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LOCALLY TREE-LIKE GRAPHS

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1. INTRODUCTION

If v is a vertex of an undirected graph G, then the set of all vertices adjacent to v in G is called the neighbourhood of v in G and is denoted by $N_G(v)$ or simply by $N(v)$. If an undirected graph G (without loops and multiple edges) has the property that for each vertex v of G the neighbourhood $N(v)$ of v induces a connected subgraph of G, then G is called locally connected. In the papers [1], [2], [3], [4] these graphs were studied. Here we shall study a special case if locally connected graphs, namely, the locally tree-like graphs. (They were also studied in [1], not being thus called.)

An undirected graph G is called locally tree-like, if for each vertex v of G the subgraph of G induced by the neighbourhood $N(v)$ is a tree. (A graph consisting of one vertex is also considered a tree.)

For the investigation of locally tree-like graphs we shall use some results concerning hypergraphs. All graphs and hypergraphs are considered finite (with the unique exception at the end of the paper).

2. PREPARATORY CONSIDERATIONS ON HYPERGRAPHS

We shall prove some assertions on r-uniform hypergraphs. A hypergraph is called r-uniform, if all of its edges have the same cardinality equal to r. (For $r = 2$ it is a graph in the usual sense.) The vertex set and the edge set of a hypergraph H will be denoted by $V(H)$ and $E(H)$, respectively. We introduce notation for some properties of r-uniform hypergraphs. In the sequel r always denotes an integer greater than or equal to 2.

An r-uniform hypergraph H is said to have the property $\mathbf{P}_1$, if for any two edges E_1, E_2 of H there exists a finite sequence $F_0, F_1, \ldots, F_k$ of edges of H such that $F_0 = E_1$, $F_k = E_2$ and $|F_i \cap F_{i+1}| = r - 1$ for $i = 0, 1, \ldots, k - 1$. If v is a vertex of H, then by $H(v)$ we denote the subhypergraph of H whose edge set consists of all edges containing the vertex v and whose vertex set consists of all vertices which are contained in these edges. If $H(v)$ has the property $\mathbf{P}_1$ for each vertex v of H, we say that H has the property $\mathbf{P}_2$. If there exists a finite sequence $E_1, \ldots, E_k$ of edges of H such that $k \geqq 4$ and $|E_i \cap E_{i+1}| = r - 1$ for $i = 1, \ldots, k - 1$, $|E_k \cap E_1| = r - 1$ and $|E_i \cap E_{i+2}| \leqq r - 2$ for $i = 1, \ldots, k - 2$, $|E_{k-1} \cap E_1| \leqq r - 2$, $|E_k \cap E_2| \leqq r - 2$, we say that H has the property $\mathbf{P}_3$. If there exists a finite sequence $E_1, \ldots, E_k$ of edges of H such that $k \geqq 3$, $|E_i \cap E_{i+1}| = r - 1$ for $i = 1, \ldots, k - 1$, $|E_k \cap E_1| = r - 1$ and $|\bigcap_{i=1}^{k} E_i| = r - 2$, the hypergraph H is said to have the property $\mathbf{P}_4$.

The following theorems are generalizations of well-known results for graphs (i.e. for $r = 2$).

Lemma 2.1. *Let H be an r-uniform hypergraph with the property $\mathbf{P}_1$ and with at least 2 edges. Then there exists an edge E of H with the property that the hypergraph obtained from H by deleting E has the property $\mathbf{P}_1$.*

Proof. Let G be the graph whose vertices are the edges of H and in which two vertices are adjacent if and only if their intersection (as of sets) has cardinality $r - 1$. Evidently H has the property $\mathbf{P}_1$ if and only if G is connected. It is well-known that in any connected graph there is a vertex whose deleting does not violate the connectedness of this graph. The edge of H corresponding to such a vertex of G is the required edge.

