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THE ISOPERIMETRIC INEQUALITY FOR A PENTAGON IN E_3 AND ITS GENERALIZATION IN E_n SPACE

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1. INTRODUCTION

Isoperimetric problems concern the connection between the curve length and the volume of its convex envelope in a certain curve class in an n-dimensional Euclidean space E_n. The result is an isoperimetric inequality — a bound for the volume of the convex envelope of the curve depending on its length L, and finding of such curves that the volume of their convex envelopes becomes maximum. These problems may be solved for both the closed and the open curves. Four isoperimetric problems may appear, with regard to the parity of the dimension of the space E_n : 1. n is even and the curve is closed; 2. n is even and the curve is open; 3. n is odd and the curve is closed; 4. n is odd and the curve is open.

The first problem in the class of convex curves was solved by I. J. Schoenberg in 1954 in the paper [7]. To prove the inequality $L^n \geqq V(\pi n)^{n/2}\, n!\, (n/2)!$ he used the method of A. Hurwitz and the Fourier series.

The second problem in the class of convex curves was solved by A. A. Nudel'man in 1975 in the paper [6]. First he solved the isoperimetric problem for the convex envelope in the class of convex N-sided polygons of a length L in E_n, and then by passing to the limit for $N \to \infty$ he got the inequality $L^n \geqq V \,.\, n!\,(n-1)!!\,.\,(\pi \,.\, n/2)^{n/2}$.

The third problem for $n = 3$ was solved by Z. A. Melzak in his works from 1960 to 1968 under the following strict restrictions on the curve: Let it be a closed smooth curve of the class C^1 with two planes, $x = 0$ and $y = 0$, as planes of its symmetry. Its projections in these planes are open convex curves, and its projection in the plane $z = 0$ is a closed convex curve. If the curve is given parametrically by functions $x(s), y(s), z(s)$, then the volume of its convex envelope is maximum if and only if it is a periodic solution of the differential equations $x'' = -xy^2$, $y'' = -yx^2$, $z' = xy$, see [3] and [4]. In his paper [5] published in 1968 he calculated numerically the best constant B in the inequality $L^3 \geqq B \,.\, V$. He calls it the Baggins constant.

The fourth problem in the class of convex curves in E_n was solved in 1949 by E. Egervary in [1]. He proved the inequality $L^3 \geqq 18\sqrt{(3)}\,\pi\,.\,V$ by converting the problem to a plane isoperimetric problem. Egervary's result was then generalized by M. G. Krein and A. A. Nudel'man in [2]. They proved the inequality $L^n \geqq (1/2)\,V\,.\,\pi^{(n-1)/2}\,.\,n!\,(n-1)!\,n^{n/2}$ on the basis of Schoenberg's results and theorems from the theory of moments.

In the present paper an isoperimetric inequality is found for $(n+2)$-polygons in E_n, the convex envelope of which is either a simplex or a pair of simplexes with a common wall.

We shall first investigate the convex envelope of $n+2$ points from E_n. Therefore, let $A = (A_1, \ldots, A_{n+2})$ be an ordered $(n+2)$-tuple of points from E_n. These points must be linearly dependent, therefore there exist real numbers $c_1, \ldots, c_{n+2}$ so that

