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RADII OF STARLIKENESS AND COEFFICIENT ESTIMATES OF A CLASS OF ANALYTIC FUNCTIONS

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1. INTRODUCTION

Let S^* denote the class of functions $f(z)$ analytic in the open unit disc $E\{z : |z| < 1\}$, normalized so that $f(0) = 0 = f'(0) - 1$ and univalently starlike in E. The properties of the elements of this class have been investigated extensively for many years. One of the more important early discoveries for the class S^* was that $f(z)$ satisfies the inequality

$$|\sqrt{(z/f(z))} - 1| > 1\,, \quad (z \in E)\,.$$

This fact may also be expressed in the form

$$\operatorname{Re} \sqrt{(f(z)/z)} \geqq \frac{1}{1 + |z|} > 1/2\,, \quad (z \in E)\,.$$

Then $f(z)/z \ll (1 + z)^{-2}$ in E (where $\ll$ denotes subordination) and there exists an analytic function $\omega(z)$, $|\omega(z)| \leqq |z| < 1$, such that

$$\text{(1.1)} \qquad \frac{f(z)}{z} = \frac{1}{(1 + \omega(z))^2}\,, \quad (z \in E)\,.$$

Proofs of this attractive result are due to Marx [4], Strohhächer [8], and to Robertson [6]. Motivated by this discovery, we introduce the class $S(\alpha, \beta)$ as follows.

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disc E is in the class $S(\alpha, \beta)$ if it satisfies the condition

$$\text{(1.2)} \qquad \frac{f(z)}{z} \ll \left[\frac{1 + (2\alpha\beta - 1)\, z}{1 + (2\beta - 1)\, z}\right]^2, \quad (z \in E)$$

where α and β are arbitrary fixed numbers, $0 \leqq \alpha < 1$, $0 < \beta \leqq 1$.

It follows from the definition of subordination that $f \in S(\alpha, \beta)$ has a representation of the form

$$\frac{f(z)}{z} = \left[\frac{1 + (2\alpha\beta - 1)\,\omega(z)}{1 + (2\beta - 1)\,\omega(z)}\right]^2, \quad (z \in E) \tag{1.3}$$

for some function ω, analytic in E, and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in E$.

A function $f \in S(\alpha, \beta)$ may not be univalently starlike in E as is easily seen from the example $f(z) = z(1 + z^2)^{-2} \in S(1/2, 1)$.

The class $S(1/2, 1)$ has been investigated by Dvořák [2], Duren and Schober [1], and Reade and Umezawa [5]. We, further, note that the class $S(1/(2\varrho), 1) \equiv S(1/(2\varrho))$, $\varrho > 1/2$, is larger than the class introduced and studied by Goel [3].

In this paper, we obtain the radii of starlikeness and coefficient estimates for the functions in the class $S(\alpha, \beta)$.

2. RADII OF STARLIKENESS

Let B denote the class of analytic functions ω in E which satisfy the conditions (i) $\omega(0) = 0$, and (ii) $|\omega(z)| < 1$ for z in E.

Theorem 1. *Let $f \in S(\alpha, \beta)$ and let r_0 be the smallest positive root of the equation*

$$(2\beta - 1)(2\alpha\beta - 1)\,r^4 - 2(2\beta - 1)(2\alpha\beta - 1)\,r^3 - \\ - 2(\beta + \alpha\beta + 2\alpha\beta^2 - 1)\,r^2 - 2r + 1 = 0\,. \tag{2.1}$$

Then

(i) *for $0 \leqq r < r_0$, f is starlike in $|z| < r_1$, where r_1 is the smallest positive root of the equation*

$$(2\beta - 1)(2\alpha\beta - 1)\,r^2 + 2(3\alpha\beta - \beta - 1)\,r + 1 = 0\,, \tag{2.2}$$

