Časopis pro pěstování matematiky

Jerzy Płonka

On some regularities of graphs. I.

Časopis pro pěstování matematiky, Vol. 107 (1982), No. 3, 231--240

Persistent URL: http://dml.cz/dmlcz/118129

Časopis pro pěstování matematiky, roč. 107 (1982), Praha

ON SOME REGULARITIES OF GRAPHS I

JERZY PŁONKA, Wrocław

(Received March 31, 1980)

§ 0

By a graph we shall mean a simple graph, i.e. a pair $G = (U; X)$ where U is a non-empty set of vertices (not necessarily finite) and X is a set of 2-element subsets of U, called the set of edges (see [1]). If u, v are two adjacent vertices, i.e. $\{u, v\} \in X$, we shall write $u \leftrightarrow v$. A graph G is said to be connected if for any two vertices $u, v \in U$, $u \neq v$ there exists a sequence $u = u_1, \ldots, u_n = v$ such that $u_i \leftrightarrow u_{i+1}$ for $i = 1, 2, \ldots, n - 1$. A maximal connected subgraph of G is called a component of G. If v is a vertex we denote by $\Gamma(v)$ the set of all vertices adjacent to v and by $\varrho(v)$ the degree of v, i.e. $\varrho(v) = |\Gamma(v)|$. Vertices of degree zero are called isolated, those of degree one are pendent. The graphs G in which $\varrho(v) = m = \text{const.}$ are called m-regular (see [1]). We denote $\varrho_\Gamma(v) = \sum_{u \in \Gamma(v)} \varrho(u)$; $\varrho_\Gamma(v) = 0$ if $\Gamma(v) = \emptyset$. In (2) we described all graphs G satisfying the condition $\varrho_\Gamma(v) = m = \text{const.}$ for all vertices v of G. For $v \in U$ let us denote $\varrho_\Gamma^+(v) = \varrho_\Gamma(v) + \varrho(v)$, $\varrho_\Gamma^-(v) = \varrho_\Gamma(v) - \varrho(v)$. We say that a vertex $v \in U$ has the arithmetical medium property if $|\Gamma(v)|$ is finite and $\varrho_\Gamma(v) = [\varrho(v)]^2$. Let $G = (U; X)$ be a graph. In § 1 we prove (Theorem 1) that $\varrho_\Gamma^+(v) = m = \text{const.}$ for any vertex $v \in U$ iff G is k-regular, where $k^2 + k = m$; k, m are non-negative integers. Further, we prove (Theorem 2) that any vertex v of G has the arithemtical medium property iff any component of G is a regular subgraph of G. In § 2 we consider $m - \Gamma^-$-regular graphs, i.e. graphs G for which $\varrho_\Gamma^-(v) = m = \text{const.}$ for any $v \in U$. The meaning of $m - \Gamma^-$-regularity is such that we count all exits from the vertices of $\Gamma(v)$ without those which lead to v. We describe (Theorem 3) connected $m - \Gamma^-$-regular graphs having pendent vertices and connected $m - \Gamma^-$-regular graphs G having a vertex v where $\varrho(v) = m$. Finally, we describe all $m - \Gamma$-regular graphs for $0 \leqq m \leqq 5$ and all $m - \Gamma^-$-regular graphs in which degrees of vertices assume exactly 2 values.

The paper does not contain a complete characterization of $m - \Gamma^-$-regular graphs, which seems to be difficult, but the research in this direction is continued by Z.

Majcher. We (with M. M. Sysło) study also some other extensions of Γ^- and Γ^+-regularities. The results of these efforts will appear in forthcoming papers (II and III).

Some generalizations of regular graphs are easily reconstructed, see [3], so that new types of regularities seem to be interesting for other aspects of graph theory.

