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ON AN EXTREMAL CHARACTERIZATION OF PARTITIONS

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In this note we are concerned with equivalence relations on a finite set, where the factor sets of these equivalence relations have a given cardinality. These equivalence relations are characterized as solutions of an extremal problem in a set of tolerances (i.e. reflexive and symmetric relations).

Let n and k be positive integers, $k \leqq n$, and let $S_n = \{s_1, s_2, \ldots, s_n\}$ be an n-element set. Let $\mathscr{R}_n$ denote the set of all reflexive and symmetric relations $\varrho \subseteq S_n \times S_n$, and let us set $\mathscr{R}_n^{(k)}$ for the set of all $\varrho \in \mathscr{R}_n$ satisfying the following condition:

$$\forall S \subseteq S_n: \text{ If } (S \times S) \cap \varrho \subseteq \{(s, s) \mid s \in S\} \text{ then } \operatorname{card}(S) \leqq k .$$

A partition of a finite set into k pair-wise disjoint nonempty subsets will be called a *k-partition*. A relation $\varrho \in \mathscr{R}_n$ will be called a *k-equivalence* if ϱ is an equivalence relation, and the factor set induced by ϱ is a k-partition of S_n.

The following theorem characterizes k-equivalences (or, equivalently speaking. k-partitions) on the set S_n as solutions of a class of minimization problems on $\mathscr{R}_n^{(k)}$,

Theorem 1. *Let $\hat{\varrho} \in \mathscr{R}_n^{(k)}$. Then the following assertions are equivalent:*

(i) *$\hat{\varrho}$ is a k-equivalence.*

(ii) *There exist positive numbers $c_1, c_2, \ldots, c_n$ such that*

$$\sum_{j=1}^{n} c_j \,.\, \operatorname{card}(\{s_j\} \times S_n) \cap \hat{\varrho}) = \min \{ \sum_{j=1}^{n} c_j \,.\, \operatorname{card}((\{s_j\} \times S_n) \cap \varrho) \mid \varrho \in \mathscr{R}_n^{(k)}\} .$$

Proof.

I. (i) $\Leftarrow$ (ii): This implication is equivalent to the b) part of Theorem 2 of [1]. Indeed, let $\mathscr{G}_n$ denote the set of all undirected graphs without loops and multiple edges $G = \langle S_n, E(G)\rangle$, having S_n for the set of vertices; $E(G)$ denotes the set of all edges of G. Further, let us denote by $\mathscr{G}_n^{(k)}$ the set of all graphs $G \in \mathscr{G}_n$ such that $\forall S \subseteq S_n$: If there is no pair of distinct adjacent vertices s, s' in S then

$$\operatorname{card}(S) \leqq k .$$

(This can be equivalently expressed by saying that $\alpha(G) \leqq k$, where $\alpha(G)$ denotes the number of stability of G, cf. [2], p. 260.)

Let us define a mapping $\varphi : \mathscr{R}_n^{(k)} \to \mathscr{G}_n^{(k)}$ as follows: $\varphi(\varrho) \overset{\text{def}}{=} G$ iff $\forall j\ \forall j'$ (s_j and $s_{j'}$ are adjacent in G iff ($j \neq j'$ and $(s_j, s_{j'}) \in \varrho$)), and observe the following facts:

(a) φ is bijective;

(b) ϱ is a k-equivalence iff $\varphi(\varrho)$ is a *k-clique graph*, i.e. a graph having exactly k connected components where each component is a complete subgraph, cf. [1];

(c) $\sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \varrho) = \sum_{j=1}^{n} c_j \,.\, d_j(\varphi(\varrho)) + \sum_{j=1}^{n} c_j$, where $d_j(G)$ denotes the degree of the vertex s_j in G.

