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ON SOLVABILITY OF EQUATIONS OF THE 4th ORDER WITH JUMPING NONLINEARITIES

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INTRODUCTION

In the study of generalized 2π-periodic solutions of the nonlinear beam equation with jumping nonlinearities

$$(1) \qquad \beta u_t + u_{tt} + u_{xxxx} = \psi(u) + h\,,$$

where β is a positive constant, ψ is a continuous function with $\lim \psi(u)/u = \mu$ if $u \to +\infty$ and $\lim \psi(u)/u = \nu$ if $u \to -\infty$ for some positive constants μ, ν and $h \in L_2(]0, 2\pi[^2)$ one can proceed by the methods of [1], [2], which have been developed for the nonlinear telegraph equation

$$(2) \qquad \beta u_t + u_{tt} - u_{xx} = \psi(u) + h\,,$$

with analogous assumptions for β, ψ and h.

It can be shown that there exists a subset A_{-1} of $]0, +\infty[^2$ such that for each $(\mu, \nu) \notin A_{-1}$ the equation (1) is solvable for any right-hand side h. The set A_{-1} is defined as the set of all pairs (μ, ν), for which there exists a nonconstant 2π-periodic function $u \in C^4(R^1)$ solving the ordinary differential equation of the fourth order

$$(3) \qquad u^{\mathrm{IV}} = \mu u^+ - \nu u^-\,,$$

where $u^+(x) = \max(u(x), 0)$ and $u^-(x) = \max(-u(x), 0)$ are the positive and negative parts of u.

The aim of this paper is to describe the set A_{-1} for the periodic problem. In 3 we will pursue a qualitative study of the boundary-value problem for the equation (3). Let us remark that the boundary-value problem for the equation (2) is solved in [3].

The cases $\mu \leqq 0$ or $\nu \leqq 0$ are trivial. If $\mu = 0$ or $\nu = 0$ then the only periodic solution of (3) is the constant one, if $\mu\nu < 0$, then there ts no nonzero periodic solution (one can see it after integrating the equation (3) over the period). In the case

$\mu < 0$, $\nu < 0$ it suffices to multiply the equation (3) by u and to integrate again over the period. We obtain again $u = 0$.

In all the paper we denote by $]x, y[$ ($[x, y]$) the open (closed) interval with bounds $x < y$, by $\boldsymbol{R}^1$ the set of all real numbers and by $\boldsymbol{N}$ the set of all natural numbers.

1. PRELIMINARIES

For further investigation it is useful to put $\mu = a^4$, $\nu = b^4$, $(a, b) \in \,]0, +\infty[^2$. The equation (3) will be written in the form

$$R(a, b): \quad u^{\text{IV}} = a^4 u^+ - b^4 u^- .$$

It is well known that the equation $R(a, b)$ for every a, b satisfies the assumptions of the theorems of existence, unicity and continuous dependence of its solutions on initial conditions and parameters. Moreover, each of its solutions is defined on $\boldsymbol{R}^1$. The solution $u \equiv 0$ will be called trivial.

(1.1) **Lemma.** *Let $\psi : \boldsymbol{R}^1 \to \boldsymbol{R}^1$ be a continuous increasing function satisfying locally the Lipschitz condition in $\boldsymbol{R}^1$ and let $u, v \in C^n(\boldsymbol{R}^1)$ be two solutions of the equation of the n-th order*

$$u^{(n)} = \psi(u) ,$$

where n is a given natural number. Let us assume that there exists $j \in \{0, 1, \ldots$ $\ldots, n - 1\}$ such that $u^{(j)}(0) > v^{(j)}(0)$ and $u^{(i)}(0) \geqq v^{(i)}(0)$ for all $i \in \{0, 1, \ldots, n - 1\}$.

Then the functions $u^{(i)}(x) - v^{(i)}(x)$ for $i \in \{0, 1, \ldots, n - 1\}$ are increasing and positive in $]0, +\infty[$.

