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GENERALIZED UNIFORMLY LINE VULNERABLE DIGRAPHS

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1. INTRODUCTION, DEFINITIONS AND NOTATION

The digraphs considered in this paper are finite without loops and multiple lines. Let C_3 be the class of all strong digraphs, i.e. strongly connected digraphs, C_2 the class of all unilateral digraphs which are not strong, C_1 the class of all weak digraphs which are not unilateral and, finally, C_0 the class of all disconnected digraphs. We will also say that a digraph $D \in C_2$ ($D \in C_1$) is strictly unilateral (strictly weak). If $D \in C_i$ and x is a line (a point) of D such that $(D - x) \in C_j$ then x is called an i, j line (point).

In [1], F. Harary, R. Norman and D. Cartwright studied the so called uniformly line vulnerable digraphs, i.e. digraphs with the property that all lines have the same i, j value. They proved that the only uniformly line vulnerable digraphs are those in which every line is 1,0 line, every line is 2,0 line or every line is 3,2 line. In order to extend the concept of uniformly line vulnerable digraphs we give the following definitions.

Let $m_{ij}(D)$ denote the number of lines of a digraph D which are not i, j lines.

Definition. *A digraph D is (i, j) minimal, $i \neq j$, if D has an i, j line and $m_{ij}(D) \leqq m_{ij}(D')$ for every digraph D' such that D' has an i, j line.*

Thus, [1] contains a description, in our terminology, of all (i, j)-minimal digraphs D with $m_{ij}(D) = 0$. Further, there is no (i, j)-minimal digraph for $i < j$ nor a (3,0)-minimal digraph as there is no i, j line for $i < j$, nor a 3,0 line in any digraph. The aim of this paper is to describe the two remaining cases, the (3,1)-minimal and (2,1)-minimal digraphs.

Next we give some other definitions and conventions. Let D be a digraph. Then $V(D)$ is the set of points and $E(D)$ is the set of lines of D, respectively. We will denote by D^* the digraph which is the condensation of D. If there is a path in D from a point u to a point v then v is said to be reachable from u. If v is reachable from u

or u is reachable from v, then u and v are said to communicate in D. In the other cases u and v are separated. Let v be a point of a digraph D. Then $t\,d(v)$ is the total degree of v and the symbol C_v denotes the set of the cycles of D on which v lies, i.e. $C_v = \{C, C \text{ is a cycle of } D, v \in V(C)\}$. Let D be a strong digraph and let x be a 3,1 line of D. Let u and v be the unique transmitter and the unique receiver of $(D - x)^*$, respectively. Then D_x is a digraph that arises from the digraph $(D - x)^*$ by adding the line $x' = vu$. It is clear that D_x is also strong and the structure of D_x is "similar" to the structure of D. Further, x' is a 3,1 line of D_x and x' lies on every cycle of D_x. Finally, let a pseudocycle be a digraph obtained from a directed path by adding one arc from its first point to its last point.

The notions not defined here will be used in the sense of [2].

2. PRELIMINARIES

In this part of our paper we give the statements useful for the description of (i, j)-minimal digraphs.

Theorem 1 ([3]). *Let D be a strong digraph with at least two points .Then for every point v of D there exists a point $u(v) \neq v$ such that $D - u(v)$ is unilateral and v can be reached from every point in $D - u(v)$.*

Corollary 1 ([3]). *Let D be a strong digraph with at least two points. Then for every point v of D there exists a line $x(v)$ such that $D - x(v)$ is unilateral and v can be reached from every point in $D - x(v)$.*

Lemma 1. *Let D be a strong digraph and let x be a line of D incident with a point v. Then, if $D - v$ is unilateral, $D - x$ is unilateral as well.*

Proof. Let $x = zv$ be a line of D, $D \in C_3$, and assume that $D - v$ is unilateral. As D is strong, v can reach every point in $D - x$, i.e. $D - x$ is also unilateral. The proof of the line $x = vz$ is analogous to the preceding one.

Lemma 2. *Let D be a strong digraph, let x be a 3,1 line of D and let u, v, w be points of D_x. Then u and v are separated in $D_x - w$ iff $C_w \supset C_u$, $C_w \supset C_v$ and $C_u \cap C_v = \emptyset$.*

Proof. Let D be a strong digraph, let x be a 3,1 line of D and let r and t be the receiver and the transmitter of $(D - x)^*$, respectively. Assume that the points u, v, w of D_x satisfy $C_w \supset C_u$, $C_w \supset C_v$, $C_u \cap C_v = \emptyset$. Since $C_u \cap C_v = \emptyset$, the points u and v are separated in $(D - x)^*$. Further, $C_w \supset C_u$, $C_w \supset C_v$ implies that neither u nor v can reach the point r, or neither u nor v are reachable from the point t in $D_x - w$. Thus u and v are separated in $D_x - w$.