Theorem 2.1. *Let H be an r-uniform hypergraph without isolated vertices and with the property $\mathbf{P}_1$. Let the number of vertices of H be n. Then H has at least $n - r + 1$ edges.*

Proof. The assertion can be expressed in other words so that if H has m edges, then it has at most $m + r - 1$ vertices. We shall proceed by induction on m. For $m = 1$ the assertion is evidently true. Let m_0 be an integer, $m_0 \geqq 2$, and suppose that the assertion is true for $m = m_0 - 1$. Let H be an r-uniform hypergraph with the property $\mathbf{P}_1$, with m_0 edges and without isolated vertices. Let E be an edge of H with the property that the hypergraph obtained from H by deleting E has the property $\mathbf{P}_1$. If the mentioned hypergraph has isolated vertices, we delete them and denote the hypergraph thus obtained by H'. According to the induction hypothesis the hypergraph H' has at most $m_0 + r - 2$ vertices. As H has the property $\mathbf{P}_1$, the edge E has $r - 1$ common vertices with some edge of H' and thus it contains at most one vertex not belonging to H'. As H has no isolated vertices, any one of its vertices belongs to H' or to E and thus the number of vertices of H is at most $m_0 + r - 1$.

Theorem 2.2. *Let H be an r-uniform hypergraph without isolated vertices, with the property $\mathbf{P}_1$, with n vertices and with $n - r + 1$ edges. Then H has not the properties $\mathbf{P}_3$ and $\mathbf{P}_4$.*

Proof. Let H be an r-uniform hypergraph without isolated vertices, with the property $\mathbf{P}_1$ and with n vertices, and suppose that H has the property $\mathbf{P}_3$. Let $E_1, \ldots, E_k$ be the edges of H fulfilling the condition from the definition of $\mathbf{P}_3$. Let H_0 be the subhypergraph of H formed by these edges and by all vertices contained in them. As $|E_1 \cap E_2| = |E_2 \cap E_3| = r - 1$, $|E_1 \cap E_3| \leqq r - 2$, any vertex of E_2 belongs either to E_1, or to E_3. The hypergraph H_0' obtained from H_0 by deleting E_2 has the same number of vertices as H_0. On the other hand, H_0' has the property $\mathbf{P}_1$, therefore it has at most $k + r - 2$ vertices. If $H = H_0$, then $n \leqq k + r - 2$ and the number of edges of H is $k \geqq n - r + 2 > n - r + 1$. If $H \neq H_0$, let l be the number of edges belonging to H and not belonging to H_0. We may use the induction according to l analogously as in the proof of Theorem 2.1 to prove the assertion that $n \geqq m + r$, where m is the number of edges of H. This implies our assertion on $\mathbf{P}_3$. Now let again H be r-uniform without isolated vertices, with the property $\mathbf{P}_1$ and with n vertices. Suppose that H has the property $\mathbf{P}_4$. Let $E_1, \ldots, E_k$ be the edges of H fulfilling the condition from the definition of $\mathbf{P}_4$. If H_0 is defined analogously to the preceding case, then evidently H_0 has $k + r - 2$ vertices. The rest of the proof is the same as in the preceding case.

Theorem 2.3. *Let H be an r-uniform hypergraph with n vertices, $n - r + 1$ edges, without isolated vertices and with the property $\mathbf{P}_1$. Then H has the property $\mathbf{P}_2$.*

Proof. Let H fulfil the conditions of the theorem. Suppose that H has not the property $\mathbf{P}_2$. Then there exists a vertex v of H such that the subhypergraph $H(v)$ of H consisting of all edges of H which contain v and all vertices which are contained in these edges has not the property $\mathbf{P}_1$. Let G be the graph whose vertices are edges of $H(v)$ and in which two vertices are adjacent if and only if their intersection has cardinality $r - 1$. As $H(v)$ has not the property $\mathbf{P}_1$, the graph G is disconnected. Let $H_1(v), \ldots, H_k(v)$ be its connected components; we have $k \geqq 2$. We delete the vertex v from H and add k new vertices $v_1, \ldots, v_k$. Each edge of $H_i(v)$ $(i = 1, \ldots, k)$ will contain v_i instead of v. The hypergraph H' thus obtained has the property $\mathbf{P}_1$, its number of vertices is $n + k - 1$ and its number of edges is $n - r + 1$. But according to Theorem 2.1 it should have at least $n + k - r$ edges, thus we have a contradiction.