§ 1

Lemma 1. *If in a graph $G = (U; X)$ we have $\varrho_\Gamma^+(v) = m$ for each $v \in U$ where m is a non-negative integer and there exist $u_1, v_1 \in U$ such that $u_1 \leftrightarrow v_1$, $\varrho(u_1) < \varrho(v_1)$, then there exist $u_2, v_2 \in U$ such that $u_2 \leftrightarrow v_2$ and $\varrho(u_2) < \varrho(u_1)$, $\varrho(v_1) \leqq \varrho(v_2)$.*

Proof. Denote $\varrho(u_1) = k$, $\varrho(v_1) = s$. By the assumption we have $m > 0 < k < s$. Let $s' = \max\{\varrho(u): u \in \Gamma(u_1)\}$ and let v_2 be a vertex from $\Gamma(u_1)$ for which $\varrho(v_2) = s'$. By the assumption we have

$$m = \varrho_\Gamma^+(u_1) \leqq k + k \,.\, s' \,. \tag{1}$$

Since $s' \geqq s > k$ we get by (1)

$$s' + s' \,.\, k > m \,. \tag{2}$$

Observe now that $u_1 \in \Gamma(v_2)$ and $\varrho(u_1) = k$. If $\varrho(v) \geqq k$ for each $v \in \Gamma(v_2)$ then (2) yields $m = \varrho_\Gamma^+(v_2) = s' + s'k > m$ – a contradiction. Thus there exists $u_2 \in \Gamma(v_2)$ where $\varrho(u_2) = k' < k < s'$ and $u_2 \leftrightarrow v_2$. Q.E.D.

Theorem 1. $\varrho_\Gamma^+(v) = m =$ const. *for any vertex $v \in U$ iff G is k-regular, where $k^2 + k = m$; k, m, are non-negative integers.*

Proof of $\Rightarrow$. If $m = 0$ then $\varrho(v) = 0$ for each $v \in U$ and G is 0-regular. Let $m > 0$ and suppose that G is not regular. Then there exist $u_1, v_1 \in U$ satisfying the assumptions of Lemma 1. Using repeatedly Lemma 1 we find that $u_n, v_n \in U$ with

$$1 = \varrho(u_n) < \varrho(v_n) = s_n \,. \tag{3}$$

We have $\varrho_\Gamma^+(u_n) = 1 + s_n = m$, $\varrho_\Gamma^+(v_n) \geqq s_n + s_n \,.\, 1 = 2s_n$. Hence $2s_n \leqq 1 + s_n$ and $s_n \leqq 1$ which contradicts (3). Thus the assumption that G is not regular leads to a contradiction. So G is k-regular for some k, but if $v \in U$ then $\varrho_\Gamma^+(v) = k + k \,.\, k = m$.

Proof of $\Leftarrow$ is obvious.

Theorem 2. *Every vertex v of G has the arithmetical medium property iff any component of G is a regular subgraph of G.*

Proof of $\Rightarrow$. Let C be a component of G and v a vertex of C such that $\varrho(v) = \min\{\varrho(u): u \in C\}$. Let $\Gamma(v) = \{u_1, \ldots, u_{\varrho(v)}\}$. We have

$$\frac{\sum_{u_i \in \Gamma(v)} \varrho(u_i)}{\varrho(v)} = \varrho(v) . \tag{4}$$

We also have $\varrho(u_i) \geqq \varrho(v)$ for $u_i \in \Gamma(v)$. If $\varrho(u_k) > \varrho(v)$ for some $u_k \in \Gamma(v)$ then we get a contradiction with (4). So $\varrho(u_i) = \varrho(v)$ for $u_i \in \Gamma(v)$ and since C is connected, $\varrho(u) = \text{const.}$ for any $u \in C$.

Proof of $\Leftarrow$ is obvious.

Remark 1. We can say that $v \in U$ has the geometrical medium property if $|\Gamma(v)|$ is finite and

$$\varrho(v) = \varrho(v) \sqrt[\varrho(v)]{[\prod_{u \in \Gamma(v)} \varrho(u)]} .$$

Then we can state a theorem similar to Theorem 2.

§ 2

A graph G is said to be a double m-star $(m \geqq 0)$ if G is of the form

$$(\{a_0, \ldots, a_m, b_0, \ldots, b_m\}; \{\{a_0, b_0\}\} \cup \{\{a_0, a_i\}; i = 1, \ldots, m\} \cup \cup \{\{b_0, b_i\}; i = 1, \ldots, m\}) .$$

Obviously a double 0-star is a single edge.

