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NONOSCILLATORY SOLUTIONS OF n-th ORDER NONLINEAR DIFFERENTIAL EQUATION

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Nonoscillatory solutions of the linear differential equation

$$y^{(n)} + p(x)\, y = 0$$

were studied in the paper [1]. The present paper extends those results to the nonlinear differential equation

$$\text{(E)} \qquad y^{(n)} \pm (-1)^n f(x, y, y', \ldots, y^{(n-1)}) = 0\,.$$

Throughout the whole paper we suppose that the function $f(x, u_0, u_1, \ldots, u_{n-1})$ is continuous and of one sign on the region

$$\text{D:} \qquad a \leqq x < \infty\,, \quad -\infty < u_i < \infty\,, \quad i = 0, 1, \ldots, n-1\,,$$

and for every point $(c_0, c_1, \ldots, c_{n-1}) \neq (0, 0, \ldots, 0)$ the function $f(x, c_0, c_1, \ldots, c_{n-1})$ does not identically equal zero in any subinterval of the interval $[a, \infty)$.

A solution $y(x)$ of (E) is said to be nonoscillatory on $[a, \infty)$ if there exists a number $b \geqq a$ such that $y(x) \neq 0$ on $[b, \infty)$. By (E^+) or (E^-) we denote equation (E) with the sign $+$ or $-$, respectively.

PRELIMINARY RESULTS

Let $q(x)$ be a continuous function on $[a, \infty)$ such that

$$(1) \qquad 0 < q(x) \leqq x \quad \text{on} \quad [a, \infty) \quad \text{and} \quad \lim_{x\to\infty} q(x) = \infty\,.$$

Let us define the following sets of nonoscillatory solution of (E): Let S_0 be the set of bounded nonoscillatory solutions of (E), let S_k, $k = 1, 2, \ldots, n-1$, be the set of nonoscillatory solutions $y(x)$ of (E) with the properties

$$(2) \qquad \lim_{x\to\infty} \frac{|y(x)|}{q(x)^{k-1}} > K \quad \text{and} \quad \lim_{x\to\infty} \frac{y(x)}{q(x)^k} = 0\,,$$

and let S_n be the set of nonoscillatory solutions $y(x)$ of (E) such that

$$(3) \qquad \lim_{x\to\infty} \frac{|y(x)|}{q(x)^{n-1}} > K$$

for a positive constant K.

Lemma 1. *Suppose* $y(x) \in C^n[b, \infty)$, $y(x) \geqq 0$ *on* $[b, \infty)$,

$$(4) \qquad \lim_{x\to\infty} \frac{y(x)}{q(x)^r} = 0$$

fore an integer r, $1 \leqq r \leqq n-1$, *and* $y^{(n)}(x) \neq 0$ *on any subinterval of* $[b, \infty)$.

If $y^{(n)} \leqq 0$ *on* $[b, \infty)$, *then*

$$(-1)^{k+1} y^{(n-k)}(x) > 0 \quad on \quad [b, \infty)$$

for $k = 1, 2, \ldots, n-r$, *and also for* $k = n-r+1$ *if* $n-r$ *is even. If* $y^{(n)} \geqq 0$ *on* $[b, \infty)$, *then*

$$(-1)^k y^{(n-k)}(x) > 0 \quad on \quad [b, \infty)$$

for $k = 1, 2, \ldots, n-r$, *and also for* $k = n-r+1$ *if* $n-r$ *is odd.*

Proof. Consider the case $y^{(n)} \geqq 0$. We need to prove $y^{(n-1)} < 0$ on $[b, \infty)$. If $y^{(n-1)}(\alpha) \geqq 0$ for some $\alpha \geqq b$, then $y^{(n-1)}(x) > K$ for a positive constant K on an interval $[\beta, \infty)$, $\beta > \alpha$. However, this implies that $y(x) > K_1 x^{n-1}$ on $[\beta_1, \infty)$ for some $\beta_1 > \beta$ and $K_1 > 0$ and also

$$\lim_{x\to\infty} \frac{y(x)}{x^{n-1}} > K_1 > 0 .$$

On the other hand,

$$\lim_{x\to\infty} \frac{y(x)}{x^{n-1}} = \lim_{x\to\infty} \frac{y(x)}{q(x)^r} \cdot \frac{q(x)^r}{x^{n-1}} \leqq \lim_{x\to\infty} \frac{y(x)}{q(x)^r} \cdot x^{r-(n-1)} = 0 ,$$

which is a contradiction. Thus $y^{(n-1)}(x) < 0$ on $[b, \infty)$. If $y^{(n-2)}(\alpha) \leqq 0$ for some $\alpha \geqq b$, then $y(x) \to -\infty$, contradicting the inequality $y(x) \geqq 0$, and so $y^{(n-2)}(x) > 0$ on $[b, \infty)$. Repeating the above arguments we complete the proof.

