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SOLUTION OF THE PROBLEM OF DIRECTLY DECOMPOSABLE HOMOMORPHISMS

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathfrak{A}_i = \langle A_i, F \rangle$, $\mathfrak{B}_i = \langle B_i, F \rangle$, $i \in I$, be algebras of the same type and let h_i be a homomorphism of $\mathfrak{A}_i$ into $\mathfrak{B}_i$ (in symbols: $h_i \in \operatorname{Hom}(\mathfrak{A}_i, \mathfrak{B}_i)$) for every $i \in I$. Then it is a trivial fact that the mapping h defined by the rule $h((a_i)_{i\in I}) = (h_i(a_i))_{i\in I}$, $(a_i)_{i\in I} \in \prod_{i\in I} A_i$, is a homomorphism of $\prod_{i\in I} \mathfrak{A}_i$ into $\prod_{i\in I} \mathfrak{B}_i$; denote it by $h = (h_i)_{i\in I}$. Homomorphisms of this form are called *directly decomposable homomorphisms*, briefly: *DDHom*. It is well-known (see e.g. [1]) that not every homomorphism of $\prod_{i\in I} \mathfrak{A}_i$ into $\prod_{i\in I} \mathfrak{B}_i$ is directly decomposable and so a natural question arises: Under which conditions is a given homomorphism of $\prod_{i\in I} \mathfrak{A}_i$ into $\prod_{i\in I} \mathfrak{B}_i$ directly decomposable?

This problem, i.e. the so called problem of DDHom, was investigated in a number of papers, see e.g. [6, 7, 8, 10] and references there. These papers include many useful results dealing with the necessary or sufficient conditions for DDHom on various types of algebras. However, it appears rather difficult to find the full characterization of this phenomenon, i.e. to state necessary and sufficient conditions for DDHom.

Recently, the problem of DDHom was solved for the products of two similar algebras and thus, evidently, for the products of any finite family of similar algebras, see [1]. The aim of the present paper is to generalize the results of [1] for the products of arbitrary families of similar algebras, i.e. to give the full characterization of DDHom.

In order to avoid interrupting the discussion later, let us first recall some preliminary concepts and results:

For any product $A = \prod_{i\in I} A_i$ of nonempty sets A_i, $i \in I$, there are binary operations $d_i : A \times A \to A$, $i \in I$, introduced by H. Werner [12] as follows:

$$pr_j\, d_i(x, y) = \begin{cases} pr_j\, x & \text{for } j \neq i \\ pr_i\, y & \text{for } j = i\,. \end{cases}$$

These operations are closely related with the canonical projections $pr_i : A \to A_i$, $i \in I$; one can easily verify that $\operatorname{Ker} pr_i = \{(x, y) \in A \times A;\ x = d_i(x, y)\}$ for each $i \in I$.

Further, let us recall that the nonindexed product $\bigotimes_{i \in I} \mathfrak{A}_i$ of algebras $\mathfrak{A}_i = \langle A_i, F_i \rangle$, $i \in I$, is the algebra $\langle \prod_{i \in I} A_i, F \rangle$, where any n-ary operation $f \in F$ corresponds to a certain sequence of n-ary polynomials p_i of $\mathfrak{A}_i$, $i \in I$, and is defined by $f((a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I}) = (p_i(a_i^1, \ldots, a_i^n))_{i \in I}$ for any elements $(a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I} \in \prod_{i \in I} A_i$; see [3], [4; p. 357], [5] and [11]. For the sake of brevity we identify the operation f with the sequence $(p_i)_{i \in I}$, i.e. we write $f = (p_i)_{i \in I}$.

The nonindexed product is defined for algebras of various similarity types. However, if algebras $\mathfrak{A}_i$, $\mathfrak{B}_i$, $i \in I$, are of the same type, then the algebras $\bigotimes_{i \in I} \mathfrak{A}_i$ and $\bigotimes_{i \in I} \mathfrak{B}_i$ are of the same type as well and so the symbol $\operatorname{Hom}(\bigotimes_{i \in I} \mathfrak{A}_i, \bigotimes_{i \in I} \mathfrak{B}_i)$ has the usual meaning.