Theorem 3. *A connected graph $G = (U, X)$ with a pendent vertex is $m - \Gamma^-$-regular $(m \geqq 0)$ iff G is a double m-star.*

Proof of $\Rightarrow$. Let a be a pendent vertex in G. Let b be the vertex adjacent to a. If $m = 0$ then $\varrho(b) = 1$ and $b \leftrightarrow a$. Suppose that $m \geqq 1$. We have $\varrho_\Gamma^-(a) = \varrho(b) - \varrho(a) = \varrho(b) - 1 = m$. Hence $\varrho(b) = m + 1$ and as $m \geqq 1$ there must exist $c \leftrightarrow b$, $c \neq a$. We shall show that if $c \leftrightarrow b$ and $\varrho(c) > 1$ then in $\Gamma(c)$ there exists a pendent vertex. In fact, otherwise we have $\varrho(x) \geqq 2$ for any $x \in \Gamma(c)$, hence $m = \varrho_\Gamma^-(c) \geqq \varrho(b) + 2(\varrho(c) - 1) - \varrho(c) = m + 1 + \varrho(c) - 2 = m + \varrho(c) - 1$. Thus $m \geqq m + \varrho(c) - 1$ – a contradiction.

We shall show now that for any $c \in \Gamma(b)$ it must be $\varrho(c) = 1$ or $\varrho(c) = m + 1$. In fact, let $\varrho(c) = k$ where $1 < k < m + 1$. Then, as we have shown above, there exists $d \in \Gamma(c)$ and d is a pendent vertex. So $\varrho_\Gamma^-(d) = k - 1$ – a contradiction. Evidently, there exists $c \in \Gamma(b)$ with $\varrho(c) > 1$ so that $\varrho(c) = m + 1$. But there exists exactly one $c \in \Gamma(b)$ such that $\varrho(c) = m + 1$. Otherwise we have $\varrho_\Gamma^-(b) = i(m + 1) + m + 1 - i - (m + 1) = mi$ for some $1 < i \leqq m + 1$, which contradicts the fact that $\varrho_\Gamma^-(b) = m$. Let $\varrho(c) = m + 1$, $d \in \Gamma(c)$ and $d \neq b$, then $m = \varrho_\Gamma^-(d) = \varrho(c) - \varrho(d) = m + 1 - \varrho(d)$. Thus $\varrho(d) = 1$ for any $d \in \Gamma(c)$ and $d \neq b$ and we have a double m-star.

Proof of $\Leftarrow$ is obvious.

Theorem 4. *If a connected graph $G = (U; X)$ without pendent vertices is $m - \Gamma^-$-regular and possesses a vertex v with $\varrho(v) = m > 2$, then G is of the following form: $U = U_1 \cup U_2$ where $U_1 \cap U_2 \neq \emptyset$, $U_1 \neq \emptyset \neq U_2$, the subgraph induced by U_2 is 1-regular (a join of disjoint complete graphs with two vertices), the subgraph induced by U_1 is 0-regular (any vertex is isolated); any vertex from U_1 is adjacent exactly to m vertices from U_2, any vertex from U_2 is adjacent exactly to one vertex from U_1, and there are no more edges in G.*

Proof. Since $\varrho(v) = m > 2$, $\varrho_\Gamma^-(v) = m$ and there are no pendent vertices in G, it must be $\varrho(x) = 2$ for any $x \in \Gamma(v)$. Fix x and let $y \in \Gamma(x)$, $y \neq v$. We have $m = \varrho_\Gamma^-(x) = m + \varrho(y) - \varrho(x) = m + \varrho(y) - 2$. Hence $\varrho(y) = 2$. Let $v' \in \Gamma(y)$, $v' \neq x$, then we can find that $\varrho(v') = m$. Since G is connected it must be $\varrho(z) = m$ or $\varrho(z) = 2$ for any $z \in U$. Putting $U_1 = \{u: \varrho(u) = m\}$, $U_2 = \{u: \varrho(u) = 2\}$ we find the required description.

