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CONNECTIONS ON THE SECOND TANGENT BUNDLE

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In [2], the author described a construction of a prolongation $F(\Gamma, \Lambda)$ of a (generalized) connection Γ on a fibred manifold $\pi : Y \to M$ with respect to an arbitrary prolongation functor F of order (r, s) (from the category $\mathcal{FM}_0$ of fibred manifolds with diffeomorphisms to the category $2\mathcal{FM}$ of 2-fibred manifolds) by means of an auxiliary linear r-th order connection Λ on the base manifold M. In the special case of a trivial fibred manifold $id : M \to M$, we obtain in this way a connection $F(\Lambda) := F(0, \Lambda)$ on FM, where 0 denotes the unique connection on $id : M \to M$.

A natural question arises, when a connection Σ on FM is of the form $\Sigma = F(\Lambda)$ for a suitable higher order linear connection Λ on M. We shall not discuss this problem in full generality, but we its solution for the functor $F = TT$, the iteration of the tangent functor T.

A prolongation functor F (for the definition, see [2], 89–90) from the category $\mathcal{M}$ of smooth manifolds and mappings to the category $\mathcal{FM}$ of smooth fibred manifolds is said to be *of order r*, if for any two maps $f, g : M \to N$, $j^r_x f = j^r_x g$ implies $Ff/F_x M = Fg/F_x M$, where $F_x M$ denotes the fibre over $x \in M$ and j^r_x means the r-jet at x. Thus for any two manifolds M, N, an r-th order functor F induces an associated map

$$F_{M,N} : FM \oplus J^r(M, N) \to FN\,,$$

where $\oplus$ denotes the Whitney sum of fibred manifolds $\pi : FM \to M$ and $\alpha : J^r(M, N) \to M$, with α being the source jet projection.

The construction of the connection $F(\Lambda)$ for a functor $F : \mathcal{M} \to \mathcal{FM}$ of order r can be described via its lifting map (see [4]) $\widetilde{F(\Lambda)} : FM \oplus TX \to TFM$. We define $\widetilde{F(\Lambda)}(z, v) = (F\zeta)(z)$ for $z \in F_x M$, $v \in T_x M$, $x \in M$, where ζ is a vector satisfying $\Lambda(v) = j^r_x \zeta$, and $F\zeta$ is its prolongation. In [4] it was proved that the value $(F\zeta)(z)$ of the prolonged field $F\zeta$ at $z \in F_x M$ depends only on $j^r_x \zeta$, and the induced map

$$FM \oplus J^r TM \to T(FM)$$

is smooth and linear with respect to $J^r TM$. We shall recall the proof here, and derive the coordinate form of $F(\Lambda)$.

Let (x^i, y^p) be a local fibre coordinate system on FM such that x^i are local coordinates on M. The flow of the vector field $F\zeta$ is defined by

$$\exp t(F\zeta) := F(\exp t\zeta) .$$

In local coordinates, $\zeta = \zeta^i(x) . \partial/\partial x^i$, $\exp t\zeta = (\varphi_t^1(x), \dots, \varphi_t^m(x))$, $m = \dim M$, and $\partial\varphi_t^i/\partial t = \zeta^i(x)$, $i = 1, \dots, m$. Let

$$(1) \qquad F_{M,M} : y^p = F^p(x^i, \bar{x}^i, \bar{x}^i_j, \dots, \bar{x}^i_{j_1 \dots j_r}, y^q)$$

be the coordinate expression of the associate map $F_{M,M}$, where $\bar{x}^i_j, \dots, \bar{x}^i_{j_1 \dots j_r}$ are the induced coordinates on $J^r(M, M)$. Then $F_{\varphi_t} = (\varphi_t^i, F^p \circ \varphi_t)$. The coefficients of $F\zeta$ with respect to the basis $\partial/\partial x^i$, $\partial/\partial y^p$ of TFM are $\partial\varphi_t^i/\partial t$ and $\partial(F^p \circ \varphi^r)/\partial t$, respectively, so that

$$(2) \qquad F\zeta = \zeta^i(x) . \frac{\partial}{\partial x^i} + \frac{\partial(F^p \circ \varphi_t)}{\partial t} . \frac{\partial}{\partial y^p} .$$