2. CHARACTERIZATION OF DDHom

The main result of this paper is the following

Theorem 1. *Let $\mathfrak{A}_i = \langle A_i, F \rangle$, $\mathfrak{B}_i = \langle B_i, F \rangle$, $i \in I$, be algebras of the same type. Then for any homomorphism $h \in \operatorname{Hom}(\prod_{i \in I} \mathfrak{A}_i, \prod_{i \in I} \mathfrak{B}_i)$, the following conditions are equivalent*:

(1) *h is directly decomposable*;

(2) $h \in \operatorname{Hom}(\bigotimes_{i \in I} \mathfrak{A}_i, \bigotimes_{i \in I} \mathfrak{B}_i)$;

(3) $h \in \operatorname{Hom}(\langle \prod_{i \in I} A_i, \{d_i;\ i \in I\} \rangle, \langle \prod_{i \in I} B_i, \{d_i;\ i \in I\} \rangle)$;

(4) *h preserves the kernels of all canonical projections. i.e. $(h \times h) \operatorname{Ker} pr_i \subseteq \operatorname{Ker} pr_i$ for all $i \in I$.*

Proof. (1) $\Rightarrow$ (2). Choose an n-ary operation $f = (p_i)_{i \in I}$ of the nonindexed product $\bigotimes_{i \in I} \mathfrak{A}_i$. By hypothesis, the homomorphism h is directly decomposable, i.e. h is of the form $h = (h_i)_{i \in I}$, where $h_i \in \operatorname{Hom}(\mathfrak{A}_i, \mathfrak{B}_i)$ for all $i \in I$. Clearly, every homomorphism h_i preserves the polynomial p_i, $i \in I$, and, consequently, $h = (h_i)_{i \in I}$ preserves the operation $f = (p_i)_{i \in I}$.

(2) $\Rightarrow$ (3). This implication follows directly from the fact that for each $i \in I$,

$$d_i = (d_{ij})_{j \in I} \quad \text{where} \quad d_{ij} = \begin{cases} e_1^2 & \text{for} \quad i \neq j \\ e_2^2 & \text{for} \quad i = j\,, \end{cases}$$

and that the trivial operations e_1^2, e_2^2 ($x = e_1^2(x, y)$, $y = e_2^2(x, y)$) are polynomials of any algebra.

(3) $\Rightarrow$ (4). Let $i \in I$ and take $(x, y) \in \mathrm{Ker}\, pr_i$. As we noted above, see Section 1, $(x, y) \in \mathrm{Ker}\, pr_i$ implies the identity $x = d_i(x, y)$. Now, by applying the hypothesis, we get $h(x) = h(d_i(x, y)) = d_i(h(x), h(y))$ and thus $(h(x), h(y)) \in \mathrm{Ker}\, pr_i$, proving the inclusion $(h \times h)\, \mathrm{Ker}\, pr_i \subseteq \mathrm{Ker}\, pr_i$.

(4) $\Rightarrow$ (1). Firstly, we claim that for an arbitrarily chosen $i \in I$, the correspondence $a_i \mapsto pr_i\, h((a_i)_{i\in I})$, $(a_i)_{i\in I} \in \prod_{i\in I} A_i$, is a mapping. Take an element $(a_i')_{i\in I} \in \prod_{i\in I} A_i$ such that $a_i = a_i'$, i.e. $(a_i')_{i\in I}$ is an element with the property $((a_i)_{i\in I}, (a_i')_{i\in I}) \in \mathrm{Ker}\, pr_i$. By applying the hypothesis we have $(h((a_i)_{i\in I}, h(a_i')_{i\in I}) \in \mathrm{Ker}\, pr_i$ or, equivalently, $pr_i\, h((a_i)_{i\in I} = pr_i\, h((a_i')_{i\in I})$. However, this proves that the correspondence $a_i \mapsto pr_i\, h((a_i)_{i\in I}$ is a mapping of A_i into B_i; denote it by h_i.

Secondly, the proof that $h_i \in \mathrm{Hom}\,(\mathfrak{A}_i, \mathfrak{B}_i)$, $i \in I$, and $h = (h_i)_{i\in I}$ is straightforward and hence omitted.

Remark 1. It is an interesting fact that for $I = \{1, \ldots, n\}$ the set of binary operations $\{d_i;\ 1 \leqq i \leqq n\}$ may be replaced by one n-ary operation d^n defined by the rule $d^n = (e_i^n)_{i\leqq n}$, where $e_1^n, \ldots, e_n^n$ are the n-ary trivial operations $(x_i = e_i^n(x_1, \ldots, x_n))$. Clearly, d^n is the operation of the nonindexed product $\bigotimes_{i\leqq n} \mathfrak{A}_i$ of any algebras $\mathfrak{A}_i = \langle A_i, F_i\rangle$, $1 \leqq i \leqq n$, such that $d^n((a_i^1)_{i\leqq n}, \ldots, (a_i^n)_{i\leqq n}) = (a_1^1, \ldots, a_n^n)$ for $(a_i^1)_{i\leqq n}, \ldots, (a_i^n)_{i\leqq n} \in \prod_{i\in I} A_i$ and thus we have the folloving relationship between d^n and d_i, $1 \leqq i \leqq n$: $d_i(x, y) = d^n(\underbrace{x, \ldots, x}_{(i-1)\text{-times}}, y, x, \ldots, x)$, $1 \leqq i \leqq n$.