Lemma 2. *If a graph $G = (U; X)$ is $m - \Gamma^-$-regular and has no isolated or pendent vertices then $m \geqq \varrho(v) \geqq 2$ for any $v \in U$.*

Proof. In fact, for any $u \in \Gamma(v)$ we have $\varrho(u) \geqq 2$, hence

$$m = \varrho_\Gamma^-(v) = \sum_{u \in \Gamma(v)} \varrho(u) - \varrho(v) \geqq 2\,\varrho(v) - \varrho(v) \geqq \varrho(v) \geqq 2\,.$$

Theorem 5. *If a graph $G = (U; X)$ is connected, $2s - \Gamma^-$-regular, $s > 1$, has no pendent vertices and possesses no vertices of degree k, where $2 < k < s + 1$, but possesses a vertex v of degree $s + 1$, then G is of the following form: $U = U_1 \cup U_2$ where $U_1 \neq \emptyset \neq U_2$, $U_1 \cap U_2 = \emptyset$, the subgraph induced by U_1 is 1-regular, the subgraph induced by U_2 is 0-regular, any vertex from U_1 is adjacent exactly to s vertices from U_2, any vertex from U_2 is adjacent to 2 vertices from U_1, and there are no more edges in G.*

Proof. Since $\varrho(v) = s + 1$ there must exist exactly one vertex $v_1 \in \Gamma(v)$ with degree $s + 1$ and all other vertices from $\Gamma(v)$ have degree equal to 2. Analogously, if $\varrho(u) = 2$ and $u_1, u_2 \in \Gamma(u)$, $u_1 \neq u_2$, then $\varrho(u_1) = \varrho(u_2) = s + 1$, since $\varrho_\Gamma^-(u) = 2s$. As G is connected, any vertex G has degree either $s + 1$ or 2. Putting $U_1 = \{u: \varrho(u) = s + 1\}$, $U_2 = \{u: \varrho(u) = 2\}$ we get the required result.

In Figure 1 we have a $2s - \Gamma^-$-regular graph with $s = 2$.

Theorems 3, 4, 5 and Lemma 2 enable us to give a full characterization of $m - \Gamma^-$-regular graphs for $0 \leqq m \leqq 5$.

Corollary 1. *A graph G is $0 - \Gamma^{-}$-regular iff any component of G is either an isolated vertex or a double 0-star.*

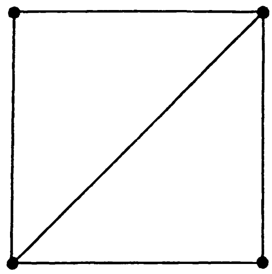

Fig. 1.

Proof of $\Rightarrow$. Let C be a component of G. In fact, if C is a component of G where $|C| > 1$ then by Lemma 2 there exist pendent vertices in G. So by Theorem 3 G is a double 0-star.

Proof of $\Leftarrow$ is obvious.

Corollary 2. *A graph G is $1 - \Gamma^{-}$-regular iff any component of G is a double 1-star.*

Proof of $\Rightarrow$ follows from Lemma 2 and Theorem 3.

Proof of $\Leftarrow$ – obvious.

Corollary 3. *A graph G is $2 - \Gamma^{-}$-regular iff any component of G is either a double 2-star or a 2-regular subgraph.*

Proof of $\Rightarrow$. Let C be a component of G. If there are pendent vertices in C then we use Theorem 3. If there are no pendent vertices then by Lemma 2 we get that $\varrho(v) = 2$ for any vertex from C.

Proof of $\Leftarrow$ is obvious.

Corollary 4. *A graph G is $3 - \Gamma^{-}$-regular iff any component of G is either a double 3-star or a graph described in Theorem 4 for $m = 3$.*

Proof of $\Rightarrow$. Let C be a component of G. If there are pendent vertices in C then we use Theorem 3. If there are no pendent vertices in C then by Lemma 2 any vertex v of C has degree either 2 or 3. But if $\varrho(v) = 2$ and $v_1 \leftrightarrow v \leftrightarrow v_2$, $v_1 \neq v_2$, then, since $\varrho_{\Gamma}(v) = 3$, it must be $\varrho(v_1) = 3$ and $\varrho(v_2) = 2$ or $\varrho(v_1) = 2$ and $\varrho(v_2) = 3$. Thus in C there exists always a vertex of degree 3 and we can use Theorem 4.

Proof of $\Leftarrow$ is obvious.