$$\sum_{i=1}^{n+2} c_i = 0\,, \quad (c_1, \ldots, c_{n+2}) \neq (0, \ldots, 0)$$

and

$$(1) \qquad c_1 A_1 + \ldots + c_{n+2} A_{n+2} = (0, \ldots, 0)\,.$$

We can evidently assume that the number of negative coefficients $c_1, \ldots, c_{n+2}$ is smaller or equal to the number of positive coefficients. We shall say that the $(n+2)$-tuple A is of the type 0, provided all points $A_1, \ldots, A_{n+2}$ are located in a certain subspace E_k of the space E_n, where $k < n$. If an $(n+2)$-tuple A is not of the type 0, $n+1$ points from the points $A_1, \ldots, A_{n+2}$ are linearly independent and, therefore, the coefficients $c_1, \ldots, c_{n+2}$ are determined, except for a non-zero factor, uniquely. Therefore we can define that an $(n+2)$-tuple A, which is not of the type 0, is of the type k, provided that exactly k coefficients $c_1, \ldots, c_{n+2}$ are negative. It is easy to prove that an $(n+2)$-tuple A is of the type 1 or 2 if and only if its convex envelope is a simplex or a pair of simplexes with a common wall, respectively. In the space E_3 there clearly exist only $(n+2)$-tuples of types 0, 1 and 2. If an $(n+2)$-tuple A is of the type 2, we shall call the points of A, which do not lie in the common wall of the corresponding pair of simplexes, opposite vertices. Clearly, a given $(n+2)$-tuple is of the type 2 if and only if it includes such two points that the segment determined by them intersects the simplex formed by the remaining points of the $(n+2)$-tuple (this includes a simplex in a hyperplane of the space E_n). The two points are then opposite vertices of the investigated group of $n+2$ points.

Let us also mention the symbols we shall use. For any points $X_1, \ldots, X_m \in E_n$ the symbol $\overline{X_1X_2}$ denotes the distance between the points X_1 and X_2, the symbol $\sphericalangle X_1X_2X_3$ the angle with X_2 at its vertex and sides containing the points X_1 and X_3, the symbol $\{X_1, \ldots, X_m\}$ is used to denote the subspace of the space E_n, spanned by the points $X_1, \ldots, X_m$, the symbol $[X_1, \ldots, X_m]$ denotes the closed m-sided polygon with the vertices $X_1, \ldots, X_m$ (i.e. a cyclically ordered set of points), the symbol $L(X_1, \ldots, X_m)$ denotes the length of this m-sided polygon (i.e. the number $\overline{X_1X_2} + \ldots$

$\ldots + \overline{X_{m-1}X_m} + \overline{X_mX_1})$, the symbol $V(X_1, \ldots, X_m)$ denotes the volume of the convex envelope of the m-sided polygon $[X_1, \ldots, X_m]$ considered in the space $\{X_1, \ldots, X_m\}$. When denoting $X = [X_1, \ldots, X_m]$ we shall also write $L(X)$ instead of $L(X_1, \ldots, X_m)$.

As regards an $(n + 2)$-sided polygon $[A_1, \ldots, A_{n+2}]$ we shall say that it is of the type k, provided the unordered $(n + 2)$-tuple of the points $(A_1, \ldots, A_{n+2})$ is of the type k.

2. FUNDAMENTAL PROPERTIES OF AN $(n + 2)$-SIDED POLYGON OF THE TYPE 2 IN E_n

The following cases may occur for an $(n + 2)$-sided polygon of the type 2:

a) the line joining the opposite vertices is its side;

b) the line joining the opposite vertices is not its side.

In the cases a) and b) we say that the $(n + 2)$-sided polygon is of the 1st and 2nd kinds, respectively.

Remark. It may occur that for an $(n + 2)$-sided polygon of the type 2 just two coefficients in Eq. (1) are negative and just two positive. Then the $(n + 2)$-sided polygon $[A_1, \ldots, A_{n+2}]$ can be expressed in two different ways as the union of two simplexes. In this case the $(n + 2)$-sided polygon $[A_1, \ldots, A_{n+2}]$ can be viewed as an $(n + 2)$-sided polygon of the 1st as well as the 2nd kind.