Since

$$\frac{\partial(F^p \circ \varphi_t)}{\partial t} = \frac{\partial F^p}{\partial \bar{x}^i} . \frac{\partial \varphi_t^i}{\partial t} + \frac{\partial F^p}{\partial \bar{x}^i_j} . \frac{\partial}{\partial t}\left(\frac{\partial \varphi_t^i}{\partial x^j}\right) + \dots + \frac{\partial F^p}{\partial \bar{x}^i_{j_1 \dots j_r}} . \frac{\partial}{\partial t}\left(\frac{\partial^r \varphi_t^i}{\partial x^{j_1} \dots \partial x^{j_r}}\right)$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial^k \varphi_t^i}{\partial x^{j_1} \dots \partial x^{j_k}}\right) = \frac{\partial^k}{\partial x^{j_1} \dots \partial x^{j_k}}\left(\frac{\partial \varphi_t^i}{\partial t}\right) = \frac{\partial^k \zeta^i}{\partial x^{j_1} \dots \partial x^{j_k}} ,$$

we have

$$(3) \quad F\zeta = \zeta^i \frac{\partial}{\partial x^i} + \left(\frac{\partial F^p}{\partial \bar{x}^i} . \zeta^i + \frac{\partial F^p}{\partial \bar{x}^i_j} . \frac{\partial \zeta^i}{\partial x^j} + \dots + \frac{\partial F^p}{\partial \bar{x}^i_{j_1 \dots j_r}} . \frac{\partial^r \zeta^i}{\partial x^{j_1} \dots \partial x^{j_r}}\right) . \frac{\partial}{\partial y^p} .$$

Any linear connection $\Lambda : TM \to J^rTM$ of order r on M can be expressed in the form

$$\Lambda : \begin{cases} \zeta^i_j = \Gamma^i_{kj}(x) . \zeta^k , \\ \quad \vdots \\ \zeta^i_{j_1 \dots j_r} = \Gamma^i_{kj_1 \dots j_r}(x) . \zeta^k , \end{cases}$$

where ζ^i are the natural fibre coordinates on TM, and $\zeta^i_j, \dots, \zeta^i_{j_1 \dots j_r}$ are the induced coordinates on J^rTM. Then the equations of $F(\Lambda) : FM \to J^1FM$ are

$$(4) \qquad F(\Lambda) : y^p_i = \Gamma^i_{ij_1 \dots j_r}(x) . \frac{\partial F^p}{\partial \bar{x}^i_{j_1 \dots j_r}} + \dots + \Gamma^i_{ij}(x) . \frac{\partial F^p}{\partial \bar{x}^i_j} + \frac{\partial F^p}{\partial \bar{x}^i} .$$

Before discussing the case $F = TT$, we introduce some useful notions and deduce some auxiliary results.

Let $F, G : \mathcal{M} \to \mathcal{FM}$ be two prolongation *functors*. We say that G is an *extension* of F, if for any manifold M, FM is a fibred submanifold of GM, and for any map

$f : M \to N$, the following diagram commutes:

$$
\begin{array}{ccc}
GM & \xrightarrow{Gf} & GN \\
\uparrow & & \uparrow \\
FM & \xrightarrow{Ff} & FN \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & N
\end{array} \tag{5}
$$

A vector field ζ on a manifold Y is called *reducible* to a submanifold Z ($\varkappa : Z \to Y$ being the imbedding), if there exists a vector field η on Z such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\zeta} & TY \\
{\scriptstyle \varkappa}\uparrow & & \uparrow{\scriptstyle T\varkappa} \\
Z & \xrightarrow{\eta} & TZ
\end{array} .
$$

Lemma 1. *Let G be an extension of F, and let ζ be a vector field on M. Then the vector field $G\zeta$ on GM si reducible to FM, and $F\zeta$ is the corresponding reduction.*

Proof. It suffices to apply the diagram (4) to the flow of ζ.