Corollary 5. *A graph G is $4 - \Gamma^{-}$-regular iff any component of G has one of the following forms:*

(5a) *C is a double* 4-*star*,

(5b) *C is a graph from Theorem* 4 *with* $m = 4$,

(5c) *C is a graph from Theorem* 5 *with* $s = 2$.

Proof of $\Rightarrow$. Let C be a component of G. If there are pendent vertices in C then C is a double 4-star by Theorem 3. If there are no pendent vertices in C and there exists a vertex with degree 4 then we use Theorem 4. Otherwise, by Lemma 2 any vertex v of C has degree either 2 or 3. But there are vertices of both kinds since if $\varrho(v) = 2$ and $v_1 \leftrightarrow v \leftrightarrow v_2$, $v_1 \neq v_2$, then as $\varrho_\Gamma^-(v) = 4$ we have either $\varrho(v_1) = 3$ or $\varrho(v_2) = 3$. Now we can use Theorem 5.

Proof of $\Leftarrow$ is obvious.

Theorem 6. *A graph* $G = (U; X)$ *is* $5 - \Gamma^{--}$*-regular iff any component C of G has one of the following forms*:

(6a) *C is a double* 5-*star*,

(6b) *C is a graph from Theorem* 4 *with* $m = 5$,

(6c) $V(C) = V(C_1) \cup V(C_2) \cup V(C_3) \cup V(C_4)$, *where* $V(C_1)$, $V(C_2)$, $V(C_3) \cup \cup V(C_4)$ *are not empty and pairwise disjoint*, C_1, C_2, C_3 *are* 0-*regular subgraphs*, C_4 *is a* 2-*regular subgraph*; *any vertex from* C_1 *is adjacent to* 3 *vertices from* C_2 *and one vertex from* $C_3 \cup C_4$; *any vertex from* C_2 *is adjacent to one vertex from* C_1 *and to one vertex from* $C_3 \cup C_4$; *any vertex from* C_3 *is adjacent to* 2 *vertices from* C_2 *and to one vertex from* C_1; *any vertex from* C_4 *is adjacent to one vertex from* C_2. *No vertex of* C_3 *is adjacent to a vertex of* C_4.

Proof of $\Rightarrow$. Let C be a component of G. If there are pendent vertices in C then we use Theorem 3. Otherwise, if there exists a vertex with degree 5 then we use Theorem 4. In the remaining case for $v \in C$ we can have $\varrho(v) = 2$, 3 or 4 by Lemma 2. But there are vertices in C of all kinds since if $\varrho(v_1) = 2$ and $v_2 \leftrightarrow v_1 \leftrightarrow v_3$, $v_2 \neq v_3$ then $\varrho(v_2) = 4$ and $\varrho(v_3) = 3$ or $\varrho(v_2) = 3$ and $\varrho(v_3) = 4$. If $\varrho(v) = 4$ then there are 3 vertices in $\Gamma(v)$ of degree 2 and one of degree 3. If $\varrho(v) = 3$, $\{v_1, v_2, v_3\} = \Gamma(v)$ and $\varrho(v_1) \geqq \varrho(v_2) \geqq \varrho(v_3)$, then we can have the following cases:

$$\varrho(v_1) = 4\,, \quad \varrho(v_2) = 2\,, \quad \varrho(v_3) = 2\,, \tag{5}$$

$$\varrho(v_1) = 3\,, \quad \varrho(v_2) = 3\,, \quad \varrho(v_3) = 2\,. \tag{6}$$

Putting $C_1 = \{v: \varrho(v) = 4\}$, $C_2 = \{v: \varrho(v) = 2\}$ and denoting C_3 the set of all vertices for which (5) holds and by C_4 the set of vertices for which (6) holds, we get our theorem.

Proof of $\Leftarrow$ is obvious.

In Figure 2 we have an example of a graph described in (6c).