Theorem 1. *Let A be an $(n + 2)$-sided polygon of the* 1st *kind in E_n. Then there exists an $(n + 2)$-sided polygon $\bar{A}$ of the* 2nd *kind such that*

$$L(\bar{A}) \leqq L(A), \quad V(\bar{A}) > V(A).$$

Proof. Let $A = [A_1, \ldots, A_{n+2}]$. We can clearly renumber the vertices of the $(n + 2)$-sided polygon $A_1, \ldots, A_{n+2}$ so that the $(n + 2)$-sided polygon will not change and that A_1 and A_2 will become opposite vertices. The remaining n points, $A_3, \ldots, A_{n+2}$, determine a hyperplane E_{n-1} in E_n. Now let us construct the plane E_2 passing through the points A_3 and A_{n+2} and perpendicular to E_{n-1}. Let the orthogonal projections of the vertices A_1 and A_2 into this plane be points A'_1 and A'_2. Clearly, the points A_i and A'_i have the same distance from the hyperplane E_{n-1} for $i = 1, 2$. Therefore, $V(A_1, \ldots, A_{n+2}) = V(A'_1, A'_2, A_3, \ldots, A_{n+2})$. Moreover, $L(A'_1, A'_2, A_3, \ldots, A_{n+2}) \leqq L(A_1, \ldots, A_{n+2})$. Let us denote $d = L(A'_1, A'_2, A_3, A_{n+2})$. In the plane E_2 we can find points B and C such that the quadrangle $[B, C, A_3, A_{n+2}]$ has the length d and the largest volume of its convex envelope of all the quadrangles in the plane E_2 which have the side A_3A_{n+2} and the length d. Then

$$V(A'_1, A'_2, A_3, A_{n+2}) < V(B, C, A_3, A_{n+2}) . \tag{2}$$

We shall now prove that

$$(3) \qquad V(A_1', A_2', A_3, \ldots, A_{n+2}) < V(B, C, A_3, \ldots, A_{n+2}).$$

We shall denote by J_1 a pyramid, the base of which is formed by the convex envelope of four points, A_1', A_2', A_3, A_{n+2}, and whose vertex is the point A_4. Now for $i = 2, \ldots, n-2$, J_i represents a pyramid with the base J_{i-1} and the vertex A_{i+3}. Let v_i be the height of the pyramid J_i. Clearly J_{n-2} is the convex envelope of the $(n+2)$-tuple $(A_1', A_2', A_3, \ldots, A_{n+2})$. Since the volume of the pyramid in E_n with the base P and the height v is Pv/n, we conclude

$$V(A_1', A_2', A_3, \ldots, A_{n+2}) = \tfrac{1}{3}V(A_1', A_2', A_3, A_{n+2})\, v_1\, \tfrac{1}{4}v_2 \cdot \ldots \cdot \frac{1}{n}\, v_{n-2}\,.$$

If we construct a finite sequence of pyramids for the $(n+2)$-tuple $(B, C, A_3, \ldots, A_{n+2})$ in a similar way, we obtain pyramids J_i', $i = 1, \ldots, n+2$. Since the space determined by the base of the pyramid J_i coincides with the space determined by the base of the pyramid J_i', the heights of the pyramids J_i and J_i' are the same. Therefore

$$(4) \qquad V(B, C, A_3, \ldots, A_{n+2}) = \tfrac{1}{3}V(B, C, A_3, A_{n+2})\, v_1\, \tfrac{1}{4}v_2 \cdot \ldots \cdot \frac{1}{n}\, v_{n-2}\,.$$

Now, the statement (3) to be proved follows from (2). The above constructed points $A_3, \ldots, A_{n+1}, B$ form a simplex in a hyperplane of the space E_n. The segment CA_{n+2} intersects this simplex and therefore, the $(n+2)$-sided polygon of the type 2 is clearly of the 2nd kind. Theorem 1 is proved.

3. ISOPERIMETRIC INEQUALITY FOR A PENTAGON IN E_3

We shall first investigate a special case: a pentagon in the three-dimensional Euclidean space E_3. The isoperimetric inequality for a pentagon in E_3 can be proved by constructing for each pentagon A in E_3 a finite sequence $A^i = [A_1^i, \ldots, A_5^i]$, $i = 1, \ldots, 5$, of pentagons in E_3, the first term of which is the pentagon A. All the pentagons of the sequence have the same length. The volume of the convex envelope of each of them, except the first term of the sequence, is larger or equal to the volume of the convex envelope of the preceding term of the sequence, and the last term of the sequence is the same for all pentagons in the space E_3. This last term of the sequence will be the required pentagon with the maximum volume of its convex envelope. Evidently, it is sufficient to restrict ourselves to pentagons A of the 2nd kind (see Theorem 1).