Lemma 2. *Suppose* $y(x) \in C^n[b, \infty)$, $y(x)$ *is bounded and* $y^{(n)}(x) \neq 0$ *on any subinterval of* $[b, \infty)$.

If $y^{(n)} \leqq 0$ *on* $[b, \infty)$, *then*

$$(-1)^{k+1} y^{(n-k)}(x) > 0 \quad on \quad [b, \infty)$$

for $k = 1, 2, \ldots, n-1$.

If $y^{(n)} \geqq 0$ *on* $[b, \infty)$, *then*

$$(-1)^k y^{(n-k)}(x) > 0 \quad on \quad [b, \infty)$$

for $k = 1, 2, \ldots, n-1$

The proof is easy and will be omitted. (It also follows from the proof of Theorem 1 in [2].)

Lemma 3. *Let* $y(x)$ *be a solution of* (E). *Then*

(5)
$$y^{(n-k)}(x) = y^{(n-k)}(c) + K_k(c) + K_k(x) \mp$$
$$\mp (-1)^n (-1)^{k-1} \frac{1}{(k+1)!} \int_c^x s^{k-1} f(s, y(s), \ldots, y^{(n-1)}(s)) \, ds$$

holds for $x \geqq c \geqq a$ *and* $1 < k \leqq n$, *where*

$$K_k(x) = -\sum_{j=1}^{k-1} (-1)^{j+1} \frac{1}{j!} x^j y^{(n-k+j)}(x) .$$

Proof. Let $y(x)$ be a solution of (E). Integrating twice over $[c, x]$ yields

$$y^{(n-2)}(x) = y^{(n-2)}(c) + y^{(n-1)}(c)\, x - y^{(n-1)}(c)\, c \mp$$
$$\mp (-1)^n \int_c^x d\xi \int_c^\xi f(s, y(s), \ldots, y^{(n-1)}(s)) \, ds .$$

Changing the order of integration we get

(6)
$$y^{(n-2)}(x) = y^{(n-2)}(c) + y^{(n-1)}(c)\, x - y^{(n-1)}(c)\, c \mp$$
$$\mp (-1)^n x \int_c^x f(s, y(s), \ldots, y^{(n-1)}(s)) \, ds \pm (-1)^n \int_c^x s f(s, y(s), \ldots, y^{(n-1)}(s)) \, ds .$$

Substituting

$$y^{(n-1)}(x) = y^{(n-1)}(c) \mp (-1)^n \int_c^x f(s, y(s), \ldots, y^{(n-1)}(s)) \, ds$$

into (6) we obtain

$$y^{(n-2)}(x) = y^{(n-2)}(c) + x y^{(n-1)}(x) - c y^{(n-1)}(c) \mp$$
$$\mp (-1)^n (-1) \int_c^x s f(s, y(s), \ldots, y^{(n-1)}(s)) \, ds ,$$

i.e. Lemma 3 holds for $k = 2$. If we repeat the above argument we obtain that Lemma 3 holds for $1 < k \leqq n - 1$.

Theorem 1. *Let a function $f(x, u_0, \ldots, u_{n-1})$ have the following properties:*

(H$_1$) $$u_0 f(x, u_0, \ldots, u_{n-1}) \geqq 0\,;$$

(H$_2$) *if $\alpha(x) \in C^n[a, \infty)$ and $\lim_{x\to\infty} \alpha(x) = L$, $0 < L < \infty$, then*

$$\operatorname{sgn} \alpha(x) \int^{\infty} x^{n-1} f(x, \alpha(x), \alpha'(x), \ldots, \alpha^{(n-1)}(x))\, dx = \infty\,.$$

Then (i) $S_0 = \emptyset$ *for the equation* (E$^+$), *i.e. every bounded solution of* (E$^+$) *is oscillatory.*

(ii) *If $y(x)$ is a solution of* (E$^-$) *and $y(x) \in S_0$, then* $\lim_{x\to\infty} y(x) = 0$.