Remark 2. It seems that an $m - \Gamma^{-}$-regular graph is probably a union of a complete n-partite graph and a graph consisting of regular subgraphs, but the greater m

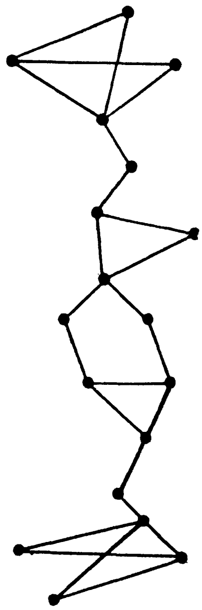

Fig. 2.

is the more complicated becomes the description. Some other results may be obtained by bounding the number of degrees of vertices in a graph $G = (U; X)$. Let us denote $D(G) = \{n: \exists_{u \in U} \varrho(u) = n\}$. Obviously, if $|D(G)| = 1$ then G is k-regular for some k and $(k^2 - k) - \Gamma^{-}$-regular. So we start studying an $m - \Gamma^{-}$-regular graph G, where $|D(G)| = 2$. We define a graph $G_{k,l} = (U_1 \cup U_2; X)$ where $U_1 \neq \emptyset \neq U_2$, $U_1 \cap U_2 = \emptyset$; for $u \in U_1$ we have $|\Gamma(u) \cap U_1| = 1$, $|\Gamma(u) \cap U_2)| = k$; for $v \in U_2$ we have $|\Gamma(v) \cap U_1| = l$, $|\Gamma(v) \cap U_2| = 0$; $k + 1 \neq l$; k and l are arbitrary non-negative integers, where $k = 0 \Leftrightarrow l = 0$. Obviously $\varrho(u) = k + 1$ for $u \in U_1$, $\varrho(v) = l$ for $v \in U_2$. We have

Theorem 7. *An $m - \Gamma^{-}$-regular graph G satisfies $|D(G)| = 2$ iff G is of the form $G_{k,1}$ where $kl = m$.*

Proof of $\Leftarrow$ is obvious.

Proof of $\Rightarrow$. Put $D(G) = \{p, q\}$. Denote $U_1 = \{u: u \in U, \varrho(u) = p\}$, $U_2 = \{v: v \in U, \varrho(v) = q\}$. Observe that if for some $u_0 \in U_i$ we have $|\Gamma(u_0) \cap U_j| = r$ then for any $u \in U_i$ it must be $|\Gamma(u) \cap U_j| = r$. In fact, let for instance $u_0 \in U_1$, $|\Gamma(u_0) \cap U_1| = r_1$ and for $u \in U_1$, $u \neq u_0$ let $|\Gamma(u) \cap U_1| = r_2$. Then $0 = m - m = \varrho_{\Gamma^-}(u_0) - \varrho_{\Gamma^-}(u) = r_1 p + (p - r_1) q - p - [r_2 p + (p - r_2) q - p] = (r_1 - r_2)(p - q)$ and $(p - q)(r_1 - r_2) = 0$ but $p \neq q$ so that $r_1 = r_2$.

For $u \in U_1$, denote $|\Gamma(u) \cap U_1| = t$, $|\Gamma(u) \cap U_2| = k$. For $v \in U_2$ denote $|\Gamma(v) \cap U_1| \Gamma(v) \cap U_1| = l$, $|\Gamma(v) \cap U_2| = s$. Thus

(7) $$k + t = p \neq q = l + s .$$

We have

(8) $$m = \varrho_{\Gamma^-}(u) = k(l + s) + t(k + t) - (k + t) ,$$

(9) $$\varrho_{\Gamma^-}(v) = l(k + t) + s(l + s) - (l + s) = m .$$

Hence

$$kl + ks + tk - k + t^2 - t = kl + lt + sl - l + s^2 - s ,$$

$$k(s + t - 1) + t^2 - t = l(s + t - 1) + s^2 - s ,$$

$$(k - l)(s + t - 1) = (s + t)(s - t) - (s - t)$$

and finally, $[(k + t) - (l + s)](s + t - 1) = 0$.

It must be $[(k + t) - (l + s)] \neq 0$ since otherwise we get $k + t = l + s$ which contradicts (7). Thus

(10) $$s + t - 1 = 0 .$$

But s and t must be non-negative integers so (10) implies:

(11) $$s = 0 , \quad t = 1 \quad \text{or}$$

(12) $$s = 1 , \quad t = 0 .$$

If (11) holds then by (7), $p = k + 1$, $q = l$ and $k + 1 \neq l$. By (8) or (9), $m = kl$. Moreover, $k = 0 \Leftrightarrow l = 0$ since if there is an edge connecting some vertex from U_1 with a vertex v in U_2 then $\varrho(v) > 0$. So in the case (11) the proof is finished. If (12) holds then the proof is analogous and it is enough to substitute in the end l by k, k by l, U_1 by U_2 and U_2 by U_1.