A connection Γ on a fibred manifold $\pi : Y \to M$ is called *reducible* to a fibred submanifold $Z \xrightarrow{\varkappa} Y$, if there exists a connection Σ on Z such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\Gamma} & J^1Y \\
{\scriptstyle \varkappa}\uparrow & & \uparrow{\scriptstyle J^1\varkappa} \\
Z & \xrightarrow{\Sigma} & J^1Z
\end{array} .
$$

The connection $\Sigma = \Gamma/Z$ will be called the *reduction* of Γ to Z.

Lemma 2. *Let G be an extension of a prolonagation functor F, and let s and r, $s \geqq r$, be the orders of G and F, respectively. For any manifold M and any linear connection Λ of order s on M, $\Lambda : TM \to J^sTM$, the prolonged connection $G(\Lambda)$ is reducible to $FM \subset GM$, and the corresponding reduction is $G(\Lambda)/FM = F(\hat{\Lambda})$, where $\hat{\Lambda} = j^s_r \circ \Lambda$ (j^s_r denotes the jet projection $J^sTM \to J^rTM$).*

Proof. This follows directly from Lemma 1.

Given two fibred manifolds $U \xrightarrow{q} Y$, $Y \xrightarrow{p} X$, the quintuple $U \xrightarrow{q} Y \xrightarrow{p} X$ is called a *2-fibred manifold.*

A prolongation functor G (of order s) is called a *prolongation* of a functor F (of order r, $r \leqq s$), if for any manifold M, $GM \to FM \to M$ is a 2-fibred manifold, and for any map $f : M \to N$, the following diagram commutes:

$$
(12)\qquad
\begin{array}{ccc}
GM & \xrightarrow{Gf} & GN \\
\downarrow & & \downarrow \\
FM & \xrightarrow{Ff} & FN \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & N
\end{array}\ .
$$

A vector *field* ζ on $\pi : Y \to X$ is said to be *projectable* (or *projectable over* η), if there exists a vector field η on the base manifold X such that $T\pi \circ \zeta = \eta \circ \pi$. In local fibre coordinates x^i, y^p on Y, the expression of a projectable vector field ζ is $\zeta(x, y) = \eta^i(x) \, . \, \partial/\partial x^i + \zeta^p(x, y) \, . \, \partial/\partial y^p$, where $\eta = \eta^i(x) \, . \, \partial/\partial x^i$ is the underlying vector field.

Lemma 3. *If G is a prolongation of F and ζ is a vector field on a manifold M, then the prolonged vector field $G\zeta$ is projectable over $F\zeta$.*

The proof is similar to the proof of Lemma 1.

A connection Γ on a 2-fibred manifold $U \xrightarrow{q} Y \xrightarrow{p} X$ is called *projectable* (more precisely *q-projectable over* Σ), if there exists a connection Σ on Y such that the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{\Gamma} & J^1U \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle J^1q} \\
Y & \xrightarrow{\Sigma} & J^1q \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \alpha} \\
X & & X
\end{array}\ .
$$

In local fibre coordinates x^i, y^p, u^α on U, the equations of Γ and Σ are

$$
\Gamma : \begin{cases} y^p_i = F^p_i(x, y) \\ z^\alpha_i = G^\alpha_i(x, y, u) \, ; \end{cases} \qquad \Sigma : y^p_i = F^p_i(x, y) \, .
$$

As a direct consequence of Lemma 3 we obtain

Lemma 4. *If G (of order s) is a prolongation of F (of order $r \leqq s$) then for any manifold M and any linear connection $\Lambda : TM \to J^sTM$, the connection $G(\Lambda)$ is projectable over $F(\hat{\Lambda})$, where $\hat{\Lambda} = j^s_r \circ \Lambda$.*

A 2-fibred manifold $U \xrightarrow{q} Y \xrightarrow{p} X$ is called a *semi-vector bundle,* if $U \xrightarrow{q} Y$ is a vector bundle. If $U \xrightarrow{q} Y \xrightarrow{p} X$ is a semi-vector bundle, then obviously $J^1U \xrightarrow{J^1q} J^1Y \xrightarrow{\alpha} X$ is a semi-vector bundle, too. A projectable connection $\Gamma : U \to \to J^1U$ over a connection $\Sigma : Y \to J^1Y$ on a semi-vector bundle $U \to Y \to X$ induces for any $y \in Y$ a map $\Gamma/U_y : U_y \to (J^1U)_{\Gamma(y)}$ of vector spaces, where U_y denotes the fibre over y. Γ is said to be *semi-linear*, if the maps Γ/U_y are linear for all $y \in Y$. In linear coordinates u^α on U, the equations of Γ are

$$\Gamma : \begin{cases} y_i^p = F_i^p(x, y)\,, \\ u_i^\alpha = G_{\beta i}^\alpha(x, y) \,.\, u^\beta\,. \end{cases}$$