Proof. Let us denote $M = \max\{s > 0, \forall x \in [0, s[, u^{(j)}(x) > v^{(j)}(x)\}$. Obviously, $M > 0$. For all $x \in \,]0, M[$ we have $u^{(i)}(x) > v^{(i)}(x)$ for $i \leqq j$; in particular $u(x) > v(x)$ for $x \in \,]0, M[$. Using the fact that ψ is increasing, we have $\psi(u(x)) > \psi(v(x))$ and thus $u^{(n)}(x) > v^{(n)}(x)$ for $x \in \,]0, M[$. Similarly, $u^{(j+1)}(x) > v^{(j+1)}(x)$ for every $x \in \,]0, M[$. If $M < +\infty$, then $u^{(j)}(M) > v^{(j)}(M)$, which is a contradiction. Therefore, $M = +\infty$, and the functions $u^{(i)} - v^{(i)}$ are positive, and hence increasing in $]0, +\infty[$ for $i \leqq n$. ■

2. PERIODIC SOLUTIONS

The notion of a periodic solution of the equation $R(a, b)$ is considered in the sense mentioned in the introduction. Let us give now some simple results.

(2.1) If u is an ω-periodic solution of $R(a, b)$ with $\omega > 0$, then for all $A \neq 0$, $\lambda \neq 0$, $\vartheta \in \boldsymbol{R}^1$ the function $\tilde{u}$ defined by the relation

$$\tilde{u}(x) = A\, u(\lambda x + \vartheta) , \quad x \in \boldsymbol{R}^1$$

is an $\omega/|\lambda|$-periodic solution of $R(|\lambda|\, a, |\lambda|\, b)$ if $A > 0$ and $R(|\lambda|\, b, |\lambda|\, a)$ if $A < 0$.

(2.2) For every nontrivial solution u of $R(a, b)$ the set $u^{-1}(0)$ has no limit point except maybe $+\infty$ or $-\infty$. Moreover, if u is an ω-periodic solution of $R(a, b)$ and $u(x_0) = 0$ at a point $x_0 \in \mathbf{R}^1$, then $u'(x_0) \cdot u'''(x_0) < 0$.

The proof of the last assertion is based on Lemma (1.1) with $u(x - x_0)$ or $u(x_0 - x)$ and $v(x) = 0$.

Consequently, each periodic solution u of $R(a, b)$ is composed of *semi-waves*, where a semi-wave is the portion of the graph of u between two successive zeros of u. The simplest periodic solution possible is the one composed of one positive and one negative semi-wave.

Let us assume that there exists such a simple ω-periodic solution u of $R(a, b)$, $u(0) = u(x_0) = u(\omega) = 0$, $0 < x_0 < \omega$. Then the function

$$u_1(x) = u(x) + u(x_0 - x)$$

is also an ω-periodic solution of $R(a, b)$, which is moreover symmetrical in the following sense:

$$u_1(0) = u_1(x_0) = u_1(\omega) = 0\,,$$
$$u_1^{(i)}(0) = (-1)^i u_1^{(i)}(x_0) = u_1^{(i)}(\omega) \quad \text{for} \quad i = 1, 2, 3\,.$$

The results of this section consists in the proof of existence of symmetrical and nonexistence of nonsymmetrical periodic solutions.

(2.3) **Lemma.** *Let $\varphi \in \,](3/4)\,\pi, \pi[$ be the smallest positive root of the equation*

$$\tan(x) + \operatorname{th}(x) = 0\,,$$

and assume that there exists a positive semi-wave of a solution u of $R(a, b)$ on $[x_1, x_2]$. Then $x_2 - x_1 \leqq 2\varphi/a$.

Proof. We have $u(x_1) = u(x_2) = 0$, $u > 0$ on $]x_1, x_2[$. Thus u is a solution of the linear equation $u^{\mathrm{IV}} = a^4 u$ in $[x_1, x_2]$, and can be written in the form

$$u(x) = A\sin(ax) + B\cos(ax) + C\operatorname{sh}(ax) + D\operatorname{ch}(ax)\,,$$

where A, B, C, D are real constants and $x \in [x_1, x_2]$. Put $x_0 = (x_1 + x_2)/2$, $r = (x_2 - x_1)/2$. Then the function

$$\tilde{u}(x) = u(x_0 + x) + u(x_0 - x)$$

is a symmetrical positive semi-wave of a solution of $R(a, b)$ on $[-r, r]$. Hence $\tilde{u}(x)$ is of the form

$$\tilde{u}(x) = A\left(\cos(ax) - \frac{\cos(ar)}{\operatorname{ch}(ar)}\operatorname{ch}(ax)\right),$$

where A is a positive constant. The conditions $\tilde{u}(x) > 0$ on $]-r, r[$ and $\tilde{u}'(-r) \geqq 0$ give us the inequality $ar \leqq \varphi$ which completes the pfoof. ■