Remark 3. For any k, l satisfying the assumptions of Theorem 7, a graph $G_{k,l}$ always exists. In fact, if $k = l = 0$ it is enough to take a graph $(\{a, b, c\}; \{\{a, b\}\})$ with exactly one edge $\{a, b\}$ and to denote $U_1 = \{a, b\}$, $U_2 = \{c\}$. If $k, l > 0$ put $U_1 = A \cup B$, $U_2 = C \cup D$ where $A = \{a_1, \ldots, a_l\}$, $B = \{b_1, \ldots, b_l\}$, $C = \{c_1, \ldots, c_k\}$, $D = \{d_1, \ldots, d_k\}$, $\Gamma(a_i) = \{b_i\} \cup C$, $\Gamma(b_i) = \{a_i\} \cup D$, $i = (1, \ldots, l)$; $\Gamma(c_i) = A$, $\Gamma(d_i) = B$, $i = (1, \ldots, k)$; the sets A, B, C, D are pairwise disjoint.

Corollary 6. *For any $m > 0$ there exists at least $d(m)$ non-isomorphic $m - \Gamma^-$-regular graphs, where $d(m)$ denotes the number of positive divisors of m.*

Remark 4. Theorems 4 and 5 can be also derived from Theorem 7, but the proofs given above are esentially shorter.

If we have $D(G) = 3$ we obtain much more possibilities. For instance, the graphs in Figures 3 and 4 are $10 - \Gamma^{-}$-regular.

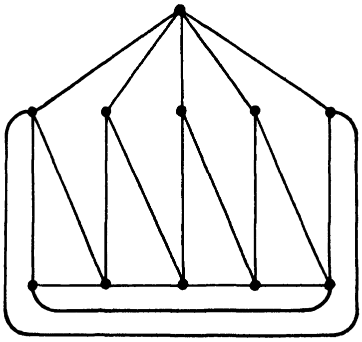

Fig. 3.

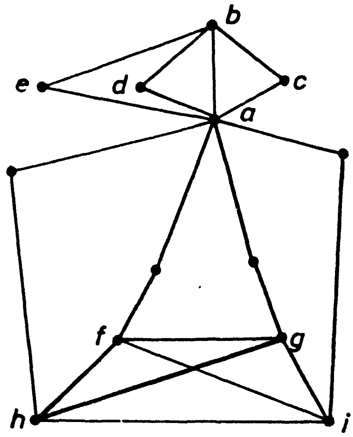

Fig.4.

Finding a representation of $m - \Gamma^{-}$-regular graphs G for which $|D(G)| = 3$ is difficult since two vertices with the same degree need not be adjacent to the same number of vertices of a given degree, as was the case with $|D(G)| = 2$. For example, the vertex b in Figure 4 is adjacent to a with $\varrho(a) = 8$ and the vertex f is adjacent to no vertex of degree 8.

Thus if $|D(G)| = 3$, the problem cannot be quickly reduced to finding non-negative solutions of some equations as in the case $|D(G)| = 2$.

One can ask if the existence of an $m - \Gamma^{-}$-regular graph with $D(G) = \{p_1, \ldots, p_n\}$ implies the existence of a finite graph G' with $D(G') = \{p_1, \ldots, p_n\}$. The next theorem gives the negative answer.