Lemma 2.2. *Let H be an r-uniform hypergraph with $n \geqq r + 1$ vertices and $n - r + 1$ edges, without isolated vertices and with the property $\mathbf{P}_1$. Then there exist at least two vertices of H, each of which is contained only in one edge of H.*

Proof. Let H fulfil the conditions. Choose an edge E_1 in H and construct a sequence $E_1, E_2, \ldots$ of edges of H with the property that $|E_i \cap E_{i+1}| = r - 1$ and $|E_i \cap E_{i+2}| \leqq r - 2$ for each i. No edges can be repeated in this sequence, because according to Theorem 1.2 the hypergraph H has not the property $\mathbf{P}_3$. As H is finite, our sequence must be also finite and thus there exists an edge E_k which is the last one. The set $E_k - E_{k-1}$ contains exactly one element v. Consider the hypergraph $H(v)$ used in the proof of Theorem 2.3. One edge of $H(v)$ is E_k; if $H(v)$ contains more than one edge, it must contain an edge E^* such that $|E^* \cap E_k| = r - 1$, because according to Theorem 1.3 it has the property $\boldsymbol{P}_1$. If $|E^* \cap E_{k-1}| = r - 1$, then $|E_{k-1} \cap \cap E_k \cap E^*| = r - 2$ and H has the property $\boldsymbol{P}_4$, which is a contradiction. If $|E^* \cap E_{k-1}| \leqq r - 2$, then E^* can be added to our sequence as E_{k+1}, which is a contradiction with the assumption that E_k 's the last term of this sequence. Thus v is contained only in the edge E_k. Now we can construct a new sequence $E'_1, E'_2, \ldots$ with the described properties, taking $E'_1 = E_k$. As $n \geqq r + 1$, the number of edges of H is greater than 1 and our sequence has more than one term (because of $\boldsymbol{P}_1$); the last term E'_l of this sequence is other than E_k and the vertex $w \in E'_l - E'_{l-1}$ is another vertex with the required properties.

Theorem 2.4. *Let $\mathfrak{T}_r$ be the class of all finite r-uniform hypergraphs H without isolated vertices, with the property $\boldsymbol{P}_1$ and such that $|V(H)| - |E(H)| = r - 1$. The class $\mathfrak{T}_r$ can be described in the following way:*

(i) *An r-uniform hypergraph with r vertices and one edge belongs to $\mathfrak{T}_r$.*

(ii) *Let $H_0 \in \mathfrak{T}_r$. Let H be a hypergraph obtained from H_0 by adding one vertex and one edge containing the new vertex and arbitrary $r - 1$ vertices which all belong to some edge of H_0. Then $H \in \mathfrak{T}_r$.*

(iii) *The class $\mathfrak{T}_r$ contains no hypergraphs except those described in* (i) *and* (ii).

Proof. (i) This is evident.

(ii) As the difference between the number of vertices and the number of edges of H_0 is equal to $r - 1$, evidently the same holds for H. For any two edges of H_0 there exists a sequence from the definition of $\boldsymbol{P}_1$ and it is also in H. Let E be the edge of H not belonging to H_0, let E' be the edge of H_0 which contains $r - 1$ vertices of E. The required sequence for E and an arbitrary edge E'' of H_0 is obtained by constructing such a sequence for E' and E'' and adding E before its first term. Thus H has also the property $\boldsymbol{P}_1$.

(iii) Let $H \in \mathfrak{T}_r$. If H has only one edge, then it is the hypergraph from (i). If H has more than one edge, then according to Lemma 2.2 there exists a vertex v of H which belongs only to one edge E. Let H_0 be the hypergraph obtained from H by deleting the vertex v and the edge E. As the difference between the number of vertices and the number of edges of H is equal to $r - 1$, the same holds for H_0. In any sequence from the definition of $\boldsymbol{P}_1$ the edge E can be evidently only the first term or the last one, therefore after deleting E all these sequences for pairs of edges different preserved and H_0 has the property $\boldsymbol{P}_1$. Hence H was obtained from H_0 according to (ii).

Now we show interrelations of the results from the preceding section with locally tree-like graphs. In the preceding section we have considered r-uniform hypergraphs generally for $r \geqq 3$. Here we shall apply the particular case $r = 3$.

Let H be a 3-uniform hypergraph. By $G(H)$ we denote the graph whose vertex set is that of H and in which two vertices are adjacent if and only if there exists an edge of H containing both of them.

The symbol K_2 denotes the complete graph with 2 vertices.

