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FUNCTORIAL PROLONGATIONS OF LIE GROUPS AND THEIR ACTIONS

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Our starting point are the following two classical results by Ehresmann, [2]. If G is a Lie group, then the space of all k^r-velocities $T_k^r G$ is also a Lie group. Moreover, if G acts on a manifold M, then $T_k^r G$ acts canonically on $T_k^r M$. We first discuss the same problems for an arbitrary prolongation functor F and we deduce analogous results under the assumption that F has the point property and is either product-preserving or linear. In both cases, FG can be expressed as a semi-direct product of G and the fiber over the unit of G. This leads to a canonical group structure on the dual vector bundle $(FG)^*$ for any linear functor F with the point property. Given any two Lie groups G and H, we introduce a natural group composition on the space of all r-jets of G into H. Taking $H = \mathbf{R}$, we obtain a "concrete" description of the group $T^r G$, where T^r means the r-th order tangent functor.

1. Let $\mathbf{M}$ be the category of smooth manifolds and smooth maps, $\mathbf{FM}$ the category of smooth fibered manifolds and smooth morphisms and $B : \mathbf{FM} \to \mathbf{M}$ the base functor. A prolongation functor means any functor $F : \mathbf{M} \to \mathbf{FM}$ satisfying $B \circ F = \mathrm{id}_{\mathbf{M}}$ and the following regularity condition: if M, N, Q are smooth manifolds and $\varphi : M \times Q \to N$ is a smooth map, then the induced map $\Phi : FM \times Q \to FN$, $\Phi(-, q) = F(\varphi(-, q))$, $q \in Q$, is also smooth. We denote by $p_M : FM \to M$ the bundle projection of FM and by $F_x M$ the fiber over $x \in M$.

Remark 1. If one replaces $\mathbf{M}$ by the subcategory $\mathbf{M}_n$ of n-dimensional manifolds and their embeddings, one gets a so-called lifting functor intensively studied by several authors, see e.g. [4]. However, in our situation it is essential that F is defined on the whole category $\mathbf{M}$, since the group composition and the group action are smooth maps of rather general type.

Let G be a Lie group and $\varphi : G \times G \to G$ its composition law. The unit of G will be interpreted as a map of a one-element set, typically denoted by pt, into G, i.e. $e_G : pt \to G$. If F preserves products, we have the prolongation $F\varphi : F(G \times G) = FG \times FG \to FG$. Assume that F has the following "point property": the prolongation of a one-element set is a one-element set. Using a standard diagram chasing, one easily deduces that FG with the composition law $F\varphi$ is also a Lie group. The unit of FG is the prolongation of the unit of G, i.e. $e_{FG} = Fe_G : pt \to FG$. Analogously, if $\psi : G \times M \to M$ is an action of G on a manifold M, then it is easy to verify that $F\psi : F(G \times M) = FG \times FM \to FM$ is also an action of FG on FM. The simplest example of a product-preserving functor is T_k^r that transforms any manifold M into the set $J_0^r(\mathrm{R}^k, M) =: T_k^r M$ of all r-jets of R^k into M with source O and any map $f : M \to N$ into a morphism $T_k^r f : T_k^r M \to T_k^r N$ defined by means of the composition of jets. In this case, we obtain the classical results by Ehresmann.

However, certain important functors of differential geometry do not preserve products. A class of such functors can be defined as follows. Put $T_k^{r*}M = J^r(M, \mathrm{R}^k)_0$ (= the set of all r-jets of M into R^k with target O). This is a vector bundle over M, the dual bundle of which will be denoted by $T^{r,k}M = (T_k^{r*}M)^*$. Any r-jet A of M into N with source x and target y determines a linear map $\tilde{A} : (T_k^{r*}N)_y \to (T_k^{r*}M)_x$ and we can construct the dual map $\tilde{A}^* : T_x^{r,k}M \to T_y^{r,k}N$. In this way, any smooth map $f : M \to N$ induces a linear morphism $T^{r,k}f : T^{r,k}M \to T^{r,k}N$ over f and we obtain a prolongation functor $T^{r,k}$ with values in the subcategory $\mathbf{VB} \subset \mathbf{FM}$ of smooth vector bundles. (The technical details of this construction are explained in [3].) For $k = 1$, $T^{r,1} =: T^r$ is the classical r-th order tangent functor. Clearly, if $r > 1$ and $\dim M$, $\dim N > 0$, then $\dim T^r(M \times N) > \dim T^rM + \dim T^rN$, so that $T^{r,k}$ does not preserve products in general.

A prolongation functor will be said to be linear if its values lie in the subcategory $\mathbf{VB} \subset \mathbf{FM}$. Given a linear functor F and two manifolds M, N, we define a map $i_{M,N} : FM \times FN \to F(M \times N)$ as follows. Consider the injections $i_y : M \approx M \times \{y\} \to M \times N$ and $i_x : N \approx \{x\} \times N \to M \times N$, $x \in M$, $y \in N$. For any $A \in F_xM$ and $B \in F_yN$, we put

$$(1) \qquad i_{M,N}(A, B) = Fi_y(A) + Fi_x(B)$$

with the vector addition on the right-hand side.