(2.4) **Theorem.** *The set S_1 of all pairs $(a, b) \in]0, +\infty[^2$ such that there exists a nontrivial 2π-periodic solution of $R(a, b)$, which is composed of two semiwaves, is a curve $(a, b(a))$, where $b(a)$ is a decreasing C^∞-function defined in $]\varphi/\pi, +\infty[$ (see (2.3)) with $\lim b(a) = \varphi/\pi$ if $a \to +\infty$. The curve S_1 is symmetrical with respect to the straight line $b = a$ and fulfils $S_1 \subset G_1$, where G_1 is the set of all pairs $(a, b) \in]\varphi/\pi, +\infty[^2$ such that*

$$b \geqq a\,,\quad \left(\frac{b}{a}\right)^2 - g\left(\pi a\left(1 - \frac{1}{2b}\right)\right) \geqq 0 \geqq \left(\frac{a}{b}\right)^2 - g\left(\pi b\left(1 - \frac{1}{2a}\right)\right),$$

or

$$b \leqq a\,,\quad \left(\frac{a}{b}\right)^2 - g\left(\pi b\left(1 - \frac{1}{2a}\right)\right) \geqq 0 \geqq \left(\frac{b}{a}\right)^2 - g\left(\pi a\left(1 - \frac{1}{2b}\right)\right)$$

and $g(z)$ is the function defined for $z \in]0, \varphi[$ by the formula

$$g(z) = \frac{\operatorname{ch}(z)\sin(z) - \operatorname{sh}(z)\cos(z)}{\operatorname{ch}(z)\sin(z) + \operatorname{sh}(z)\cos(z)}\,.$$

Proof. Let us consider the positive and negative semi-waves, u_1, u_2,

$$u_1(x) = A\left(\cos(ax) - \frac{\cos(ar)}{\operatorname{ch}(ar)}\operatorname{ch}(ax)\right), \quad r > 0\,,\quad A > 0\,,\quad x \in [-r, r]\,,$$

$$u_2(x) = -B\left(\cos(bx) - \frac{\cos(bs)}{\operatorname{ch}(bs)}\operatorname{ch}(bx)\right), \quad s > 0\,,\quad B > 0\,,\quad x \in [-s, s]\,,$$

$$r + s = \pi\,.$$

The necessary and sufficient condition for u_1, u_2 to be the semiwaves of a solution of $R(a, b)$ is that of continuity:

$$u_1^{(i)}(r) = u_2^{(i)}(-s)\,,\quad i = 1, 2, 3\,. \tag{2.5}$$

We have $u_1'(r) \neq 0$, $u_2'(-s) \neq 0$ (see (2.2)). Thus we can divide the last two equations of (2.5) by the first one. Let the function $g(z)$ be defined as above and put

$$f(z) = \frac{\operatorname{ch}(z)\cos(z)}{\operatorname{ch}(z)\sin(z) + \operatorname{sh}(z)\cos(z)}\,,\quad z \in]0, \varphi[\,.$$

Then the condition (2.5) together with the assumption $r + s = \pi$ is equivalent to the system

$$\begin{aligned} a\ f(ar) &= -b\ f(bs)\,,\\ a^2\,g(ar) &= \ \ b^2\,g(bs)\,,\\ r + s &= \ \ \pi\,. \end{aligned}$$

Let us define the mapping $h : \mathbf{R}^4 \to \mathbf{R}^3$ with components $h^i : \mathbf{R}^4 \to \mathbf{R}^1$,

$$h^1(a, b, r, s) = a \ f(ar) + b \ f(bs),$$

$$h^2(a, b, r, s) = a^2 g(ar) - b^2 g(bs),$$

$$h^3(a, b, r, s) = r + s - \pi,$$

where the domain of definition of h is the set

$$D(h) = \{(a, b, r, s) \in \mathbf{R}^4 \mid a > 0, \ b > 0, \ 0 < r < \pi, \ 0 < s < \pi, \ ar < \varphi, \ bs < \varphi\}.$$

Let $J_{b,r,s}(a, b, r, s)$ denote the determinant of the Jacobi matrix of h at a point $(a, b, r, s) \in h^{-1}(0)$ with respect to the variables b, r, s (and analogously $J_{a,r,s}(a, b, r, s)$). Knowing that $f'(z) = -g(z) - 2f^2(z)$ for every $z \in]0, \varphi[$, and consequently $a^2 f'(ar) = b^2 f'(bs)$ for every $(a, b, r, s) \in h^{-1}(0)$, one finds $J_{b,r,s}(a, b, r, s) = (f(bs) + bs f'(bs)) (a^3 g'(ar) + b^3 g'(bs))$.