Theorem 3.1. *A connected undirected graph G is locally connected if and only if either $G \cong K_2$, or $G \cong G(H)$ for some 3-uniform hypergraph H with the properties $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ and without isolated vertices.*

Proof. The case $G \cong K_2$ is trivial. Let G be connected and locally connected with more than two vertices. Let e be an edge of G, let u, v be its end vertices. If u has the degree 1, then u is an isolated vertex in the graph induced by $N(v)$; this is possible only if $N(v)$ has only one vertex and the degree of v is also 1. As G is connected, we have $G \cong K_2$ and this is a contradiction with the assumption that G has more than two vertices. If u has a degree greater than 1, then the subgraph induced by $N(u)$ has more than one vertex and, as it is connected, there exists a vertex w of this graph which is adjacent to v; the vertices u, v, w form a triangle containing the edge e. We have proved that each edge of G is contained in a triangle. Let H be the 3-uniform hypergraph whose vertex set is that of G and in which any three vertices form an edge if and only if they form a triangle in G; then evidently $G \cong G(H)$ and H has no isolated vertices. It remains to prove that H has the properties $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. Let E_1, E_2 be two edges of H such that $E_1 \cap E_2 \neq \emptyset$, let $u \in E_1 \cap E_2$. Let v_{11}, v_{12} (or v_{21}, v_{22}) be the vertices of E_1 (or E_2, respectively) distinct from u. Then in G there are edges $e_1 = v_{11}v_{12}$, $e_2 = v_{21}v_{22}$. As the subgraph of G induced by $N(u)$ is connected, there exists a sequence $f_0, f_1, \ldots, f_k$ of edges of this graph such that $f_0 = e_1, f_k = e_2$ and f_i, f_{i+1} have a common end vertex for $i = 0, 1, \ldots, k - 1$. For each $i = 0, 1, \ldots, k$ the edge f_i and the edges joining its end vertices with u form a triangle; let F_i be the edge of H corresponding to it. Then $F_0, F_1, \ldots, F_k$ form a sequence from the definition of $\boldsymbol{P}_1$. The subhypergraph of H consisting of edges containing u and of vertices contained in them has the property $\boldsymbol{P}_1$; as u was chosen arbitrarily, H has the property $\boldsymbol{P}_2$. Now let E_1, E_2 be arbitrary two edges of H. If $E_1 \cap E_2 \neq \emptyset$, we have already proved that for them there exists a sequence of edges with the required properties. Thus let $E_1 \cap E_2 = \emptyset$. As G is connected, there exists a simple path P in G connecting a vertex $u_1 \in E_1$ with a vertex $u_2 \in E_2$ and such that no inner vertex of P belongs to $E_1 \cup E_2$. Let the edges of P be $e'_1, \ldots, e'_k$, let e'_1 be incident with u_1 and e'_k with u_2. For $i = 1, \ldots, k$ let E'_i be an arbitrary edge of H containing both the end vertices of e'_i. Further, put $E'_0 = E_1$, $E'_{k+1} = E_2$. We have $E'_i \cap E'_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, k$, thus the required sequence $\mathscr{S}_i$ exists for E'_i and E'_{i+1}. We take the sequence formed by writing subsequently $\mathscr{S}_0, \mathscr{S}_1, \ldots, \mathscr{S}_k$; this is evidently the required sequence for E_1 and E_2 and H has the property $\mathbf{P}_1$.

Now let H be a 3-uniform hypergraph with the properties $\mathbf{P}_1$ and $\mathbf{P}_2$ and without isolated vertices. Consider the graph $G(H)$. Let v be a vertex of H. Let $H(v)$ be the subhypergraph of H formed by all edges containing v and all vertices contained in them. Let u_1, u_2 be two vertices of $H(v)$. Let E_1 (or E_2) be an edge of $H(v)$ containing u_1 (or u_2, respectively). As H has the property $\mathbf{P}_2$, there exists a sequence of edges $F_0, F_1, \ldots, F_k$ of $H(v)$ such that $F_0 = E_1$, $F_k = E_2$ and $|F_i \cap F_{i+1}| = 2$ for $i = 0, 1, \ldots, k - 1$. Let f_i be the edge of $G(H)$ joining the vertices of F_i distinct from v for $i = 0, 1, \ldots, k$. Then f_i and f_{i+1} have a common end vertex for $i = 0, 1, \ldots, k - 1$ and there exists a path connecting u_1 with u_2 in the subgraph of $G(H)$ induced by $N(v)$. As v, u_1, u_2 were chosen arbitrarily, $G(H)$ is locally connected. The connectedness of $G(H)$ can be analogously derived from the property $\mathbf{P}_1$ of H.