Lemma 1. *$i_{M,N}$ is an injective immersion.*

Proof. By the regularity condition, the map $i_1 : FM \times N \to F(M \times N)$, $(A, y) \mapsto Fi_y(A)$ is smooth, as well as the similar map $i_2 : M \times FN \to F(M \times N)$. Obviously, $i_{M,N}$ is the composition of the fiber product of i_1 and i_2 and the vector addition in $F(M \times N)$, so that $i_{M,N}$ is smooth. Taking into account the product projections p_1, p_2 of $M \times N$, the prolongations $Fp_1 : F(M \times N) \to FM$ and $Fp_2 : F(M \times N) \to FN$ induce a smooth map $j_{M,N} : F(M \times N) \to FM \times FN$ satisfying $j_{M,N} \circ i_{M,N} = 1_{FM \times FN}$. This implies that $i_{M,N}$ is an injective immersion. QED.

2. Let G be a Lie group with a composition law $\varphi : G \times G \to G$ and F a linear functor having the point property. We define $\mathbf{F}\varphi := F\varphi \circ i_{G,G} : FG \times FG \to FG$. To prove that $(FG, \mathbf{F}\varphi)$ is also a Lie group, we need several steps.

Assume first that (H, φ) is a smooth monoid. If we put

(2) $$\mathbf{F}\varphi := F\varphi \circ i_{H,H} : FH \times FH \to FH$$

as above, we prove by a standard diagram chasing that $(FH, \mathbf{F}\varphi)$ is also a smooth monoid. Its unit is the prolongation of the unit $e : pt \to H$ of H. Obviously, $p_H : FH \to H$ is a monoid homorphism, so that we have an exact sequence of monoids

(3) $$0 \to F_e H \to FH \overset{p_H}{\to} H \to 0 .$$

As F is a linear functor, $F_e H$ is a vector space.

Lemma 2. *The composition law in $F_e H$ given by (2) coincides with the vector addition.*

Proof. For any $A, B \in F_e H$, we have $\mathbf{F}\varphi(A, B) = (F\varphi \circ i_{H,H})(A, B) = F\varphi(Fi_e(A) + Fi_e(B)) = A + B$, since $\varphi \circ i_e$ is the identity map by the definition of a unit. QED.

Denote by $O_H : H \to FH$ the zero section. By (2), O_H is a monoid homomorphism satisfying $p_H \circ O_H = 1_H$, i.e. O_H is a splitting of (3). If the monoid in question is a group G, then both G and $F_e G$ in (3) are groups. Then the elementary algebraic theory of semi-direct products indicates that FG is also a group. Given an action $\psi : G \times M \to M$ of G on a manifold M, we introduce $\mathbf{F}\psi := F\psi \circ i_{G,M} : FG \times FM \to FM$ and we easily verify that $\mathbf{F}\psi$ is an action of FG on FM. Thus, we have proved

Theorem 1. *For any linear functor with the point property, $(FG, \mathbf{F}\varphi)$ is also a Lie group and $\mathbf{F}\psi$ is an action of FG on FM.*

Remark 2. Our construction is based on the map $i_{M,N} : FM \times FN \to F(M \times N)$ determined by means of the vector addition. One can derive a similar result for an arbitrary prolongation functor F with the point property under the assumption that there are a priori given some maps $i_{M,N} : FM \times FN \to F(M \times N)$ with suitable functorial properties. Such a generalization is straightforward, but too technical, that is why we do not go into details here.

Every $g \in G$ determines a diffeomorphism $\tilde{g} : M \to M$, $\tilde{g}(x) = \psi(g, x)$, which is prolonged into $F\tilde{g} : FM \to FM$. On the other hand, the zero vector $O_g \in FG$ similarly determines a diffeomorphism $\tilde{O}_g : FM \to FM$.

Lemma 3. *We have $F\tilde{g} = \tilde{O}_g$.*

Proof. Clearly, we can write $\tilde{g} = \psi \circ i_g$. Then $\tilde{O}_g = F\psi \circ i_{G,M} \circ i_{O_g} = F\psi \circ Fi_g = F\tilde{g}$. QED.

According to the general theory of semi-direct products, [1], every $\gamma \in F_g G$ is identified with the pair $(g, O_{g^{-1}}\gamma) \in G \times F_e G$. If we put

$$\varrho(g)(A) = O_{g^{-1}} A O_g \quad g \in G, \quad A \in F_e G, \tag{4}$$

the product on the right-hand side being in FG, we get a right action ϱ of G on $F_e G$. Then FG is equal to the semi-direct product $G \times^- F_e G$ with the composition law

$$(g_1, A_1)(g_2, A_2) = (g_1 g_2, \varrho(g_2)(A_1) + A_2). \tag{5}$$

Lemma 4. *ϱ is a linear representation of G on $F_e G$.*

Proof. By Lemma 3, $\varrho(g)$ is a composition of two linear maps. QED.