Proof. (i) From Lemma 3 it follows that every solution of (E$^+$) satisfies the equation

$$(7)\quad y(x) = y(c) + K_n(c) + K_n(x) + \frac{1}{(n-1)!}\int_c^x s^{n-1} f(s, y(s), \ldots, y^{(n-1)}(s))\, ds\,.$$

The proof is by contradiction. Suppose $S_0 \neq \emptyset$, i.e. there exists a bounded non-oscillatory solution $y(x)$. Let $y(x) > 0$, let n be even. Then $y^{(n)} = -f(x, y, \ldots \ldots, y^{(n-1)}) \leqq 0$ and Lemma 2 implies that the sum in (7) is positive, therefore

$$(8)\qquad y(x) \geqq y(c) + K_n(c) + \frac{1}{(n-1)!}\int_c^x s^{n-1} f(s, y(s), \ldots, y^{(n-1)}(s))\, ds\,.$$

From Lemma 2 it follows that $y'(x) > 0$, therefore $y(x)$ is increasing. Since $y(x)$ is bounded, $\lim_{x\to\infty} y(x)$ exists and is positive. Hence, by the assumption (H$_2$), the right-hand side diverges to ∞ which contradicts the boundedness of $y(x)$. When $y(x) < 0$, or n is odd, the proof is similar.

(ii) Let $y(x)$ be a solution of (E$^-$), $y(x) \in S_0$ and $\lim_{x\to\infty} y(x) = c \neq 0$. If $y(x) > 0$, then, by Lemma 2 and Lemma 3, it satisfies the inequality

$$y(x) \leqq y(c) + K_n(c) - \frac{1}{(n-1)!}\int_c^x s^{n-1} f(s, y(s), \ldots, y^{(n-1)}(s))\, ds\,.$$

The right-hand side tends to $-\infty$, while the left-hand side is bounded, which is a contradiction.

The proofs of the other cases are similar.

Let $S = S_0 \cup S_2 \cup \ldots \cup S_n$ if n is even and $S = S_0 \cup S_2 \cup \ldots \cup S_{n-1}$ if n is odd for equation (E$^+$).

For equation (E$^-$) let $S = S_1 \cup S_3 \cup \ldots \cup S_n$ if n is odd, and $S = S_1 \cup S_3 \cup \ldots \ldots \cup S_{n-1}$ if n is even.

The following theorem generalizes Theorem 1.

Theorem 2. *Suppose $f(x, u_0, \ldots, u_{n-1})$ has the properties*

(h_1) *there exists a continuous function $p(x) \geqq 0$ on $[a, \infty)$ such that*

$$\operatorname{sgn}\{u_0\} \cdot f(x, u_0, \ldots, u_{n-1}) \geqq p(x)\,|u_0| \quad \textit{for all} \quad (x, u_0, \ldots, u_{n-1}) \in D\,,$$

(h_2)
$$\int^{\infty} q(x)^{n-1}\, p(x)\,\mathrm{d}x = \infty\,.$$

Then $S = \emptyset$.

Proof. Consider equation (E^+) and n even, i.e. consider the equation $y^{(n)} + f(x, y, \ldots, y^{(n-1)}) = 0$. Suppose on the contrary that $S \neq \emptyset$. Let $y(x) \in S$, $y(x)$ eventually positive. If $(h_1), (h_2)$ hold, then $(H_1), (H_2)$ hold as well and therefore $S_0 = \emptyset$. Now we show that $S_n = \emptyset$. If $y(x) \in S_n$, then

$$\lim_{x \to \infty} \frac{y(x)}{q(x)^{n-1}} > K > 0$$

and so $y(x) > K\,q(x)^{n-1}$ on an interval $[b, \infty)$, $b > a$. Since $f(x, y(x), \ldots, y^{(n-1)}(x)) \geqq 0$, then $y^{(n)}(x) \leqq 0$ and $y^{(n-1)}(x) > 0$ on $[a, \infty)$. It follows from (h_1) that

$$f(x, y(x), \ldots, y^{(n-1)}(x)) \geqq y(x)\,p(x) > K\,q(x)^{n-1}\,p(x)$$

on $[b, \infty)$. Consequently,

$$y^{(n-1)}(x) = y^{(n-1)}(c) - \int_c^x f(s, y(s), \ldots, y^{(n-1)}(s))\,\mathrm{d}s \leqq$$
$$\leqq y^{(n-1)}(c) - K \int_c^x q^{n-1}(s)\,p(s)\,\mathrm{d}s\,.$$