Theorem 3.2. *A connected undirected graph G is locally tree-like if and only if either $G \cong K_2$, or $G \cong G(H)$ for some 3-uniform hypergraph H with the properties $\mathbf{P}_1$, $\mathbf{P}_2$, without the property $\mathbf{P}_4$ and without isolated vertices.*

Proof. The case $G \cong K_2$ is trivial. Let G be a connected locally tree-like graph with more than two vertices. Then it is locally connected and according to Theorem 3.1, $G \cong G(H)$ for some 3-uniform hypergraph H with the properties $\mathbf{P}_1, \mathbf{P}_2$ and without isolated vertices. This hypergraph H is constructed in the same way as in the proof of Theorem 3.1. It remains to prove that H has not the property $\mathbf{P}_4$. Suppose that H has the property $\mathbf{P}_4$. Then there exists a finite sequence $E_1, \ldots, E_k$ of edges of H such that $k \geqq 3$, $|E_i \cap E_{i+1}| = 2$ for $i = 1, \ldots, k - 1$, $|E_k \cap E_1| = 2$ and $\left|\bigcap_{i=1}^{k} E_i\right| = 1$. Thus there exists a vertex v contained in all edges $E_1, \ldots, E_k$. For $i = 1, \ldots, k$ let e_i be the edge of G joining the vertices of E_i distinct from v. The edges e_i, e_{i+1} for $i = 1, \ldots, k - 1$ have a common end vertex and so have e_k, e_1; therefore there exists a closed trail and thus also a circuit in the subgraph of G induced by $N(v)$. Hence this subgraph is not a tree and G is not locally tree-like, which is a contradiction.

Now let H be a 3-uniform hypergraph with the properties $\mathbf{P}_1, \mathbf{P}_2$, without the property $\mathbf{P}_4$ and without isolated vertices. The graph $G(H)$ is connected and locally connected according to Theorem 3.1. Suppose that it is not tree-like. Then there exists a vertex v of G such that the subgraph of G induced by $N(v)$ is not a tree. As it is connected, it contains a circuit with the vertices $u_1, \ldots, u_k$ and edges u_iu_{i+1} for $i = 1, \ldots, k - 1$ and u_ku_1. Then for $i = 1, \ldots, k - 1$ the hypergraph H contains the edge $E_i = \{u_i, u_{i+1}, v\}$ and the edge $E_k = \{u_k, u_1, v\}$. We have $|E_i \cap E_{i+1}| = 2$ for $i = 1, \ldots, k - 1$ and $|E_k \cap E_1| = 2$, $\left|\bigcap_{i=1}^{k} E_i\right| = 1$, which is a contradiction with the assumption that H has not the property $\mathbf{P}_4$.

Theorem 3.3. *Let G be a locally tree-like graph with n vertices. Let m be the number of edges of G and t the number of triangles in G. Then*

$$t = (2m - n)/3 .$$

Proof. Let v be a vertex of G. The subgraph of G induced by $N(v)$ is a tree with $\delta(v)$ vertices, where $\delta(v)$ is the degree of v in G. Thus it has $\delta(v) - 1$ edges. Each triangle containing v contains exactly one edge of this tree and each edge of this tree is contained in exactly one such triangle; therefore the number of trinagles in G which contains v is $\delta(v) - 1$. As each triangle contains three vertices, we compute the number of triangles in G by taking the sum of the numbers of triangles containing v for all $v \in V(G)$ and dividing it by 3. Hence

$$t = \tfrac{1}{3} \sum_{v \in V(G)} (\delta(v) - 1) = \tfrac{1}{3} \sum_{v \in V(G)} \delta(v) - \tfrac{1}{3} n .$$

On the other hand, it is well-known that

$$m = \tfrac{1}{2} \sum_{v \in V(G)} \delta(v) .$$

From these two equations we obtain

$$t = (2m - n)/3 .$$

Theorem 3.4. *The least number of edges of a connected locally connected graph G with n vertices is $2n - 3$. If G is a finite connected locally connected graph with n vertices and $2n - 3$ edges, then G is locally tree-like and $G \cong K_2$ or $G \cong G(H)$, where $H \in \mathfrak{T}_3$.*