(ii) *for $r_0 \leqq r < 1$, f is starlike in $|z| < r_2$, where r_2 is the smallest positive root of the equation*

$$(16\alpha\beta - 9 - \alpha)\,r^4 - 2(8\alpha\beta + 3 - 3\alpha)\,r^2 + (9\alpha - 1) = 0\,. \tag{2.3}$$

The bounds for $|z|$ in (i) *and* (ii) *are sharp.*

Proof. If $f(z) = z + a_2 z^2 + \ldots$ and

$$p(z) = \left(\frac{f(z)}{z}\right)^{1/2}, \tag{2.4}$$

then $p(z)$ is analytic in E and $p(0) = 1$. Thus (1.3) may be rewritten as

$$p(z) = \frac{1 + (2\alpha\beta - 1)\,\omega(z)}{1 + (2\beta - 1)\,\omega(z)}, \tag{2.5}$$

where $\omega \in B$. Taking logarithmic derivatives of (2.5), we find that

$$(2.6)\quad \operatorname{Re}\left\{\frac{z\,p'(z)}{p(z)}\right\} = -2\beta(1-\alpha)\operatorname{Re}\left\{\frac{z\,\omega'(z)}{(1+(2\beta-1)\,\omega(z))\,(1+(2\alpha\beta-1)\,\omega(z))}\right\}.$$

From (2.4) we may write

$$(2.7)\qquad \operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} = 1 + 2\operatorname{Re}\left\{\frac{z\,p'(z)}{p(z)}\right\}.$$

Combining (2.6) and (2.7), we get

$$(2.8)\quad \operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} = 1 - 4\,\beta(1-\alpha)\frac{z\,\omega'(z)}{(1+(2\beta-1)\,\omega(z))\,(1+(2\alpha\beta-1)\,\omega(z))}.$$

It is well known [7] that if $\omega \in B$, then for all $z \in E$,

$$(2.9)\qquad \left|z\,\omega'(z) - \omega(z)\right| \leqq \frac{|z|^2 - |\omega(z)|^2}{1-|z|^2}.$$

Equation (2.8) yields in conjunction with (2.9),

$$(2.10)\quad \operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} \geqq 1 + \frac{1}{\beta(1-\alpha)}\left[\operatorname{Re}\left\{(2\beta-1)\,p(z) + \frac{2\alpha\beta-1}{p(z)}\right\}\right] - $$
$$- \frac{2(\beta+\alpha\beta-1)}{\beta(1-\alpha)} - \frac{r^2|(2\beta-1)\,p(z)-(2\alpha\beta-1)|^2 - |1-p(z)|^2}{\beta(1-\alpha)\,(1-r^2)\,|p(z)|},$$

where $r = |z|$, $z \in E$.

Noting that the transformation (2.5) maps the disc $|\omega(z)| \leqq r$ onto the disc $|\omega(z) - a| < d$, where

$$a = \frac{1-(2\beta-1)\,(2\alpha\beta-1)\,r^2}{1-(2\beta-1)^2\,r^2}, \quad d = \frac{2\beta(1-\alpha)\,r}{1-(2\beta-1)^2\,r^2},$$

we set $p(z) = a + u + iv$ and $R = |p(z)|$ in (2.10). Taking $M(u, v)$ as the expression on the right hand side of (2.10), we get

$$(2.11)\qquad M(u,v) = \frac{1}{\beta(1-\alpha)}\left[(2-\beta-3\alpha\beta) + (2\beta-1)\,(a+u) + \right.$$
$$\left. + \frac{(2\alpha\beta-1)\,(a+u)}{R^2} - \frac{(1-(2\beta-1)^2\,r^2)}{1-r^2}\,\frac{(d^2-u^2-v^2)}{R}\right].$$