Now let us turn our attention to the functor TT. Let $p_N : TN \to N$ denote the bundle projection of TN. For a given manifold M, choose a local coordinate system on TTM

$$x^i, \xi^i, X^i, \Xi^i \tag{14}$$

in the usual way, i.e. $\xi^i = \mathrm{d}x^i$ on TM and $X^i = \mathrm{d}x^i$, $\Xi^i = \mathrm{d}\xi^i$ on TTM. On TTM, there exists a canonical involution $i_M : TTM \to TTM$, $i_M^2 = id$ (see [1]). In our coordinates, $i_M(x^j, \xi^j, X^j, \Xi^j) = (x^j, X^j, \xi^j, \Xi^j)$. Further, there are two projections $p_1 = p_{TM}$, $p_2 = T_{p_M}$ of TTM on TM, with the following coordinate expressions:

$$p_1(x^j, \xi^j, X^j, \Xi^j) = (x^j, \xi^j)\,,$$
$$p_2(x^j, \xi^j, X^j, \Xi^j) = (x^j, X^j)\,.$$

Obviously, $p_2 = p_1 \circ i_M$ and (TTM, TM, p_1, p_2) is a double fibred manifold in the sense of [2], p. 88.

Given any morphism $f : M \to N$ and a local coordinate system y^p, η^p, Y^p, H^p on TTN, chosen as above, the coordinate forms of the maps f, $Tf : TM \to TN$ and $TTf : TTM \to TTN$ are

$$TTf : \begin{cases} Tf : \begin{cases} f : y^p = f^p(x)\,, \\ \qquad \eta^p = \dfrac{\partial f^p}{\partial x^i} \,.\, \xi^i\,, \end{cases} \\ \qquad Y^p = \dfrac{\partial f^p}{\partial x^i} \,.\, X^i\,, \\ \qquad H^p = \dfrac{\partial^2 f^p}{\partial x^i \, \partial x^j} \,.\, \xi^i \,.\, X^j + \dfrac{\partial f^p}{\partial x^i} \,.\, \Xi^i\,. \end{cases}$$

Hence the functor TT (of the second order) is a prolongation of the first-order functor T.

Denote by KM the common kernel of both projections p_1 and p_2. KM is a fibred manifold over M, for which the space $K_xM = \{(x^i, 0, 0, \Xi^i)\}$ of all vertical vectors at 0 is the fibre over x, and p_1/KM is the projection. Clearly, $K_xM \approx T_xM$. Thus KM is a fibred submanifold of TTM, and $KM \approx TM$. For any $f : M \to N$, $y^p = f^p(x)$, define a map $Kf : KM \to KN$ by

$$Kf : \begin{cases} y^p = f^p(z)\,, \\ \eta^p = Y^p = 0\,, \\ H^p = \dfrac{\partial f^p}{\partial x^i} \,.\, \Xi^i\,. \end{cases}$$

Obviously, K is a functor isomorphic to T, and TT is an extension of K.

Theorem. *Let M be a smooth manifold, and let Γ be a (generalized) connection on TTM, i.e. $\Gamma : TTM \to J^1TTM$ is a smooth section. Then there exists on M a linear second-order connection $\Lambda : TM \to J^2TM$ such that $TT(\Lambda) = \Gamma$ iff the following conditions are satisfied:*

(A) *There exists a linear connection of the first order on M, $\bar{\Lambda} : TM \to J^1TM$, such that*

(i) *Γ is p_j-projectable over the connection $T(\bar{\Lambda})$ for $j = 1, 2$.*

(ii) *Γ is reducible to KM, the reduction being $\Gamma/KM = T(\bar{\Lambda})$.*

(iii) *Γ is semi-linear on the 2-fibred manifold $TTM \overset{p_j}{\to} TM \overset{p_M}{\to} M$ over $T(\bar{\Lambda})$ for $j = 1, 2$.*