Analogously,

$$J_{a,r,s}(a, b, r, s) = (f(ar) + ar f'(ar)) (a^3 g'(ar) + b^3 g'(bs)).$$

For every $z \in]0, \varphi[$ we have $f(z) + z f'(z) < 0$, $g'(z) > 0$. Hence, by the implicit function theorem, in a neighbourhood of an arbitrary point of $h^{-1}(0)$ there exist C^∞-functions $r(a)$, $s(a)$, $b(a)$, $b'(a) < 0$, such that $(a, b(a), r(a), s(a)) \in h^{-1}(0)$.

Put $A = \{a \in]0, +\infty[\mid \exists(b, r, s) \in \mathbf{R}^3, \ h(a, b, r, s) = 0\}$. From the above argument it follows that A is open. Let us investigate the closedness of A.

If $\{a_n, n \in \mathbf{N}\} \subset A$, then we find the corresponding b_n, r_n, s_n such that $h(a_n, b_n, r_n, s_n) = 0$ for every $n \in \mathbf{N}$. Let $a_n \to a$, $0 < a < +\infty$. If the sequence (a_n, b_n, r_n, s_n) has a limit point in $D(h)$, then $a \in A$, because $h^{-1}(0)$ is closed in $D(h)$. If this is not the case, then the sequence (a_n, b_n, r_n, s_n) has a limit point at the boundary $\partial D(h)$. It is easy to see that there are only two symmetrical cases possible:

1. $r_n \to \pi, \ s_n \to 0, \ a_n r_n \to \varphi, \ b_n s_n \to 0, \ b_n \to +\infty, \ a_n \to \varphi/\pi;$

2. $r_n \to 0, \ s_n \to \pi, \ a_n r_n \to 0, \ b_n s_n \to \varphi, \ b_n \to \varphi/\pi, \ a_n \to +\infty.$

Therefore, A is closed in the set $]0, \varphi/\pi[\cup]\varphi/\pi, +\infty[$. On the other hand, $1 \in A$ (because $h(1, 1, \pi/2, \pi/2) = 0$), hence $]\varphi/\pi, +\infty[\subset A$. If $A \cap]0, \varphi/\pi[\neq \emptyset$, then there would exist a decreasing function $b(a)$ in $]0, \varphi/\pi[$ such that $\lim b(a) = +\infty$ if $a \to \varphi/\pi$, which is a contradiction.

Thus, we have established the existence of function $b(a)$, $r(a)$, $s(a)$, defined for $a \in A =]\varphi/\pi, +\infty[$ and such that $b(a)$ is decreasing in A and $h(a, b(a), r(a), s(a)) = 0$ for every $a \in A$. Moreover, $h^{-1}(0) = \{(a, b(a), r(a), s(a)), \ a \in A\}$, because $(1, 1, \frac{1}{2}\pi, \frac{1}{2}\pi)$ is the only element of $h^{-1}(0)$ with $a = b$. Thus S_1 is the curve $(a, b(a))$, $a \in A$. The symmetry of S_1 follows e.g. from (2.1).

Now, let $(a, b, r, s) \in h^{-1}(0)$, $b \geqq a$. Then $bs \leqq \frac{1}{2}\pi \leqq ar$ and $g(ar) = (b/a)^2 g(bs) \leqq (b/a)^2 g(\frac{1}{2}\pi) = (b/a)^2$, and analogously $g(bs) \geqq (a/b)^2$. Therefore

$$\frac{\pi}{2a} \leqq r \leqq \frac{1}{a} g^{-1}\left(\left(\frac{b}{a}\right)^2\right), \quad \frac{1}{b} g^{-1}\left(\left(\frac{a}{b}\right)^2\right) \leqq s \leqq \frac{\pi}{2b}.$$

It suffices to add these two inequalities (notice that $r + s = \pi$). The case $b \leqq a$ is quite similar. Thus we obtain the inclusion $S_1 \subset G_1$ which completes the proof. ∎

(2.6) **Lemma.** *Let $(a, b) \in]0, +\infty[^2$. Then the symmetrical solution is the only periodic solution of $R(a, b)$.*

Proof. The existence of the symmetrical solution follows from Theorem (2.4) and from (2.1), where λ is such that $(\lambda a, \lambda b) \in S_1$. Now let $u : [0, r_0] \to \boldsymbol{R}^1$ be a positive semi-wave of a solution of $R(a, b)$. There exist real constants A, B, C, D such that for $x \in [0, r_0]$ we have

$$u(x) = A \cos(ax) + B \sin(ax) + C \operatorname{ch}(ax) + D \operatorname{sh}(ax).$$