Notice that d^n is the well-known n-dimensional canonical diagonal operation, see [2], [9].

Combining these facts with the preceding theorem, we readily obtain

Theorem 2. *Let $\mathfrak{A}_i = \langle A_i, F\rangle$, $\mathfrak{B}_i = \langle B_i, F\rangle$, $1 \leqq i \leqq n$, be algebras of the same type. Then for any homomorphism $h \in \mathrm{Hom}\,(\prod_{i\leqq n} A_i, \prod_{i\leqq n} B_i)$ the following conditions are equivalent:*

(1) *h is directly decomposable;*

(2) $h \in \mathrm{Hom}\,(\bigotimes_{i\leqq n} \mathfrak{A}_i, \bigotimes_{i\leqq n} \mathfrak{B}_i)$;

(3′) $h \in \mathrm{Hom}\,(\langle \prod_{i\leqq n} A_i, d^n\rangle, \langle \prod_{i\leqq n} B_i, d^n\rangle)$;

(3) $h \in \mathrm{Hom}\,(\langle \prod_{i\leqq n} A_i, \{d_1, \ldots, d_n\}\rangle, \langle \prod_{i\leqq n} B_i, \{d_1, \ldots, d_n\}\rangle)$;

(4) $(h \times h)\, \mathrm{Ker}\, pr_i \subseteq \mathrm{Ker}\, pr_i$, $1 \leqq i \leqq n$.

Proof. Immediate from Theorem 1 and Remark 1.

3. APPLICATIONS

For rings with 1 and for lattices with 0, 1, the results of Section 2 enable us to derive more detailed conditions characterizing the DDHom. In this section we write $\bar{a}$ instead of $(a_i)_{i\in I}$ if $a_i = a$ for every $i \in I$.

Corollary 1. *Let R_i, S_i, $i \in I$, be arbitrary rings with 1. Then for any homomorphism $h \in \mathrm{Hom}\,(\prod_{i\in I} R_i, \prod_{i\in I} S_i)$, the following conditions are equivalent:*

(1) *h is directly decomposable;*

(2) *h preserves the elements $d_i(\bar{0}, \bar{1})$, $i \in I$, i.e. $h(d_i(\bar{0}, \bar{1})) = d_i(\bar{0}, \bar{1})$ for each $i \in I$;*

(2′) *h preserves the elements $d_i(\bar{1}, \bar{0})$, $i \in I$.*

Proof. (1) ⇒ (2) is evident.

The equivalence (2) ⇔ (2′) follows from the fact that $d_i(\bar{1}, \bar{0}) + d_i(\bar{0}, \bar{1}) = \bar{1}$ for every $i \in I$.

(2′) ⇒ (1). It can be easily seen that $d_i(x, y) = x\, d_i(\bar{1}, \bar{0}) + y\, d_i(\bar{0}, \bar{1})$. By applying the hypothesis (2′) and, equivalently, (2), we get $h(d_i(x, y)) = h(x\, d_i(\bar{1}, \bar{0}) + y\, d_i(\bar{0}, \bar{1})) = h(x)\, d_i(\bar{1}, \bar{0}) + h(y)\, d_i(\bar{0}, \bar{1}) = d_i(h(x), h(y))$. By virtue of Theorem 1(3), condition (1) follows.

Corollary 2. *Let L_i, M_i, $i \in I$, be lattices with nullary operations 0, 1. Then for any homomorphism $h \in \mathrm{Hom}\,(\prod_{i\in I} L_i, \prod_{i\in I} M_i)$, the following conditions are equivalent:*

(1) *h is directly decomposable;*

(2) *h preserves the elements $d_i(\bar{1}, \bar{0})$ and $d_i(\bar{0}, \bar{1})$, $i \in I$.*

Proof. (1) ⇒ (2) is quite clear.

(2) ⇒ (1). Obviously, $d_i(x, y) = (x \wedge d_i(\bar{1}, \bar{0})) \vee (y \wedge d_i(\bar{0}, \bar{1}))$ and so $h(d_i(x, y)) = h((x \wedge d_i(\bar{1}, \bar{0})) \vee (y \wedge d_i(\bar{0}, \bar{1}))) = (h(x) \wedge d_i(\bar{1}, \bar{0})) \vee (h(y) \wedge d_i(\bar{0}, \bar{1})) = d_i(h(x), h(y))$ for every $i \in I$, entalling the direct decomposability of h.

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