By differentiating (2.11) partially with respect to v, we obtain

$$\frac{\partial M(u,v)}{\partial v} = \frac{vR^{-4}N(u,v)}{\beta(1-\alpha)},$$

where

$$N(u, v) = 2(1 - 2\alpha\beta)(a + u) + \frac{(1 - (2\beta - 1)^2 r^2)(d^2 - u^2 - v^2) R}{1 - r^2} + $$

$$+ \frac{2(1 - (2\beta - 1)^2 r^2) R^3}{1 - r^2}.$$

It is easily seen that $N(u, v) > 0$, and so the minimum of $M(u, v)$ on every chord $u =$ constant is attained on the diameter $v = 0$. Taking $v = 0$ in (2.11), we get

$$L(R) \equiv M(R, 0) = \frac{2 - \beta - 3\alpha\beta}{\beta(1 - \alpha)} + \frac{2}{\beta(1 - \alpha)(1 - r^2)} \cdot$$

$$\cdot \{\beta(1 - (2\beta - 1) r^2) R + \alpha\beta(1 - (2\alpha\beta - 1) r^2) R^{-1} - a(1 - (2\beta - 1)^2 r^2)\},$$

where $a - d \leqq R \leqq a + d$. Now it is easy to see that the absolute minimum of $L(R)$ in $(0, \infty)$ is attained at

$$R_0 = \left(\frac{\alpha(1 - (2\alpha\beta - 1) r^2)}{1 - (2\beta - 1) r^2}\right)^{1/2}, \tag{2.12}$$

and equals

$$L(R_0) = 1 + \frac{2\mu(r, \alpha, \beta)}{(1 - \alpha)(1 - r^2)}, \tag{2.13}$$

where

$$\mu(r, \alpha, \beta) = 2(\alpha(1 - (2\beta - 1) r^2)(1 - (2\alpha\beta - 1) r^2))^{1/2} - $$
$$- (1 + \alpha) + (4\alpha\beta - \alpha - 1) r^2.$$

We note that $R_0 < a + d$. However, R_0 may not always be greater than $a - d$. Hence, when $R_0 \in (0, a - d]$, the minimum of $L(R)$ is attained at

$$R_1 = a - d = \frac{1 + (2\alpha\beta - 1) r}{1 + (2\beta - 1) r}, \tag{2.14}$$

and is equal to

$$L(R_1) = 1 - \frac{4\beta(1 - \alpha) r}{(1 + (2\beta - 1) r)(1 + (2\alpha\beta - 1) r)}. \tag{2.15}$$

The two minima given by (2.13) and (2.15) coincide for such values of α, β $(0 \leqq \alpha < 1, 0 < \beta \leqq 1)$ for which $R_0 = R_1$, which implies (2.1). We thus conclude that

$$\operatorname{Re} \frac{z f'(z)}{f(z)} \geqq \begin{cases} 1 + \dfrac{2\mu(r, \alpha, \beta)}{(1 - \alpha)(1 - r^2)}, & R_0 \geqq R_1, \\ 1 - \dfrac{4\beta(1 - \alpha) r}{(1 + (2\beta - 1) r)(1 + (2\alpha\beta - 1) r)}, & R_0 \leqq R_1. \end{cases} \tag{2.16}$$

Therefore the function f is starlike if

$$2\mu(r, \alpha, \beta) + (1-\alpha)(1-r^2) > 0\,, \quad R_0 \geqq R_1\,, \tag{2.17}$$

$$(1 + (2\beta - 1)\,r)(1 + (2\alpha\beta - 1)\,r) - 4\beta(1-\alpha)\,r > 0\,, \quad R_0 \leqq R_1\,. \tag{2.18}$$

Now it is easy to see that (2.18) and (2.17) are satisfied, respectively, for $|z| < r_1$ and $|z| < r_2$, where r_1 and r_2 are the smallest positive roots of the equations (2.2) and (2.3). This completes the proof of the theorem.

