_1) \int_I f_\mu \leqq \sum_{j=1}^{m} \int_{I^j} f_{k_j} \leqq (M_1) \int_I f_\nu \leqq K .$$

Thus the assumption (iv), Theorem 5, is fulfilled and the assertion of Theorem 6 immediately follows.

Remark 10. Replacing (v) in Theorem 6 by

(v*) $$\lim_{k \to \infty} (M_1) \int_I f_k \quad \text{exists}$$

we obtain the Levi-type monotone convergence theorem for the M_1-integral in the form given by J. Mawhin in [1] for the GP-integral (cf. Definition 1).

Theorem 7. *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of real functions $f_k : I \to \mathbb{R}$, $I \subset \mathbb{R}^n$ an interval, such that* (i), (ii) *for Theorem* 4 *holds and*

(vii) *there exist M_1-integrable functions $g, h : I \to \mathbb{R}$ such that $g(x) \leqq f_k(x) \leqq$ $\leqq h(x)$ for all $k \in \mathbb{N}$ and all $x \in I$.*

Then the function $f : I \to \mathbb{R}$ is M_1-integrable over I and (12) *holds.*

Proof. Assume again that $\{I^1, \ldots, I^m\}$ is a partition of I and $k_1, \ldots, k_m$ a finite sequence of positive integers. Set $K = \max(|(M_1) \int_I g|, |(M_1) \int_I h|)$. Using again the monotonicity property (cf. Remark 7) and (vii) we obtain

$$(M_1) \int_{I^j} g \leqq (M_1) \int_{I^j} f_{k_j} \leqq (M_1) \int_{I^j} h$$

for $j = 1, \ldots, m$ and, consequently,

$$-K \leqq (M_1) \int_I g \leqq \sum_{j=1}^{m} (M_1) \int_{I^j} f_{k_j} \leqq (M_1) \int_I h \leqq K .$$

Thus (iv), Theorem 5, holds and the assertion of Theorem 7 follows.

Remark 11. Let us point out that our proof of Theorem 5 is strongly based on Theorem 2 that fails to hold for the GP-integral, as was shown in Example 2. Hence the proof cannot be applied to the GP-integral. Since we have not been able to find a counterexample, either, the problem whether a Lebesgue-type dominated convergence theorem holds for the GP-integral remains still open. (Cf. Remark 3 in [1].)

The three definitions of integral introduced above suggest the following general scheme.

Let $\mathscr{I}, \mathscr{J}$ be some families of subsets of a metric space (G, d). A finite collection

$$\Delta = \{(t_j, J_j);\ j = 1, \ldots, m\} \tag{20}$$

is called an *abstract partition* of $I \in \mathscr{I}$ if $t_j \in I$, $J_j \in \mathscr{J}$ for $j = 1, \ldots, m$. Let $\mathrm{CP}(I)$ be a collection of abstract partitions of I for $I \in \mathscr{I}$.

(For example, for $\mathscr{I}$ we can take the set $\mathscr{I}_1$ of all nondegenerate compact intervals in $\mathbb{R}^n$, put $\mathscr{J}_1 = \mathscr{I}_1$ and define $\mathrm{CP}_1(I)$ as the set of all P-partitions of I and $\mathrm{CP}_2(I)$ as the set of all L-partitions of I.

Let $\Sigma(\Delta) \in [0, \infty]$ for every $\Delta \in \mathrm{CP}(I)$. (Thus Σ is a nonnegative function defined for all $\Delta \in \bigcup_{I \in \mathscr{I}} \mathrm{CP}(I)$ and possibly for some other Δ's as well – in particular, we may take the functions $\Sigma_0, \Sigma_1, \Sigma_2$ from the definitions of the GP-, M_1- and M_2-integrals.)

A function $\delta : I \to (0, \infty)$ is again called a gauge (on I). We say that Δ is δ-fine if the abstract partition Δ satisfies the following condition:

$$J_j \subset B(t_j, \delta(t_j)), \quad j = 1, \ldots, m.$$

The following assumption on $\mathrm{CP}(I)$ plays a fundamental role:

Assumption. For every $I \in \mathscr{I}$ there exists such a $C > 0$ that for every gauge δ on I there is such a $\Delta \in \mathrm{CP}(I)$ that Δ is δ-fine and $\Sigma(\Delta) \leqq C$.

Finally, let $m : \mathscr{J} \to \mathbb{R}$ and denote

$$S(I, f, \Delta, m) = \sum_{j=1}^{m} f(t_j)\, m(J_j)$$

for $I \in \mathscr{I}, f : I \to \mathbb{R}$, $\Delta \in \mathrm{CP}(I)$ defined by (20).