For $r \in [0, r_0]$ put

$$\begin{aligned} A(r) &= A \cos(ar) + B \sin(ar), \\ B(r) &= B \cos(ar) - A \sin(ar), \\ C(r) &= C \operatorname{ch}(ar) + D \operatorname{sh}(ar), \\ D(r) &= D \operatorname{ch}(ar) + C \operatorname{sh}(ar). \end{aligned}$$

Then

$$u(x) = A(r)\cos(a(x - r)) + B(r)\sin(a(x - r)) + C(r)\operatorname{ch}(a(x - r)) + D(r)\operatorname{sh}(a(x - r)), \quad r \in [0, r_0], \quad x \in [0, r_0].$$

Since $u'(0) = a(B(0) + D(0)) > 0$, it follows from (2.2) that $u'''(0) < 0$, and therefore $B(0) > 0$. Similarly we can show that $B(r_0) < 0$. Consequently, there exists an $r_1 \in]0, r_0[$ such that $B(r_1) = 0$. Obviously $A(r_1) > 0$, because $A(r_1) \leqq 0$, $u(r_1) \geqq 0$ imply $C(r_1) \geqq |A(r_1)|$ and the condition $u'(0) > 0$ implies $D(r_1) \geqq 0$, hence u is increasing, which is a contradiction. Now put $u_1(x) = (A(r_1))^{-1} u(x + r_1)$, $x \in [-r_1, r_0 - r_1]$. Then $u_1(x) = \cos(ax) + \gamma \operatorname{ch}(ax) + \delta \operatorname{sh}(ax)$, where

$$\gamma = C(r_1)(A(r_1))^{-1}, \quad \delta = D(r_1)(A(r_1))^{-1}.$$

Let $u_0 : [-\varrho, \varrho] \to \boldsymbol{R}^1$ be the positive semi-wave of the symmetrical solution of $R(a, b)$,

$$u_0(x) = \cos(ax) - \beta \operatorname{ch}(ax), \quad \text{where} \quad \beta = \frac{\cos(a\varrho)}{\operatorname{ch}(a\varrho)}.$$

Put $\varepsilon = \min\{r_1, r_0 - r_1, \varrho\}$. For $x \in [-\varepsilon, \varepsilon]$ we have

$$u_1(x) = u_0(x) + (\beta + \gamma)\operatorname{ch}(ax) + \delta \operatorname{sh}(ax).$$

The proof now follows from Lemma (1.1) for $u_1(x)$, $u_0(x)$ if sign $(\mathrm{b} + \gamma) =$ sign (δ) and for $u_1(-x)$, $u_0(-x)$ in the other case. ∎

Let us collect the results of this section in the following theorem.

(2.7) **Theorem.** *The set* $\tilde{A}_{-1}$ *of all* $(a, b) \in]0, +\infty[^2$, *for which there exists a nontrivial periodic solution of* $R(a, b)$ *of period* 2π, *is the system* $\{S_k, k \in N\}$ *of* C^∞*-curves, where* S_1 *is described in Theorem* (2.4), $S_k = \{(a, b) \in]0, +\infty[^2, (a/k, b/k) \in S_1\}$, *and* $S_k \subset G_k$, *where* $G_k = \{(a, b) \in]0, +\infty[^2, (a/k, b/k) \in G_1\}$. *In particular,* $A_{-1} \subset \bigcup_{k=1}^{\infty} G_k$. *For* $(a, b) \in S_k$ *the corresponding* 2π*-periodic solution has exactly* $2k$ *semi-waves in an interval of length* 2π. *This solution is unique if translations and positive multiples are not considered.*