The last integral diverges to $-\infty$ which contradicts $y^{(n-1)}(x) > 0$. Hence $S_n = \emptyset$. Now suppose that $S_r \neq \emptyset$, $r = 2, 4, \ldots, n-2$, and let $y(x) \in S_r$. It follows from Lemma 1 that

(9) $$(-1)^{k+1}\,y^{(n-k)}(x) > 0 \quad \text{for} \quad k = 1, 2, \ldots, n-r,\ n-r+1\,.$$

We apply Lemma 3 to $y(x)$ and obtain for $k = n - r + 1$,

(10) $$y^{(r-1)}(x) = y^{(r-1)}(c) + K_{n-r+1}(c) + K_{n-r+1}(x) -$$
$$- \frac{1}{(n-r)!} \int_c^x s^{n-r} f(s, y(s), \ldots, y^{(n-1)}(s))\,\mathrm{d}s\,.$$

It follows from (9) that $K_{n-r+1}(x)$ is negative and hence

$$y^{(r-1)}(x) \leqq y^{(r-1)}(c) + K_{n-r+1}(c) -$$
$$- \frac{1}{(n-r)!} \int_c^x s^{n-r} f(s, y(s), \ldots, y^{(n-1)}(s))\,\mathrm{d}s\,.$$

Since $y(x) \in S_r$, it follows from (1), (2) and (h_1) that

$$s^{n-r} f(s, y(s), \ldots, y^{(n-1)}(s)) \geqq q(s)^{n-r} \cdot p(s) \cdot K \cdot q(s)^{r-1},$$

and therefore

$$y^{(r-1)}(x) \leqq y^{(r-1)}(c) + K_{n-r+1}(c) - \frac{K}{(n-r)!} \int_c^x q(s)^{n-1} p(s) \, ds .$$

The right-hand side diverges to $-\infty$, while the left-hand side is, by (9), positive. This contradiction proves that $S_r = \emptyset$ for $r = 2, 4, \ldots, n-2$ as well.

If $y(x)$ is eventually negative then (h_1) implies that $f(x, y(x), \ldots, y^{(n-1)}(x)) \leqq 0$, so $y^{(n)} \geqq 0$. Then $-y \geqq 0$, $(-y)^{(n)} \leqq 0$. By applying Lemma 1 we obtain $(-1)^k y^{(n-k)}(x) > 0$ for $k = 1, 2, \ldots, n-r, n-r+1$. Further, by a similar method as above we obtain a contradiction. Proofs for the other cases are similar.

From the definition of S_k it is evident that $S_i \cap S_j = \emptyset$, $i \neq j$, $i, j = 0, 1, \ldots, n$, except for $S_0 \cap S_1$ which consists of a bounded solution $y(x)$, such that $\lim_{x \to \infty} y(x) = M \neq 0$. However, if ($H_1$), ($H_2$) are satisfied, then by Theorem 1 every nonoscillatory solution of (E) either is unbounded or approaches zero, i.e. $S_0 \cap S_1$ is empty.

Let $S' = S_1 \cup S_3 \cup \ldots \cup S_{n-1}$ if n is even and $S' = S_1 \cup S_3 \cup \ldots \cup S_n$ if n is odd for equation (E^+). For equation (E^-) let $S' = S_0 \cup S_2 \cup \ldots \cup S_{n-1}$ if n is odd and $S' = S_0 \cup S_2 \cup \ldots \cup S_n$ if n is even.

Theorem 3. *Let the conditions* (h_1) *and* (h_2) *be satisfied. Let the condition*

(h_3): *If* $y(x)$ *is a nonoscillatory solution of* (E), *then*

$$\lim_{x \to \infty} \frac{|y(x)|}{q(x)^k} \quad \textit{exists (fiinite or equal to } \infty) \textit{ for } k = 0, 1, \ldots, n-1$$

be satisfied. Then every nonoscillatory solution belongs to S'.

Proof. If the conditions (h_1), (h_2) are satisfied, then by Theorem 2 the set S is empty. Therefore it is sufficient to prove that the sets $S_0, S_1, \ldots, S_n$ form a partition of the set of nonoscillatory solutions of (E) provided (h_3) is satisfied.