To prove the necessity, let u and v be separated in $D_x - w$. Assume that there is a path from u to v in $(D - x)^*$. Then there exists a cycle in D_x containing both u and v, i.e. u communications with v in $D_x - w$ for every point w of D_x, $u \neq w \neq v$. Thus $C_u \cap C_v = \emptyset$. Suppose now that there exists a cycle C of D_x such that $C \in C_u$, $C \notin C_w$. As the line $x' = rt$ belongs to every cycle of D_x and $C \notin C_w$, it means that v communicates with u. But this is a contradiction. Therefore $C_u \subseteq C_w$. Analogously we have $C_v \subseteq C_w$. Let $C_u = C_w$. In fact, this case cannot occur because $C_v \subseteq C_w$ and $C_u \cap C_v = \emptyset$. Thus $C_u \subset C_w$, $C_v \subset C_w$ and the proof is complete.

Theorem 2. *Let D be a strong digraph with a* 3,1 *line. Then there are points v_1, v_2 of D such that $D - v_i$ is unilateral, $i = 1, 2$, and v_1, v_2 are not adjacent.*

Proof. Let D be a strong digraph, and assume that x is a 3,1 line of D. Following Lemma 1 there exists a point in D which is a 3,1 point and hence D_x includes a 3,1 point as well. Put

$$k = \min \{|C_w|, w \text{ is a 3,1 point of } D_x\} .$$

Assume that a point z of D_x satisfies $|C_z| = k$. The digraph $D_x - z$ is strictly weak, therefore there are points z_1, z_2 of D_x which are separated in $D_x - z$ and clearly are not adjacent. According to Lemma 2 we get $C_z \supset C_{z_i}$, i.e. $|C_{z_i}| < k$, and $D_x - z_i$ is unilateral, $i = 1, 2$. Thus in D_x there exists a line $y' = z_1 w$, where the point w can reach every point in $D_x - z_1$. Let us denote by S_1 the strong component of $D - x$ corresponding to z_1 and by $y = ab$ the line of $D - x$ corresponding to y'. According to Theorem 1 there is a point v_1 of S_1 such that $S_1 - v_1$ is unilateral and a can be reached from every point in $S_1 - v_1$. Clearly, $D - v_1$ is also unilateral. Analogously, there is v_2 in S_2 such that $G - v_2$ is unilateral (S_2 is the strong component of $D - x$ corresponding to z_2). Since z_1, z_2 are not adjacent, v_1, v_2 have the same property. Thus, Theorem 2 is proved.

The next theorem follows immediately from Lemma 1 and Theorem 2.

Theorem 3. ([1]). *Any strong digraph with a* 3,1 *line has at least four lines that are not* 3,1.

3. MAIN RESULTS

Let M_0 be a digraph in Figure 1. Let M_n be the digraph that arises from M_0 by adding a path of length n, not containing the points u, v, from t to w. It is clear that there are exactly four lines in M_n which are not 3,1.

Theorem 4. *A necessary and sufficient condition for a digraph D to be* (3,1)-*minimal is that $D \cong M_n$, $n > 0$.*

Proof. According to Theorem 3 it follows that a digraph D is (3,1)-minimal iff there is a 3,1 line in D and exactly four lines of D are not 3,1 lines. Obviously, the digraph M_n, $n \in N$, has the property. To prove the necessity, let D be a (3,1)-minimal digraph and assume that x is a 3,1 line of D. In order to show that $D \cong M_n$ we will first of all prove that $D_x \cong M_n$, $n > 0$. The line x' corresponding to the line x of D is a 3,1 line of D_x, thus D_x has a 3,1 line. Assume that a line $y = vz$ of D_x is not 3,1. Then a line of $D - x$ corresponding to y and going from the strong component S_v

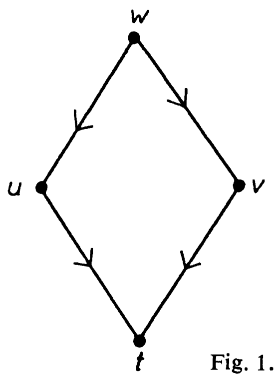

Fig. 1.