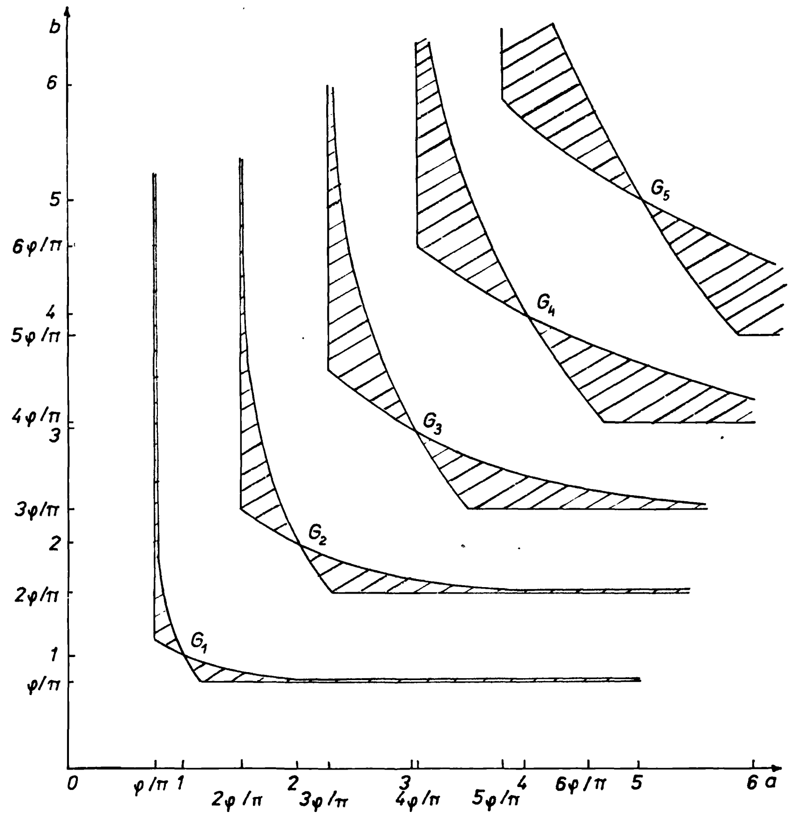

Fig. 1.

3. BOUNDARY-VALUE PROBLEM

The reasons for which the boundary value problem for the equation $R(a, b)$ is more difficult than the periodic one consist in the fact that its solutions with different numbers of semi-waves are essentially different. Nevertheless, we shall arrive at some existence results which will be summarized in Theorems (3.7), (3.8).

(3.1) Lemma. *Let $(a, b) \in]0, +\infty[^2$, and let $u : [0, +\infty[\to R^1$ be a nontrivial solution of $R(a, b)$, $u(0) = 0$. Then the following four conditions are equivalent:*

(i) *u is unbounded;*
(ii) *the set $u^{-1}(0)$ is bounded;*
(iii) $\lim u(x) = +\infty$ *or* $-\infty$ *if* $x \to +\infty$;
(iv) *there exists $x_0 \in [0, +\infty[$ such that $u^{(i)}(x_0)$ have the same sign for all $i = 0, 1, 2, 3$.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are obvious. Thus let us suppose non (ii): let $x_i \to +\infty$ be a sequence such that $u(x_i) = 0$ for each $i \in N$. Denote by $\xi_i \in]x_i, x_{i+1}[$ the point where $|u(\xi_i)| = \max \{|u(x)| \mid x \in]x_i, x_{i+1}[\}$. Then $u'(\xi_i) = 0$. Notice that the mapping

$$u \mapsto -2u'u''' + (u'')^2 + a^4(u^+)^2 + b^4(u^-)^2$$

is the first integral of $R(a, b)$. Consequently, the values of $|u(\xi_i)|^2$ are bounded by a multiple of the value of the first integral, i.e. non (i).

(3.2) Lemma. *Let α, β be real numbers. For $t \in R^1$ denote by u_t the solution of $R(a, b)$ with the initial conditions*

$$u_t(0) = 0\,, \quad u_t'(0) = \alpha\,, \quad u_t''(0) = \beta\,, \quad u_t'''(0) = t\,.$$

Then there exists $t_1 \in R^1$ such that u_{t_1} is bounded on $[0, +\infty[$.

Proof. Denote:

$$U^+ = \{t \in R^1 \mid \lim u_t(x) = +\infty \text{ if } x \to +\infty\}\,,$$
$$U^- = \{t \in R^1 \mid \lim u_t(x) = -\infty \text{ if } x \to +\infty\}\,.$$

By virtue of (3.1), (1.1) and the theorem of continuous dependence on initial conditions, U^+, U^- are open disjoint intervals, $U^+ =]t_+, +\infty[$, $U^- =]-\infty, t_-[$, $t_- \leqq t_+$. The proof now follows from (3.1). ∎

Consider now a solution $u : [0, +\infty[\to R^1$ of $R(a, b)$ and denote by

(3.3) $$x_1 < x_2 < x_3 < \ldots$$

the sequence (finite or infinite) of all its zeros. Notice that $u'(x_i) \neq 0$ for $i = 1, 2, \ldots$ (if $u'(x_i) = 0$, then (see (2.2)) x_i is the first or the last element of the sequence (3.3), and it will not be considered).