The concept of the M-integral is associated with the quadruple $(\mathscr{I}, \mathrm{CP}, \Sigma, m)$; we write

$$\mathrm{M} = (\mathscr{I}, \mathrm{CP}, \Sigma, m).$$

Definition 5. Let $I \in \mathscr{I}, f : I \to \mathbb{R}$. If $\gamma \in \mathbb{R}$ is such that for every $\varepsilon > 0$ and $C > 0$ there is a gauge δ on I such that for every δ-fine $\Delta \in \mathrm{CP}(I)$ with $\Sigma(\Delta) \leqq C$ the inequality

$$|\gamma - S(I, f, \Delta, m)| \leqq \varepsilon$$

holds, then γ is called the *M-integral of f over I, f* is said to be *M-integrable* and we write $\gamma = (\mathrm{M}) \int_I f \, \mathrm{d}m$.

By **Int** (M, I) we denote the set of all $f : I \to \mathbb{R}$ that are M-integrable over I.

Example 6. We obtain the integrals from Sections 1, 2, 3 by setting, respectively,

$\mathrm{M}_0 = (\mathscr{I}_1, \mathrm{CP}_1, \Sigma_0, m)$ (the GP-integral),

$M_1 = (\mathscr{I}_1, CP_1, \Sigma_1, m)$,

$M_2 = (\mathscr{I}_1, CP_2, \Sigma_2, m)$.

Further, if we put $\Sigma_3(\Delta) = 1$ for all Δ's,

$M_3 = (\mathscr{I}_1, CP_1, \Sigma_3, m)$,

$M_4 = (\mathscr{I}_1, CP_2, \Sigma_3, m)$,

then the M_3-integral is the Perron integral and the M_4-integral is the Lebesgue integral. (In all the above formulas, m stands for the Lebesgue measure in $\mathbb{R}^n$.)

Even this very general setting allows to prove the following result.

Theorem 8. *Let* $\mathscr{I}$, CP_3, CP_4, Σ_4, Σ_5 *and* $c > 0$ *be given, let*

$M_5 = (\mathscr{I}, CP_3, \Sigma_4, m)$,

$M_6 = (\mathscr{I}, CP_4, \Sigma_5, m)$

and assume that

$$CP_4(I) \supset CP_3(I) \quad \textit{for} \quad I \in \mathscr{I}\,, \tag{21}$$

$$\Sigma_5(\Delta) \leqq c\,\Sigma_4(\Delta) \quad \textit{for} \quad \Delta \in CP_3(I)\,, \quad I \in \mathscr{I}\,. \tag{22}$$

Then

$$\mathbf{Int}\,(M_6, I) \subset \mathbf{Int}\,(M_5, I)$$

and

$$(M_6)\int_I f\,\mathrm{d}m = (M_5)\int_I f\,\mathrm{d}m$$

for $f \in \mathbf{Int}\,(M_6, I)$.

Proof. Let $f \in \mathbf{Int}\,(M_6, I)$, $\varepsilon > 0$, $C > 0$. For ε, Cc find such a gauge δ on I that $\Delta \in CP_4(I)$, Δ is δ-fine, $\Sigma_5(\Delta) \leqq Cc$ implies

$$\left|(M_6)\int_I f\,\mathrm{d}m - S(I, f, \Delta, m)\right| \leqq \varepsilon\,. \tag{23}$$

If $\Delta^* \in CP_3(I)$, Δ^* is δ-fine, $\Sigma_4(\Delta^*) \leqq C$, then $\Delta^* \in CP_4(I)$ by (21), $\Sigma_5(\Delta^*) \leqq Cc$ by (22), hence (23) holds with Δ replaced by Δ^* and $f \in \mathbf{Int}\,(M_5, I)$.

Corollary. *Let* (22) *and*

$$c^{-1}\,\Sigma_4(\Delta) \leqq \Sigma_5(\Delta) \quad \textit{for} \quad \Delta \in CP_3(I)\,, \quad I \in \mathscr{I}$$

hold. Then

$$\mathbf{Int}\,((\mathscr{I}, CP_3, \Sigma_4, m), I) = \mathbf{Int}\,((\mathscr{I}, CP_3, \Sigma_5, m), I)$$

and the corresponding integrals coincide.

Corollary. $\mathbf{Int}\,((\mathscr{I}_1, CP_1, \Sigma_2, m), I) = \mathbf{Int}\,((\mathscr{I}_1, CP_1, \Sigma_1, m), I) = \mathbf{Int}\,(M_1, I)$ *and the corresponding integrals coincide.*

The latter corollary implies that we could have used Σ_2 instead of Σ_1 when introducing the M_1-integral in Section 2.

Obviously, $\mathbf{Int}(M_1, I) \supset \mathbf{Int}(M_2, I)$ and the corresponding integrals coincide. (By Example 3, we have even the strict inclusion.)

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