Therefore, let $A = [A_1, \ldots, A_5]$ be a pentagon of the 2nd kind in E_3. Clearly, we can choose the numbering of the vertices so that the vertices A_2 and A_4 are opposite. Let us denote $L(A) = L_0$.

We shall say that a pentagon of the 2nd kind $X = [X_1, \ldots, X_5]$ satisfies the conditions a), b), c), d) and e), respectively, provided

a) $L(X) = L_0$,

b) the vertices X_2 and X_4 are opposite,

c) the triangles $[X_1, X_2, X_3]$ and $[X_3, X_4, X_5]$ are isosceles with the bases X_1X_3 and X_3X_5 and their planes are perpendicular to the plane $\{X_1, X_3, X_5\}$,

d) $\sphericalangle X_2X_1X_3 = \sphericalangle X_4X_5X_3$,

e) $\overline{X_1X_3} = \overline{X_5X_3}$.

Now we shall construct the finite sequence of pentagons mentioned above:

1) We put $A^1 = A$.

2) We construct a pentagon A^2, satisfying the conditions a), b), c), so that $A_i^2 = A_i$ for $i = 1, 3, 5$ and $L(A_{i-1}^2, A_i^2, A_{i+1}^2) = L(A_{i-1}^1, A_i^1, A_{i+1}^1)$ for $i = 2, 4$. Clearly $V(A^1) \leqq V(A^2)$.

3) We construct a pentagon A^3, satisfying the conditions a), b), c), d), so that $A_i^3 = A_i$ for $i = 1, 3, 5$. Let us denote by v_i^j the height of the triangle $[A_{i-1}^j, A_i^j, A_{i+1}^j]$ for $j = 2, 3$, $i = 2, 4$. Further, let us denote $a = (1/2)\overline{A_1A_3}$ and $b = (1/2)\overline{A_3A_5}$. Let us construct the following points, defined by Cartesian coordinates in the auxiliary plane $E_2 : B = [0, 0]$, $B_1^j = [a, v_2^j]$, $B_2^j = [a + b, v_2^j + v_4^j]$, $j = 2, 3$. We find that $\overline{BB_1^j} + \overline{B_1^jB_2^j} = (1/2)(L_0 - \overline{A_1A_5})$, $j = 2, 3$. Since $\sphericalangle A_2^3A_1A_3 = \sphericalangle A_4^3A_5A_3$, the points B, B_1^3, B_2^3 lie on a straight line. This implies that $v_2^3 + v_4^3 \geqq v_2^2 + v_4^2$ and, therefore, $V(A^2) \leqq V(A^3)$.

4) We construct a pentagon A^4, satisfying the conditions a), b), c), d) and e), so that $L(A_1^4, A_3^4, A_5^4) = L(A_1, A_3, A_5)$ and $A_i^4 = A_i$ for $i = 1, 5$. Let us denote by v_i^4 the height of the triangle $[A_{i-1}^4, A_i^4, A_{i+1}^4]$ for $i = 2, 3$. In the same way as sub 3) we can prove that $v_2^4 + v_3^4 = v_2^4 + v_3^3$. Since $V(A_1, A_3, A_5) \leqq V(A_1^4, A_3^4, A_5^4)$, we get $V(A^4) \leqq V(A^3)$.