Put $z_i = u'''(x_i)/u'(x_i)$, $y_i = u''(x_i)/u'(x_i)$, $i = 1, 2, \ldots$.

(3.4) Lemma. *Let x_i, x_{i+1} be two successive points of the sequence (3.3). Then*

(i) *z_{i+1}, y_{i+1} are continuous functions of z_i, y_i,*
(ii) *z_{i+1}, y_{i+1} as functions of z_i, y_i are decreasing in both variables in their domains of definition.*

Proof. Part (i) is a consequence of the existence, unicity and continuous dependence of the solution on initial conditions. For proving part (ii) it is necessary to consider a semi-wave $u : [x_i, x_{i+1}] \to R^1$ of a solution of $R(a, b)$, and suppose $u'(x_i) > 0$ (the negative case is analogous). For the sake of simplicity put $x_i = 0$, $x_{i+1} = r$, $v_i = a^{-2}z_i$, $v_{i+1} = a^{-2}z_{i+1}$, $w_i = a^{-1}y_i$, $w_{i+1} = a^{-1}y_{i+1}$. Obviously, r can be considered a continuous function of v_i, w_i. For $x \in [0, r]$ we have

$$u(x) = \frac{1}{2a} u'(0) [(\mathrm{sh}(ax) + \sin(ax)) + w_i(\mathrm{ch}(ax) - \cos(ax)) + \\ + v_i(\mathrm{sh}(ax) - \sin(ax))]$$

and

$$u(x) = \frac{1}{2a} u'(r) [-(\mathrm{sh}(a(r-x)) + \sin(a(r-x))) + w_{i+1}(\mathrm{ch}(a(r-x)) - \\ - \cos(a(r-x))) - v_{i+1}(\mathrm{sh}(a(r-x)) - \sin(a(r-x)))] .$$

From the relations

$$u(0) = 0, \quad u(r) = 0, \quad v_{i+1} = u'''(r)/a^2 u'(r), \quad w_{i+1} = u''(r)/a\, u'(r)$$

we deduce the following system of three equations:

$$(A) \quad w_i = -P(r) v_i - Q(r),$$

$$(B) \quad w_{i+1} = P(r) v_{i+1} + Q(r),$$

$$(C) \quad -\frac{Q'(r)}{P'(r)} = \frac{v_i v_{i+1} - 1}{v_i + v_{i+1}},$$

where

$$P(r) = \frac{\mathrm{sh}(ar) - \sin(ar)}{\mathrm{ch}(ar) - \cos(ar)}, \quad Q(r) = \frac{\mathrm{sh}(ar) + \sin(ar)}{\mathrm{ch}(ar) - \cos(ar)}.$$

Now (A) yields

$$\frac{\partial r}{\partial v_i} = -\frac{P(r)}{P'(r) v_i + Q'(r)}.$$

But $P'(r) v_i + Q'(r) < 0$ for every v_i, w_i, because if $P'(r) v_i + Q'(r) \geqq 0$, then (note that $P'(r) > 0$)

$$v_i \geqq -\frac{Q'(r)}{P'(r)} = \frac{v_i v_{i+1} - 1}{v_i + v_{i+1}}, \quad \text{and} \quad v_i < 0, \quad v_{i+1} < 0,$$

hence $v_i^2 \leqq -1$, which is a contradiction.

Consequently, we have $\partial r/\partial v_i > 0$, and analogously $\partial r/\partial w_i > 0$. From (C) we obtain

$$\frac{\partial v_{i+1}}{\partial v_i} < 0, \quad \frac{\partial v_{i+1}}{\partial w_i} < 0,$$

from (B)

$$\frac{\partial w_{i+1}}{\partial v_i} < 0 \,, \quad \frac{\partial w_{i+1}}{\partial w_i} < 0 \,,$$

and the proof follows immediately. ■

Let us now consider the boundary-value problem for $R(a, b)$ on $[0, T]$ with boundary conditions

$$(3.5) \qquad \begin{aligned} u(0) = u(T) &= 0 \,, \\ \lambda\, u'(0) + \varkappa\, u''(0) &= 0 \,, \\ \sigma\, u'(T) + \tau\, u''(T) &= 0 \,, \end{aligned}$$

$$|\lambda| + |\varkappa| > 0 \,, \quad |\sigma| + |\tau| > 0 \,, \quad \varkappa \geqq 0 \,, \quad \tau \geqq 0 \,, \quad T > 0 \,.$$