to the strong component S_z (there exists at least one) is not a 3,1 line of D. As D is (3,1)-minimal, D_x is also (3.1)-minimal. From Theorem 2 we get that there are points u, v in D_x which are not 3,1 and, further, u and v are not adjacent. No line of D_x adjacent with u or v is a 3,1 line (Lemma 1). Since D_x is (3,1)-minimal, every point w of D_x, $u \neq w \neq v$, is a 3,1 point and $t\,d(v) = t\,d(u) = 2$. By Lemma 2 we get that u and v form the unique pair of separated points in $D_x - z$, z is a 3,1 point (in the opposite case there would exist another point which were not 3,1, and this is impossible). Therefore, $C_v \cap C_u = \emptyset$ and $C_v \subset C_z$, $C_u \subset C_z$, $z \in V(D_x)$, $u \neq z \neq v$.

Let $y = wv$ and $y' = w'u$ be the lines of D_x. Since $C_w \supset C_v$, $C_{w'} \supset C_u$ and the digraph $(D - x)^*$ does not contain a cycle we get $w = w'$. Analogously, there is a point t of D_x such that vt and ut are lines of D_x. D_x is (3,1)-minimal, thus the subdigraph of D_x induced by the points u, v, w, t is isomorphic to M_0 or M_1. If this subdigraph is isomorphic to M_0 then there is a path P_n from t to w, because D_x is strong. There is no other point z in D_x because, if it were, then u and v would be mutually reachable in $D_x - z$ and this is a contradiction. By similar reasoning there is no other line in D_x. Thus, $D_x \cong M_n$, $n > 0$. It remains to show that $D \cong D_x$, i.e. we will prove that every strong component of $D - x$ is isomorphic to K_1. Let S_z be a strong component of $D - x$ corresponding to a point z of D_x, where $z \neq w$. Suppose that $z'z''$ is a line of D, where $z'' \notin S_z$. By Corollary 1 we get that there is a line k of S_z such that the point z' can be reached from every point of $S_z - k$ and $S_z - k$ is unilateral. Then $D - k$ is also unilteral. But this is a contradiction. Thus, $S_z \cong K_1$. Analogously, by using the statements dual to Corollary 1 we get that $S_w \cong K_1$. Therefore, $D_x \cong D$, i.e. $D \cong M_n$, $n > 0$.

It is known that the strong components of a strictly unilateral digraph can be ordered in such a way that S_i can be reached from S_j iff $i > j$. We use this arrangement of the strong components in the following theorem.

Theorem 5. *Let D be a digraph with at least three points and with n strong components $S_1, S_2, \ldots, S_n$. Then D is (2,1)-minimal iff the following three conditions are fulfilled:*

a) *the condensation of D is a pseudocycle;*

b) *there is at most one line from S_i to S_j;*

c) *$S_1 \cong S_n \cong K_1$ and either $S_i \cong K_1$ or there is no line x in S_i such that $S_i - x$ is unilateral and the point v_i can reach every point in $S_i - x$ and the point u_i can be reached from every point in $S_i - x$, where $u_{i-1}v_i$ is a line from S_{i-1} to S_i, for $i = 2, \ldots, n - 1$.*

Proof. According to [1], in every strictly unilateral digraph there exists at least one line which is not 2,1. As the pseudocycle has exactly one line that is not 2,1, a digraph D with q lines, $q \geqq 2$, is (2,1)-minimal iff it contains $q - 1$ lines of the type 2,1. Let now D be a (2,1)-minimal digraph. Assume that there are two lines from S_i to S_j. Then each of them is a 2,2 line, which is a contradiction. There must be a line x from S_1 to S_n. In the opposite case lines from S_i to S_{i+1} would be of type 2,0. There cannot be another line y from S_i to S_j, $j \neq i + 1$, because, if it were, then both x and y would be 2,2 lines. Thus, D is a pseudocycle. The strong component S_1 has only one point. In the opposite case, Corollary 1 implies that in S_1 there is a line which is a 2,2 line in D. Analogously for the strong component S_n. If S_i, $2 \leqq i \leqq n - 1$, is not isomorphic to K_1, then in S_i there is no line with the property given in c) because it would be a 2,2 line in D.

If D is a digraph satisfying the assumptions a)–c) then one can easily verify that a line from S_1 to S_n is a 2,2 line and the other lines are 2,1. Thus, the proof is complete.

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