5) To each number $c \in R$, $0 < c < (1/2) L_0$, we can now construct the set P_c of all pentagons $[X_1, \ldots, X_5]$, satisfying the conditions a), b), c), d), e) and also the condition $\overline{X_1X_5} = c$. From the set P_c it is easy to select a pentagon with the maximum volume of its convex envelope. If we put $L' = L_0 - c$, we arrive at a pentagon in which the distance of the point X_3 from the line X_1X_5 is

$$v = \frac{1}{2\sqrt{2}} \sqrt{(L'^2 - c^2)} \,. \tag{5}$$

The volume of its convex envelope is

$$V_c = c(L'^2 - c^2)/3 \,.\, 2^4 \,. \tag{6}$$

It is easy to prove that the function V_c of the variable c (where $L' = L_0 - c$) reaches its maximum for $c = (1/4) L_0$. For this value V_c will be equal to $L_0^3/(3.2^7)$. From (5) we can easily determine the shape of the pentagon whose convex envelope has the volume $L_0^3/(3.2^7)$. This pentagon is the last term A^5 of our sequence of pentagons.

Theorem 2. *Let $A = [A_1, \ldots, A_5]$ be a pentagon (in E_3) with the length L_0 and the volume of its convex envelope V_0. Then*

$$L_0^3 - 3.2^7 V_0 \geqq 0 . \tag{7}$$

The equality sing in (7) is true if and only if the pentagon A is of the 2nd kind. If we number its vertices so that the vertices A_2 and A_4 are opposite, then

1) *The pentagon A satisfies the conditions* a) *to* e);
2) $\overline{A_1 A_5} = (1/4) L_0$;
3) *the distance of the point A_3 from the line $A_1 A_5$ is equal to $(1/4) L_0$;*
4) *the distance of the point A_2 or A_4 from the line $A_1 A_3$ or $A_3 A_5$, respectively, is equal to $L_0/8$.*

Proof. Theorem 2 follows from Theorem 1 and from the construction of the finite sequence of pentagons carried out above. Properties 3) and 4) of the pentagon with the maximum volume of its convex envelope can be easily checked by simple calculation.

4. ISOPERIMETRIC INEQUALITY FOR $(n+2)$-SIDED POLYGONS OF TYPE 2 IN E_n

The isoperimetric inequality for $(n + 2)$-sided polygons of type 2 in E_n can be proved in the same way as its special case (the isoperimetric inequality for a pentagon in E_3) by constructing a finite sequence $A^1, A^2, \ldots$ of $(n + 2)$-sided polygons of the 2nd type. Let us put $A^i = [A_1^i, \ldots, A_{n+2}^i]$, $i = 1, 2, \ldots$ and let $A = [A_1, \ldots, A_{n+2}]$ be an $(n + 2)$-sided polygon of the 2nd type in E_n. According to Theorem 1 we are able to restrict ourselves to the case that the $(n + 2)$-sided polygon A is of the 2nd kind. We are clearly able to number the vertices of the $(n + 2)$-sided polygon A so that the vertices A_2 and A_k $(k > 3)$ are opposite.

We first put $A^1 = A$. In the second step we shall construct an $(n + 2)$-sided polygon A^2 so that the triangles $[A_1^1, A_2^1, A_3^1]$ and $[A_{k-1}^1, A_k^1, A_{k+1}^1]$ are replaced by isosceles triangles constructed in planes perpendicular to the hyperplane $\{A_1^1, A_3^1, \ldots, A_{k-1}^1, A_{k+1}^1, \ldots, A_{n+2}^1\}$. If $k > 4$, we denote by ϱ the plane of symmetry of the pair of points A_{k-2}^2, A_{k+1}^2. Now the point symmetric to the point A_k^2 or to A_{k-1}^2 with respect to the hyperplane ϱ will be denoted by A_{k-1}^3 and A_k^3, respectively. The other points remain, i.e., for $j = 1, \ldots, n + 2$, $k - 1 \neq j \neq k$, we put $A_j^3 = A_j^2$. In this way we have constructed an $(n + 2)$-sided polygon A^3, for which $L(A^2) = L(A^3)$ and $V(A^2) = V(A^3)$, and which has A_2^3 and A_{k-1}^3 as its opposite vertices. If we repeat this procedure another $(k - 5)$-times, we arrive at an $(n + 2)$-sided polygon A^{k-2}