The functions given by

$$f_1(z) = z\left\{\frac{1 + (2\alpha\beta - 1)\,z}{1 + (2\beta - 1)\,z}\right\}^2,$$

$$f_2(z) = z\left\{\frac{1 - 2\alpha\beta bz + (2\alpha\beta - 1)\,z^2}{1 - 2\beta bz + (2\beta - 1)\,z^2}\right\},$$

where b is determined by the relation

$$\frac{1 - 2\alpha\beta br + (2\alpha\beta - 1)\,r^2}{1 - 2\beta br + (2\beta - 1)\,r^2} = R_0 = \left\{\frac{\alpha(1 - (2\alpha\beta - 1)\,r^2)}{1 - (2\beta - 1)\,r^2}\right\}^{1/2},$$

show, respectively, that the bounds in $|z|$ for (i) and (ii) are sharp for all admissible values of α, β $(0 \leqq \alpha < 1,\ 0 < \beta \leqq 1)$.

The following Corollary arises from Theorem 1 by an easy computation.

Corollary. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *analytic in E, satisfy the inequality*

$$\operatorname{Re}\sqrt{(f(z)/z)} > \frac{1}{2\varrho}, \quad (\varrho > 1/2)$$

for all z in E. Let $\varrho_0 > 1/2$ *denote the smallest positive root of the equation*

$$32\varrho^3 - 104\varrho^2 + 98\varrho - 27 = 0\,.$$

Then

(i) *for* $1/2 < \varrho \leqq \varrho_0$, *f is starlike in*

$$|z| < \left\{\frac{8\sqrt{(4\varrho - 2)} - (6\varrho + 5)}{18\varrho - 17}\right\}^{1/2},$$

(ii) *for* $\varrho \geqq \varrho_0$, *f is starlike in*

$$|z| < \frac{\sqrt{(20\varrho^2 - 28\varrho + 9)} - (4\varrho - 3)}{2(\varrho - 1)}\,.$$

These bounds for $|z|$ *are sharp for the functions given by*

$$f_1(z) = z\left\{\frac{1 + (1/\varrho - 1)\, z}{1 + z}\right\}^2,$$

$$f_2(z) = z\left\{\frac{1 - (1/\varrho)\, bz + (1/\varrho - 1)\, z^2}{1 - 2bz + z^2}\right\}^2,$$

where b is determined by the equation

$$\frac{1 - (1/\varrho)\, br + (1/\varrho - 1)\, r^2}{1 - 2br + r^2} = \left\{\frac{1 - (1/\varrho - 1)\, r^2}{2\varrho(1 - r^2)}\right\}^{1/2}.$$

Goel [3] has proved the above result for the case of $\varrho \geqq 1$.

3. COEFFICIENT ESTIMATES

Theorem 2. *Let* $f(z) = z + a_2z^2 + \ldots$ *be in* $S(\alpha, \beta)$. *Then*

$$|a_n| \leqq 4\beta(1 - \alpha)\{1 - 2\beta(1 - \alpha) + \beta(1 - \alpha)\, n\}, \quad (n \geqq 2) \tag{3.1}$$

for all values of α, β $(0 \leqq \alpha < 1, 0 < \beta \leqq 1)$. *The result is sharp.*

Proof. Letting

$$p(z) = \frac{1 + (2\alpha\beta - 1)\, \omega(z)}{1 + (2\beta - 1)\, \omega(z)} = 1 + p_1z + \ldots, \tag{3.2}$$

we may rewrite (1.3) as

$$z + z_2z^2 + \ldots = z[1 + p_1z + \ldots]^2.$$

Equating the coefficients of z^{2m} and z^{2m+1}, we get

$$a_{2m+1} = p_m^2 + 2p_{2m} + 2\sum_{r+s=2m} p_rp_s, \tag{3.3}$$

and

$$a_{2m+2} = 2p_{2m+1} + 2\sum_{r+s=2m+1} p_rp_s, \quad (m = 1, 2, \ldots). \tag{3.4}$$