If a nonoscillatory solution $y(x)$ is bounded, then it belongs to S_0. Let $y(x)$ be unbounded. If

$$\lim_{x \to \infty} \frac{|y(x)|}{q(x)^{n-1}} > K > 0 ,$$

then $y(x)$ belongs to S_n. Otherwise, there exists m which is the largest positive integer $m < n$ such that

$$\lim_{x \to \infty} \frac{|y(x)|}{q(x)^{m-1}} > L > 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{|y(x)|}{q(x)^m} = 0 .$$

Hence $y(x) \in S_m$. This shows that any nonoscillatory solution of (E) belongs to some S_k, $0 \leqq k \leqq n$.

Corollary. *If the conditions* (h_1), (h_2) *are satisfied and* $q(x) = x$ *in* (1), *then every nonoscillatory solution of* (E) *belongs to* S'.

Proof. It is sufficient to prove that (h_3) holds provided $q(x) = x$. Suppose on the contrary that

$$0 \leqq A \leqq \liminf_{x \to \infty} \frac{|y(x)|}{x^k} < \limsup_{x \to \infty} \frac{|y(x)|}{x^k} = B \leqq \infty$$

for a certain nonoscillatory solution of (E). Let $y(x) > 0$. Then there exists a number N, $A < N < B$, and a sequence $\{x_n\}$ such that the function $g_k(x) = g(x) - Nx^k$ has an infinite number of zeros x_n. Therefore $g_k^{(n-1)}(x) = y^{(n-1)}(x) - N^0(n-1)!$, where $N^0 = N$ if $k = 0, 1, \ldots, n-2$ and $N^0 = 0$ if $k = n-1$, has an infinite number of zeros, which contradicts $y^{(n)}(x) \geqq 0$ or $y^{(n)}(x) \leqq 0$.

For the existence theorems for nonoscillatory solutions of (E), see [3] and [4].

Example. Consider the equation

$$(\bar{E}^-) \qquad y''' + f(x, y, y', y'') = 0\,,$$

where the function f has the properties

$$(\bar{h}_1) \qquad \operatorname{sgn}\{u_0\}\, f(x, u_0, u_1, u_2) \geqq p(x)\,|u_0|\,, \quad p(x) \geqq 0\,,$$

$$(\bar{h}_2) \qquad \int^\infty x\, p(x)\, dx = \infty\,.$$

Then every nonoscillatory solution of $(\bar{E}^-)$ approaches zero as $x \to \infty$.

Proof. First of all we notice that if $\int^\infty x\, p(x)\, dx = \int^\infty (\sqrt{x})^2\, p(x)\, dx = \infty$, then $\int^\infty x^2\, p(x)\, dx = \infty$ as well. Let $S_i^{\sqrt{x}}$ and S_i^x, $i = 0, 1, 2$, be the sets defined by (1) corresponding to the functions $q(x) = \sqrt{x}$ and $q(x) = x$ respectively. It follows from Theorem 2 that $S_1^{\sqrt{x}} \cup S_3^{\sqrt{x}} = \emptyset$ and $S_1^x \cup S_3^x = \emptyset$. Applying Corollary we obtain that

$$\lim_{x \to \infty} \frac{|y(x)|}{x^k}\,, \quad k = 0, 1, 2\,,$$

exists (finite or ∞). This implies that

$$\lim_{x \to \infty} \frac{|y(x)|}{(\sqrt{x})^k}\,, \quad k = 0, 1, 2\,,$$

exists (finite or ∞) as well. Indeed:

If $\lim_{x \to \infty} |y(x)| = L < \infty$, then $\lim_{x \to \infty} \frac{|y(x)|}{\sqrt{x}} = 0$.

If $\lim_{x\to\infty} |y(x)| = \infty$, then $\lim_{x\to\infty} \frac{|y(x)|}{x} \neq 0$ because $S_1^x = \emptyset$ and therefore $\lim_{x\to\infty} \frac{|y(x)|}{\sqrt{x}} = \infty$.

It follows from Theorem 3 that every nonoscillatory solution belongs to $S_0^x \cup S_2^x$ and does not belong to $S_3^{\sqrt{x}}$, i.e. there exists no $y(x)$ such that $\lim_{x\to\infty} y(x) > K$. Hence $S_2^x = \emptyset$. Consequently, every nonoscillatory solution of $(\bar{\mathrm{E}}^-)$ belongs to S_0, i.e. it is bounded and by Theorem 1 it converges to zero as $x \to \infty$.

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