with the opposite vertices A_2^{k-2} and A_4^{k-2}. Now, let us denote by E_3 the subspace of the space E_n, containing the points A_1^{k-2}, A_3^{k-2}, A_5^{k-2} and the direction perpendicular to the hyperplane $\{A_1^{k-2}, A_3^{k-2}, A_5^{k-2}, A_6^{k-2}, \ldots, A_{n+2}^{k-2}\}$. Clearly $[A_1^{k-2}, \ldots, A_5^{k-2}] \subset \subset E_3$. As in the proof of Theorem 1 we find that the convex envelope of the $(n + 2)$-sided polygon A^{k-2} can be constructed by establishing a finite sequence of pyramids by adding further vertices to the base which is the convex envelope of the pentagon $[A_1^{k-2}, \ldots, A_5^{k-2}]$. This implies that, if we put $A_i^{k-1} = A_i^{k-2}$ for $i = 1, 5, \ldots, n + 2$ and if we construct points A_j^{k-1}, $j = 2, 3, 4$, so that the pentagons $[A_1^{k-2}, \ldots, A_5^{k-2}]$ and $[A_1^{k-1}, \ldots, A_5^{k-1}]$ are isometrical and the subspaces $\{A_1^{k-1}, \ldots, A_5^{k-1}\}$ and $\{A_1^{k-1}, A_5^{k-1}, \ldots, A_{n+2}^{k-1}\}$ are perpendicular to each other, then $V(A^{k-1}) \geqq V(A^{k-2})$.

If we now put $V_1 = V(A_1^{k-1}, A_5^{k-1}, \ldots, A_{n+2}^{k-1})$, $V_2 = V(A_1^{k-1}, \ldots, A_5^{k-1})$, $V' = = V(A_1^{k-1}, \ldots, A_{n+2}^{k-1})$, $c = \overline{A_1^{k-1} A_5^{k-1}}$, we find, as the result of constructing the finite sequence of pyramids, that

$$V' = \frac{2 \,.\, 3}{n(n-1)} \cdot \frac{1}{c} V_1 V_2 \,. \tag{8}$$

If we further put $L_1 = L(A_1^{k-1}, A_5^{k-1}, \ldots, A_{n+2}^{k-1}) - c$, $L_2 = L(A_1^{k-1}, \ldots, A_5^{k-1}) - c$, then, according to [6],

$$V_1 \leqq \frac{c \sqrt{((L_1^2 - c^2)^{n-3})}}{(n-2)! \sqrt{((n-2)^{n-2} (n-3)^{n-3})}} \tag{9}$$

and, with regard to Eq. (6),

$$V_2 \leqq \frac{c(L_2^2 - c^2)}{3 \,.\, 2^4} \,. \tag{10}$$

We can now obtain an estimate for the volume V' from the relations (8), (9), (10). Since we know that $L_1 + L_2 = L(A_1^{k-1}, \ldots, A_{n+2}^{k-1})$, by investigating the extremes of the function of three variables L_1, L_2 and c, we obtain the following theorem:

Theorem 3. *Let A be an $(n + 2)$-sided polygon in E_n of the type 1 or 2 with $L(A) = = L_0$ and $V(A) = V_0$. Then*

$$L_0^n - \frac{2^2 \, n^{n/2} n! \, (n+1)^{(n+1)/2}}{3 \sqrt{3}} V_0 \geqq 0 \,. \tag{11}$$

Remark. We have proved the inequality (11) for an $(n + 2)$-sided polygon of the type 2 in the way described above. Clearly this inequality is also valid for $(n + 2)$-sided polygons of the type 1. By investigating the individual steps of constructing the sequence of $(n + 2)$-sided polygons $A^1, A^2, \ldots$ one can easily (but not briefly) describe the $(n + 2)$-sided polygons for which the equality sign in relation (11) is true.

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