Theorem 8. *If* $G = (U; X)$ *is an* $m - \Gamma^{-}$*-regular graph,* $m > 5$ *and* $D(G) = \{2, 3, m - 1\}$, *then* G *is infinite.*

Proof. Let $U_1 = \{u: \varrho(u) = m - 1\}$, $U_2 = \{v: \varrho(v) = 3\}$, $U_3 = \{w: \varrho(w) = 2\}$. Let $u \in U_1$ and let S be the connected component which contains u. It must be

$$(13) \qquad |\Gamma(u) \cap U_1| = 0\,, \quad |\Gamma(u) \cap U_2| = 1\,, \quad |\Gamma(u) \cap U_3| = m - 2\,.$$

Otherwise we have $\varrho_{\Gamma^-}(u) > 3 + (m - 2) 2 = m$ — a contradiction. Thus there exists exactly 1 vertex $v \in U_2 \cap S$ adjacent to u. Let $v' \in U_2 \cap S$. It must be $|\Gamma(v') \cap \cap U_1| \geqq 1$. Otherwise, even if $|\Gamma(v') \cap U_2| = 3$, we have $\varrho(v') = 6$ which is a contradiction in the case $m > 6$. If $m = 6$, $|\Gamma(v') \cap U_2| = 3$, then it is easy to check that any vertex v'' adjacent to v' satisfies $|\Gamma(v'') \cap U_2| = 3$ and consequently any vertex in S has this property, which contradicts (13). So it must be

(14) $|\Gamma(v') \cap U_1 \cap S| = 1, \quad |\Gamma(v') \cap U_2 \cap S| = 0, \quad |\Gamma(v') \cap U_3 \cap S| = 2.$

We again see that in S there exists exactly 1 vertex u' adjacent to v' with $\varrho(u') = m - 1$. Now we can state

(15) $$|U_1 \cap S| = |U_2 \cap S| .$$

If $w \in U_3 \cap S$ then it must be $|\Gamma(w) \cap U_1 \cap S| = 1$, $|\Gamma(w) \cap U_2 \cap S| = 1$, $|\Gamma(w) \cap U_3 \cap S| = 0$. Thus if $|U_1 \cap S| = |U_2 \cap S| = k < \infty$ then $|U_3| = k(m - 2) = k \,.\, 2$ by (13), (14), (15). So $m - 2 = 2$, $m = 4$ – a contradiction.

Remark 5. There exists an infinite $m - \Gamma^-$-regular graph $G = (U; X)$ with $D(G) = \{2, 3, m - 1\}$ for any $m > 5$. In fact, put $U = U_1 \cup U_2 \cup U_3$, where U_1, U_2, U_3 are pairwise disjoint, $U_1 = \{a_1, a_2, \ldots\}$, $U_2 = \{b_1, b_2, \ldots\}$, $U_3 = \{c_1, c_2, \ldots\}$. $\Gamma(a_i) = \{b_i, c_{f(i)+1}, \ldots, c_{f(i)+m-2}\}$ $(f(i) = (i - 1)(m - 2))$, $\Gamma(b_i) = \{a_i, c_{g(i)+1}, c_{g(i)+2}\}$ $(g(i) = (e - 1)\, 2)$, $\Gamma(c_i) = \{a_{d(i)}, b_{e(i)}\}$ where $d(i) = \min\{k : k(m - 2) \geqq i\}$, $e(i) = \min\{r : 2r \geqq i\}$.

Remark 6. If $m = 5$ then Theorem 8 is not true which is shown by the graph in Figure 2.

From our considerations it is seen that finding a description of all Γ^- regular graphs is not simple. However, we can state some a little easier problems the solution of which can be perhaps helpful for answering the general question.

Problem 1. Describe all Γ^--regular graphs G in which $|D(G)| = 3$.

Problem 2. Let $m > 5$, $1 < k < m$, $1 < q_i < m$ $(i = 1, \ldots, k)$; $\sum_{i=1}^{k} q_i = k + m$. Find an algorithm of constructing a finite $m - \Gamma^-$-regular graph G having a vertex u such that $\varrho(u) = k$, $\Gamma(u) = \{u_1, \ldots, u_k\}$, $\varrho(u_i) = q_i$ $(i = 1, \ldots, k)$.

References

[1] *F. Harary:* Graph theory. London. Addison-Wesley 1969.

[2] *J. Płonka:* On $m - \Gamma$-regular graphs. Coll. Math. (to appear).

[3] *S. Rama Chandran:* Nearly regular graphs and their reconstruction. Graph Theory Newsletter *8* (1978), 3.

Author's address: ul. Ślęzna 9 m 6, 53-301 Wrocław.