Further, (3.2) gives

$$(2\beta(1 - \alpha) + \sum_{k=1}^{\infty}(2\beta - 1)\, p_kz^k)\, \omega(z) = -\sum_{k=1}^{\infty} p_kz^k. \tag{3.5}$$

We observe that the coefficient p_n on the right of (3.5) depends only on $p_1, p_2, \ldots, p_{n-1}$ on the left of (3.5). Hence for $n \geqq 1$, it follows that

$$\{2\beta(1 - \alpha) + \sum_{k=1}^{n-1}(2\beta - 1)\, p_kz^k\}\, \omega(z) = -\sum_{k=1}^{n} p_kz^k - \sum_{k=n+1}^{\infty} d_kz^k,$$

where $\sum_{k=n+1}^{\infty} d_k z^k$ converges in E. Then

$$(3.6) \qquad \left|2\beta(1-\alpha) + \sum_{k=1}^{n-1}(2\beta-1)\, p_k z^k\right| \geqq \left|\sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} d_k z^k\right|.$$

Squaring both sides of (3.6), integrating round $|z| = r$, $0 < r < 1$, and finally taking the limit as $r \to 1$, we get

$$4\beta^2(1-\alpha)^2 + \sum_{k=1}^{n-1}(2\beta-1)^2 |p_k|^2 \geqq |p_n|^2 + \sum_{k=1}^{n-1}|p_k|^2.$$

Simplifying and using the relation $0 < \beta \leqq 1$, we obtain

$$(3.7) \qquad |p_n| \leqq 2\beta(1-\alpha), \quad (n \geqq 1).$$

Using (3.7) in (3.3) and (3.4), we obtain

$$(3.8) \qquad |a_{2m+1}| \leqq 4\beta(1-\alpha) + 8\beta^2(1-\alpha)^2 \left(\frac{2m+1-2}{2}\right),$$

$$(3.9) \qquad |a_{2m+2}| \leqq 4\beta(1-\alpha) + 8\beta^2(1-\alpha)^2 \left(\frac{2m+2-2}{2}\right).$$

Combining (3.8) and (3.9) we have

$$|a_n| \leqq 4\beta(1-\alpha) + 8\beta^2(1-\alpha)^2 \left(\frac{n-2}{2}\right),$$

which yields (3.1).

The equality in (3.1) holds for the function given by

$$f(z) = z\left[\frac{1-(2\alpha\beta-1)\,z}{1-(2\beta-1)\,z}\right]^2.$$

Remark. Setting $\alpha = 1/2$ and $\beta = 1$ in Theorem 2, we get $|a_n| \leqq n$, $(n \geqq 2)$. This result was obtained by Dvořák [2]. Further, replacing α by $1/(2\alpha)$ and setting $\beta = 1$ in Theorem 2 we have a result obtained in [3].

References

[1] *P. L. Duren, G. E. Schober:* On a class of schlicht functions. Michigan Math. J. *18* (1971), 353—356.

[2] *O. Dvořák:* Über Schlichte Funktionen, I. Čas. pěst. mat. *92* (1967), 162—189.

[3] *R. M. Goel:* On a class of analytic functions. J. Australian Math. Soc. *20* (1975), 46—53.

[4] *A. Marx:* Untersuchungen über schlichte Abbildungen. Math. Ann. *107* (1932), 40—67.

[5] *M. O. Reade, T. Umezawa:* An inequality for univalent functions due to Dvořák. Čas. pěst. mat. *96* (1971), 265—267.

[6] *M. S. Robertson:* On the theory of univalent functions. Annal of Math. *37* (1936), 374—408.

[7] *V. Singh, R. M. Goel:* On radii of convexity and starlikeness of some classes of functions. J. Math. Soc. Japan *29* (1971), 323—339.

[8] *E. Strohhäcker:* Beitrage zur Theorie der schlichten Funktionen. Math. Z. *37* (1933), 356—380.

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