Let u_t be the solution of $R(a, b)$ on $[0, +\infty[$ with the initial conditions

$$(3.6) \qquad u_t(0) = 0 \,, \quad u_t'(0) = \varkappa k \,, \quad u_t''(0) = -\lambda k \,, \quad u_t'''(0) = t \,,$$

$$k \neq 0 \,, \quad t \in \boldsymbol{R}^1 \quad (\text{see } (3.2)) \,.$$

It is possible to define the sequence (3.3) of all zeros of u_t and to put $y_i(t) = u_t''(x_i)/u_t'(x_i)$ and $z_i(t) = u_t'''(x_i)/u_t'(x_i)$ as in (3.4). From (3.4) it follows easily that y_i, z_i are strictly monotone continuous functions of t. By virtue of (1.1) and (3.2) their domain of definition D_i is a bounded open non-void interval for every $i \in \boldsymbol{N}$.

Let $t \to t_0$, $t \in D_i$. If $\lim_{t \to t_0} y_i(t)$ is finite, then obviously $\lim_{t \to t_0} z_i(t)$ is finite, and the continuous dependence on initial conditions implies that $t_0 \in D_i$.

Hence, y_i is a one-to-one continuous mapping from D_i onto $\boldsymbol{R}^1$. In particular, the equation

$$y_i(t) = -\frac{\sigma}{\tau} \quad \text{for} \quad \tau \neq 0$$

has a unique solution t_k. For $\tau = 0$, t_k is the suitable extreme point of D_i.

In fact, we have constructed two nontrivial solutions of the boundary-value problem (3.5) for $R(a, b)$ with $T = x_i(t_k)$, the first one corresponding to the case $k > 0$, the second one to the case $k < 0$.

The next theorem summarizes the results of this section.

(3.7) **Theorem.** *Let $(a, b) \in]0, +\infty[^2$. Then for each $i \in \boldsymbol{N}$ there exist two positive numbers T_i^+, T_i^- and two nontrivial solutions (together with their positive multiples) u_1, u_2 of the boundary-value problem (3.5) for the equation $R(a, b)$ with $T = T_i^+$ for u_1 and $T = T_i^-$ for u_2, and both u_1 and u_2 have in $[0, T]$ exactly $i + 1$ zeros.*

In a special case it is possible to prove the following theorem analogous to (2.7).

(3.8) **Theorem.** *Let $T > 0$ be fixed. Then the set of all $(a, b) \in]0, +\infty[^2$ such that there exists a nontrivial solution u of the boundary-value problem (3.5) for $R(a, b)$ with $\lambda = 0$, $\sigma = 0$, i.e.*

$$u(0) = u(T) = u''(0) = u''(T) = 0\,, \tag{3.9}$$

is a system of continuous curves $\{S_i^+, S_i^-, i \in N\}$ such that

(i) *for $(a, b) \in S_i^+$ (S_i^-) the solution u is of the type (3.6) with $k > 0$ ($k < 0$, respectively). This solution is uniquely determined by the choice of the constant k and it has in $[0, T]$ exactly $i + 1$ zeros,*

(ii) *S_i^+ is symmetrical to S_i^- with respect to the straight line $a = b$. If i is even, then $S_i^+ = S_i^-$.*

(iii) *for each $i \in N$ we have $(S_i^+ \cup S_i^-) \cap (S_{1+i}^+ \cup S_{i+1}^-) = \emptyset$.*

Proof. Fix $i \in N$. On each straight line $b = pa$, $p > 0$ there is a unique point (a_T, b_T) such that the solution u of the boundary-value problem (3.9) for $R(a_T, b_T)$ of the type (3.6) with $k > 0$, the existence of which is proved in (3.7), fulfils $T_i^+ = T$ (the argument is analogous to (2.1)), and the same is valid for $k < 0$. Hence, part (i) is a consequence of the continuous dependence of the solutions of $R(a, b)$ on the parameters a, b. Part (ii) is obvious. For proving part (iii), let us assume that there exists $(a, b) \in (S_i^+ \cup S_i^-) \cap (S_{i+1}^+ \cup S_{i+1}^-)$ and two corresponding solutions u_i, u_{i+1}. It is possible to choose the constants k in such a way that these solutions differ only in the value of the third derivative either at 0, or at T, which leads to a contradiction, and the proof is complete. ∎

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