Remark 3. A semi-direct decomposition of FG takes place even in the case of a product-preserving functor F with the point property. Any $x \in M$ can be interpreted as a map $pt \to M$, the prolongation of which determines a distinguished element $O_x \in F_x M$. By the regularity condition, we get a smooth section $O_M : M \to FM$. If G is a Lie group, then $O_G : G \to FG$ is a group homomorphism that splits FG into a semi-direct product of G and $F_e G$.

In general, if V is a vector space and ϱ a right linear representation of G on V, we have the semi-direct product $G \times^- V$ with the composition law

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, \varrho(g_2)(v_1) + v_2). \tag{6}$$

The dual left linear representation σ of G on V^* is given by

$$\langle \varrho(g)(v), w \rangle = \langle v, \sigma(g)(w) \rangle \quad v \in V, \quad w \in V^*. \tag{7}$$

The corresponding semi-direct product $G \times^- V^*$ with

$$(g_1, w_1)(g_2, w_2) = (g_1 g_2, \sigma(g_2^{-1})(w_1) + w_2) \tag{8}$$

will be called the dual group to $G \times^- V$. In particular, if F is a linear functor having the point property, then FG is the semi-direct product of G and $F_e G$ with respect to a linear representation of G on $F_e G$. The dual vector bundle $(FG)^*$ admits a similar decomposition transforming any $\gamma \in (FG))_g^*$ into $(g, F\tilde{g}^{-1*}(\gamma)) \in G \times (F_e G)^*$. Then (7) and (8) define a group structure on $(FG)^*$. In the case of the tangent functor T, the group T^*G, introduced by an ad hoc formula, was already used in some concrete problems in differential geometry. However, if G acts on a manifold M, it is not known whether $(FG)^*$ acts canonically on $(FM)^*$, not even in the case $F = T$.

3. In the special case $F = T^{r,k}$, we obtain a "functorial" definition of the group $T^{r,k}G$. We give another description of $T^{r,k}G$ based on the following original construction of a group structure on the space $J^r(G, H)$ of all r-jets between two Lie groups G and H. The composition in H being denoted by a dot and the composition in G by superposition only, we define an operation $*$ on $J^r(G, H)$ by

$$(j^r_a\lambda) * (j^r_b\mu) := j^r_{ab}[\lambda(xb^{-1}) \,.\, \mu(a^{-1}x)]\,, \tag{9}$$

where x belongs to a neighbourhood of $ab \in G$. Let $\hat{h}$ be the constant map $G \to H$, $g \mapsto h,\ h \in H$.

Theorem 2. *For any Lie groups G and H, $J^r(G, H)$ with the composition law (9) is a Lie group. The source or target jet projection $\alpha : J^r(G, H) \to G$ or $\beta : J^r(G, H) \to H$ is a group homomorphism and the map $g \mapsto j^r_g\hat{e}$ or $h \mapsto j^r_e\hat{h}$ is its splitting, respectively.*

Proof is straightforward.

Taking $H = \mathbf{R}^k$, which is an Abelian group, we get a group $J^r(G, \mathbf{R}^k)$. As $\beta : J^r(G, \mathbf{R}^k) \to \mathbf{R}^k$ is a group homomorphism, $J^r(G, \mathbf{R}^k)_0$ is a subgroup of $J^r(G, \mathbf{R}^k)$. The splitting $g \mapsto j^r_g\hat{O}$ determines a decomposition of $J^r(G, \mathbf{R}^k)_0$ into a semi-direct product of G and $J^r_0(G, \mathbf{R}^k)_0 = (T^{r,k}_eG)^*$. The following assertion gives another characterization of the group $T^{r,k}G$.

Theorem 3. *$T^{r,k}G$ and $J^r(G, \mathbf{R}^k)_0$ are dual groups.*

Proof. Since the decomposition of $J^r(G, \mathbf{R}^k)_0$ into a semi-direct product $G \times^- J^r_0(G, \mathbf{R}^k)_0$ is determined by the splitting $g \mapsto j^r_g\hat{O}$, the corresponding left action σ of G on $J^r_0(G, \mathbf{R}^k)_0$ is given by $\sigma(g)(j^r_e\gamma) = j^r_g\hat{O} * j^r_e\gamma * j^r_{g^{-1}}\hat{O} = j^r_e\gamma(g^{-1}xg)$. The right action ϱ of G on $T^{r,k}_e$ is given by (4). We shall apply Lemma 3. Let $L_g : G \to G$ be the left translation determined by $g \in G$. Since $T^{r,k}L_g$ is defined by dualization, the value of the linear form $(T^{r,k}L_g)(B)$, $B \in T^{r,k}_{\bar{g}}G$, $\bar{g} \in G$, on $j^r_{g^{-1}\bar{g}}\gamma \in J^r_{g^{-1}\bar{g}}(G, \mathbf{R}^k)_0$ is equal to the value of B on $j^r_{\bar{g}}\gamma(gx)$. This implies that the value of the linear form $O_{g^{-1}}AO_g$ of (4) on $j^r_e\gamma$ is equal to the value of A on $j^r_e\gamma(g^{-1}xg)$. Hence ϱ and σ are dual representations in the sense of (7). QED.

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