Proof. Let G be a finite connected locally connected graph. The case $G \cong K_2$ is trivial. Let G have at least three vertices and construct the 3-uniform hypergraph H such that $G \cong G(H)$ (this construction is described in the proof of Theorem 3.1). According to Theorem 2.1 the hypergraph H has at least $n - 2$ edges, therefore G for the number t of triangles of G we have $t \geqq n - 2$. From Theorem 3.3 we obtain that for the number m of edges of G we have

$$m = (3t - n)/2 \geqq 2n - 3 .$$

We also see that G has $2n - 3$ edges if and only if it has $n - 2$ triangles and H has $n - 2$ edges, which means $H \in \mathfrak{T}_3$. Thus suppose that G has n vertices and $2n - 3$ edges. According to Theorem 2.2 the hypergraph H has not the property $\mathbf{P}_4$ and according to Theorem 2.3 it has the property $\mathbf{P}_2$. Then according to Theorem 3.2 the graph G is locally tree-like.

Theorem 3.5. *Let $\mathfrak{U}$ be the class of all connected locally tree-like graphs such that $m = 2n - 3$, where m is the number of edges and n is the number of vertices. The class $\mathfrak{U}$ can be described in the following way:*

(i) $K_2 \in \mathfrak{U}$.

(ii) *Let* $G_0 \in \mathfrak{U}$. *Let* G *be a graph obtained from* G *by adding one vertex and joining it with two adjacent vertices of* G_0. *Then* $G \in \mathfrak{U}$.

(iii) *The class* $\mathfrak{U}$ *contains no graphs except those described in* (i) *and* (ii).

This is an immediate consequence of Theorems 2.4 and 3.4.

Thus we have a lower bound for the number of edges of a finite connected locally tree-like graph in the form of a linear function of its number of vertices. A question may be posed, whether an analogous upper bound can be found. The negative answer is given by the next theorem.

Theorem 3.6. *For any positive integer* q *there exists a connected locally tree-like graph in which the minimal degree of a vertex is greater than* q.

Proof. Let q be a positive integer, let p be such an integer that $p > q$ and there exists a finite projective geometry $PG(p)$ in which each line is incident with $p + 1$ points and each point is incident with $p + 1$ lines. Now we construct the fan graph F in such a way that we take a path P of the length $2p + 1$ and a vertex v not belonging to P and join each vertex of P with v by an edge. Let the vertices of P be $u_1, \ldots, u_{2p+2}$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, 2p + 1$. By $M(F)$ we denote the set of all edges $u_i u_{i+1}$ where i is odd; evidently $|M(E)| = p + 1$. Now we take $p^2 + p + 1$ pairwise disjoint copies F_i $(i = 1, \ldots, p^2 + p + 1)$ of F and choose a one-to-one correspondence between them and the lines of $PG(p)$. For each copy F_i of F we choose a one-to-one correspondence between the edges of $M(F_i)$ and the points which are incident with the line corresponding to F_i. Then for each point of $PG(p)$ there are $p + 1$ edges corresponding to it. We identify all of them. (To identify two edges means to identify an arbitrary end vertex of one of them with an arbitrary end vertex of the other and then to identify the remaining end vertices.) If we do this for each point of $PG(p)$, we obtain a graph G. The reader may verify himself that this graph is connected and locally tree-like and the degrees of all of its vertices exceed p.

We see that the number of edges of G is greater than $\frac{1}{2}qn$ and q was chosen arbitrarily. Hence there exists no upper bound for the number of edges of a locally tree-like graph in the form of a linear function of its number of vertices.

Problem. *Find an upper bound for the number* m *of edges of a finite connected locally tree-like graph with* n *vertices in the form of a quadratic function of* n.

Such an upper bound certainly exists, because the number of edges of a complete graph is a quadratic function of its number of vertices.

If we take an infinite fan graph (obtained from a two-way infinite path by adding one vertex and joining it with all vertices of this path) and a countably infinite projective geometry, we obtain a locally tree-like graph in which all vertices have infinite degrees.

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