The b) part of Theorem 2 in [1] can the stated as follows: If $c_1, c_2, \ldots, c_n$ are positive numbers and if $G \in \mathscr{G}_n^{(k)}$ is a graph such that

$$\sum_{j=1}^{n} c_j\, d_j(G) = \min \{ \sum_{j=1}^{n} c_j\, d_j(G) \mid G \in \mathscr{G}_n^{(k)} \}$$

then G is a k-clique graph. By using this fact and the properties (a), (b), (c) of φ the implication (i) $\Leftarrow$ (ii) immediately follows.

II. (i) $\Rightarrow$ (ii): Let $\hat{\varrho}$ be a k-equivalence on S_n and let $\{V_1, V_2, \ldots, V_k\}$ be the k-element factor set induced by $\hat{\varrho}$. Let us set:

$$c_j \overset{\text{def}}{=} (\text{caıd}\,(V_\varkappa))^{-2} \quad \text{iff} \quad s_j \in V_\varkappa$$

$(j = 1, 2, \ldots, n;\ \varkappa = 1, 2, \ldots, k)$.

We shall show that for c_j $(j = 1, 2, \ldots, n)$ defined in this way and for all $\varrho \in \mathscr{R}_n^{(k)}$,

$$\text{(1)} \qquad \sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \hat{\varrho}) \leqq \sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \varrho)\,.$$

Indeed, let $\varrho^* \in \mathscr{R}_n^{(k)}$ be a relation minimizing the function

$$\varrho \mapsto \sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \varrho) \quad (\varrho \in \mathscr{R}_n^{(k)})\,,$$

i.e. we have

$$\text{(2)} \qquad \sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \varrho^*) \leqq \sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \varrho)$$

for all $\varrho \in \mathscr{R}_n^{(k)}$.

Because of the proved (i) $\Leftarrow$ (ii) part of this theorem ϱ^* is a k-equivalence; let $\{W_1, W_2, \ldots, W_k\}$ be the corresponding factor set. Now, it is sufficient to verify the inequality (1) for $\varrho = \varrho^*$. We have

$$\text{(3)} \quad \sum_{j=1}^{n} c_j \,.\, \text{card}\,((\{s_j\} \times S_n) \cap \hat{\varrho}) = \sum_{\varkappa=1}^{k} \sum_{s_j \in V_\varkappa} (\text{card}\,(V_\varkappa))^{-2} \,.\, \text{card}\,(\{s_j\} \times V_\varkappa) =$$

$$= \sum_{\varkappa=1}^{k} \sum_{s_j \in V_\varkappa} (\text{card}(V_\varkappa))^{-1} = \sum_{\varkappa=1}^{k} 1 = k .$$

Furthermore, for $x_j = \text{card}((\{s_j\} \times S_n) \cap \varrho^*)$ $(j = 1, 2, \ldots, n)$ we obtain

$$(4) \qquad \sum_{j=1}^{n} x_j^{-1} = \sum_{\varkappa=1}^{k} \sum_{s_j \in W_\varkappa} x_j^{-1} = \sum_{\varkappa=1}^{k} \sum_{s_j \in W_\varkappa} (\text{card}(W_\varkappa))^{-1} = \sum_{\varkappa=1}^{k} 1 = k .$$

Now, by using (3), (4) and the Cauchy-Lagrange inequality we have

$$(5) \quad \sum_{j=1}^{n} c_j \, . \, \text{card}((\{s_j\} \times S_n) \cap \varrho^*) = \sum_{j=1}^{n} c_j x_j = k^{-1} . (\sum_{j=1}^{n} x_j^{-1}) . (\sum_{j=1}^{n} c_j x_j) \geqq$$

$$\geqq k^{-1} . (\sum_{j=1}^{n} \sqrt{c_j})^2 = k^{-1} . (\sum_{\varkappa=1}^{k} \sum_{s_j \in V_\varkappa} (\text{card}(V_\varkappa))^{-1})^2 = k .$$

By combining (3) and (5) we complete the proof. □

This theorem shows that each k-equivalence $\varrho \in \mathscr{R}_n^{(k)}$ can be obtained as a solution of the extremal problem

$$(6) \qquad \text{minimize} \sum_{j=1}^{n} c_j \, . \, \text{card}((\{s_j\} \times S_n) \cap \varrho) \;\; \textit{w.r.t.} \;\; \varrho \in \mathscr{R}_n^{(k)} ,$$

for appropriately chosen positive numbers c_j $(j = 1, 2, \ldots, n)$.