(B) *Γ is invariant with respect to the canonical involution i_M on TTM, i.e. $J^1(i_M^{-1}) \circ \Gamma \circ i_M = \Gamma$.*

Proof. Let $\Gamma = TT(\Lambda)$. In local coordinates (14), the expression of Λ is of the form

$$\Lambda : \begin{cases} \xi^i_k = \Gamma^i_{jk}(x) \,.\, \xi^j\,, \\ \xi^i_{ljk} = \Gamma^i_{ljk}(x) \,.\, \xi^l\,. \end{cases}$$

The equations of $TT(\Lambda)$ are

$$TT(\Lambda) : \begin{cases} \xi^i_k = \Gamma^i_{kj}(x) \,.\, \xi^j\,, \\ X^i_k = \Gamma^i_{kj}(x) \,.\, X^j\,, \\ \Xi^i_l = \Gamma^i_{ljk}(x) \,.\, \xi^j \,.\, X^k + \Gamma^i_{lj}(x) \,.\, \Xi^j\,. \end{cases} \tag{15}$$

Setting $\bar{\Lambda} = j^2_1 \circ \Lambda$, i.e.

$$\bar{\Lambda} : \xi^i_k = \Gamma^i_{jk}(x) \,.\, \xi^j\,,$$

we have

$$T(\bar{\Lambda}) : \xi^i_k = \Gamma^i_{kj}(x) \,.\, \xi^j$$

and it is easy to see that the conditions (A) and (B) are satisfied.

Conversely, let us assume that Γ satisfies (A) and (B). Then the expression of a connection $\bar{\Lambda}$ from (A) is

$$\bar{\Lambda} : \xi^i_k = \bar{\Gamma}^i_{jk}(x) \,.\, \xi^j\,,$$

and

$$T(\bar{\Lambda}) : \xi^i_k = \bar{\Gamma}^i_{kj}(x) \,.\, \xi^j$$

is the connection conjugate to $\bar{\Lambda}$. According to (i),

$$\Gamma : \begin{cases} \xi^i_k = \bar{\Gamma}^i_{kj}(x) \, . \, \xi^j \, , \\ X^i_k = \bar{\Gamma}^i_{kj}(x) \, . \, X^j \, , \\ \Xi^i_k = G^i_k(x, \xi, X, \Xi) \, . \end{cases}$$

The reducibility condition (ii) implies that $G^i_k(x, 0, 0, \Xi) = \bar{\Gamma}^i_{kj}(x) \, . \, \Xi^j$. The condition (iii) implies the existence of functions $f^i_{ij}(x, X)$ and $g^i_{ij}(x, \xi)$ satisfying

$$\Xi^i_i = f^i_{ij}(x, X) \, . \, \xi^j + \bar{\Gamma}^i_{ij}(x) \, . \, \Xi^j \, ,$$

and

$$\Xi^i_i = g^i_{ij}(x, \xi) \, . \, X^j + \bar{\Gamma}^i_{ij}(x) \, . \, \Xi^j \, .$$

This yields

$$\Xi^i_i = \bar{\Gamma}^i_{ijk}(x) \, . \, \xi^j \, . \, X^k + \bar{\Gamma}^i_{ij}(x) \, . \, \Xi^j \, .$$

From (B) we finally deduce that the functions $\bar{\Gamma}^i_{ijk}$ are symmetric in j, k. Thus Γ is of the form (15), i.e. $\Gamma = TT(\bar{\Lambda})$. QED.

References

[1] *C. Godbillon:* Géometrie différentielle et mécanique analytique, Paris 1969.

[2] *I. Kolář:* Structure morphisms of prolongation functors, Math. Slovaca *30* (1980) No. 1, 83—93.

[3] *I. Kolář:* Connections in 2-fibred manifolds, Arch. Math. 1, Scripta Fac. Sci. Mat. UJEP Brunensis XVII: 23—30, 1981.

[4] *I. Kolář:* Prolongations of generalized connections, to appear in: Proceedings of Colloquium on Differential Geometry, Budapest 1979.

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