We conclude this note by describing a special case when the extremal problem (6) has a unique solution.

Theorem 2. *Let a k-partition $\{V_1, V_2, \ldots, V_k\}$ of S_n satisfy the following condition:*

$$\text{card}(V_\varkappa) \neq \text{card}(V_{\varkappa'}) \quad \textit{if} \quad \varkappa \neq \varkappa' ,$$

and let us set

$$c_j \overset{\text{def}}{=} (\text{card}(V_\varkappa))^{-2} \quad \textit{iff} \quad s_j \in V_\varkappa$$

$(j = 1, 2, \ldots, n;\ \varkappa = 1, 2, \ldots, k)$.

Then the extremal problem (6) *has a unique solution*

$$\varrho = \{(s, s') \in S_n \times S_n \mid \exists \varkappa \ (s \in V_\varkappa \text{ and } s' \in V_\varkappa)\} .$$

Proof. Let ϱ^* be any solution of the extremal problem (6) and let $\{W_1, W_2, \ldots, W_k\}$ denote the corresponding factor set induced by ϱ^*. By keeping the notation of the proof of Theorem 1 we must have the equality in (5), and hence

$$(\sum_{j=1}^{n} x_j^{-1}) . (\sum_{j=1}^{n} c_j x_j) = (\sum_{j=1}^{n} \sqrt{c_j})^2 .$$

Thus, n-tuples (vectors) $((\sqrt{x_1})^{-1}, (\sqrt{x_2})^{-1}, \ldots, (\sqrt{x_n})^{-1})$ and $(\sqrt{(c_1 x_1)}, \sqrt{(c_2 x_2)}, \ldots$

..., $\sqrt{(c_n x_n)}$) must be linearly dependent, and hence there exists a $\lambda > 0$ such that

$$x_j^{-1} = \lambda c_j x_j \quad (j = 1, 2, \ldots, n)$$

or, equivalently speaking,

$$x_j^{-1} = \sqrt{\lambda} \, . \sqrt{c_j} \quad (j = 1, 2, \ldots, n) \, . \tag{7}$$

Substitution of (7) into (4) yields

$$k = \sum_{j=1}^{n} x_j^{-1} = \sqrt{\lambda} \, . \sum_{j=1}^{n} \sqrt{c_j} = \sqrt{\lambda} \, . \, k \, ,$$

whence

$$\lambda = 1 \, .$$

Now, let $V_\varkappa \cap W_{\varkappa'} \neq \emptyset$. Then there exists an element $s_j \in V_\varkappa \cap W_{\varkappa'}$, and by using (7) for $\lambda = 1$ we have

$$\operatorname{card}(V_\varkappa) = \sqrt{c_j^{-1}} = x_j = \operatorname{card}((\{s_j\} \times S_n) \cap \varrho^*) = \operatorname{card}(W_{\varkappa'}) \, .$$

Thus, we have proved

$$\forall \varkappa \, \forall \varkappa' (V_\varkappa \cap W_{\varkappa'} \neq \emptyset \Rightarrow \operatorname{card}(V_\varkappa) = \operatorname{card}(W_{\varkappa'})) \, .$$

By combining this conclusion with the condition of the theorem we obtain

$$\forall \varkappa \, \exists \varkappa' (V_\varkappa \supseteq W_{\varkappa'}) \, . \tag{8}$$

Since $\{V_1, V_2, \ldots, V_k\}$ and $\{W_1, W_2, \ldots, W_k\}$ are k-partitions of the same set S_n we obtain from (8) that

$$\{V_1, V_2, \ldots, V_k\} = \{W_1, W_2, \ldots, W_k\} \, ,$$

which completes the proof. □

References

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[2] *C. Berge:* Graphes et hypergraphes. DUNOD, Paris, 1970.

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