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Časopis pro pěstování matematiky, Vol. 106 (1981), No. 2, 170--185

Persistent URL: `http://dml.cz/dmlcz/118087`

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BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN ANISOTROPIC SOBOLEV SPACES

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(Received May 14, 1979)

0. INTRODUCTION

The theory of weak solutions of (generally nonlinear) partial differential equations deals with differential operators of the type

$$(0.1) \qquad (Au)(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha a_\alpha(x; \delta_k u(x)), \quad x \in \Omega,$$

where Ω is a domain in $\mathbb{R}^N$, α is an N-dimensional multiindex (i.e., $\alpha \in \mathbb{N}_0^N$) and $\delta_k u$ is the so-called *generalized gradient of the k-th order*:

$$\delta_k u = \{D^\beta u;\ |\beta| \leq k\}.$$

If the functions $a_\alpha(x; \xi)$ have a "polynomial growth", i.e. if, e.g.,

$$|a_\alpha(x; \xi)| \leq c(1 + \sum_{|\beta| \leq k} |\xi_\beta|^{p-1}), \quad p > 1,$$

for a.e. $x \in \Omega$, then one can seek a weak solution of the given boundary value problem for the operator A from (0.1) in the Sobolev space

$$W^{k,p}(\Omega).$$

This paper concerns the possibility of modifying the results known for operators of the type (0.1) to the case of more general operators of the form

$$(0.2) \qquad (Au)(x) = \sum_{\alpha \in E} (-1)^{|\alpha|} D^\alpha a_\alpha(x; \delta_E u(x)), \quad x \in \Omega,$$

where E is a certain fixed set of N-dimensional multiindices and

$$(0.3) \qquad \delta_E u = \{D^\beta u;\ \beta \in E\}.$$

1. FORMULATION OF THE BOUNDARY VALUE PROBLEM

1.1. Let E be a fixed finite subset of the set $\mathbb{N}_0^N$ of all N-dimensional multiindices and let us denote by

$$W^{E,p}(\Omega), \quad p > 1, \tag{1.1}$$

the set of all functions $u = u(x)$, $x \in \Omega$, such that

$$D^\alpha u \in L^p(\Omega) \quad \text{for every} \quad \alpha \in E.$$

Let $\theta = (0, 0, \ldots, 0)$ and suppose that

$$\theta \in E. \tag{1.2}$$

Then $W^{E,p}(\Omega)$ is a *separable reflexive Banach space* if equipped with the norm

$$\|u\|_{E,p} = \sum_{\alpha \in E} \|D^\alpha u\|_p, \tag{1.3}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. Further, let us introduce the space

$$W_0^{E,p}(\Omega) \tag{1.4}$$

as the closure of the set $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact supports in Ω with respect to the norm (1.3).

1.2. Let M be the number of elements of the set E.

1.3. Consider the operator A from (0.2) and suppose that

(i) the functions $a_\alpha(x; \xi)$ are defined for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^M$,

(ii) they fulfil the Carathéodory conditions,

(iii) they fulfil the following growth conditions:

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^M$, it is

$$|a_\alpha(x; \xi)| \leqq g_\alpha(x) + c_\alpha \sum_{\beta \in E} |\xi_\beta|^{p-1}, \quad p > 1, \tag{1.5}$$

where $g_\alpha \in L^q(\Omega)$ with $q = p/(p-1)$ and $c_\alpha \geqq 0$.

1.4. Under the assumptions mentioned in Section 1.3, the operator $\mathscr{A}$ defined by the formula

$$\langle \mathscr{A}u, v \rangle = \sum_{\alpha \in E} \int_\Omega a_\alpha(x; \delta_E u(x)) D^\alpha v(x) \, dx \tag{1.6}$$

maps the space $W^{E,p}(\Omega)$ into its dual space $(W^{E,p}(\Omega))^*$. *)

1.5. (i) Let V be a Banach space such that

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*) For X a Banach space, the symbol $\langle ., . \rangle$ denotes the duality pairing between X and its dual space X^*.

$$W_0^{E,p}(\Omega) \subset V \subset W^{E,p}(\Omega);$$

the space V is equipped again with the norm (1.3).

(ii) Let Q be a Banach space of functions defined on Ω and such that the set $C_0^\infty(\Omega)$ is dense in Q and that

(1.7) $$V \hookrightarrow Q. \text{ *)}$$

(iii) Let a function φ be given,

$$\varphi \in W^{E,p}(\Omega).$$

(iv) Let a functional f be given,

$$f \in Q^*.$$

(v) Let a functional g be given,

$$g \in V^*,$$

such that

(1.8) $$\langle g, v \rangle = 0 \quad \text{for every} \quad v \in C_0^\infty(\Omega).$$

The spaces $W^{E,p}(\Omega)$, V, Q, the function φ, the functionals f, g and the operator A from (0.2) (i.e., its "coefficients" $a_\alpha(x; \xi)$) are together called the *data* of the boundary value problem (A, V, Q).

1.6. Definition. The function $u \in W^{E,p}(\Omega)$ is called a *weak solution of the b.v.p.* (A, V, Q), if

(i) $u - \varphi \in V$;

(ii) for every $v \in V$, it is

(1.9) $$\langle \mathscr{A}u, v \rangle = \langle f, v \rangle + \langle g, v \rangle.$$

1.7. Remarks. (i) If $E = \{\alpha \in \mathbb{N}_0^N; |\alpha| \leq k\}$ with $k \in \mathbb{N}$, then $W^{E,p}(\Omega)$ is the "usual" Sobolev space $W^{k,p}(\Omega)$ and the b.v.p. (A, V, Q) is the "usual" boundary value problem for the operator A from (0.1). There is a number of results concerning the existence of weak solutions of such b.v.p. – via variational methods, the theory of monotone operators etc. (see e.g. [2], [3]). Since the b.v.p. (A, V, Q) for a *general* set E is a direct analogue of the "usual" b.v.p., we are concerned with the modification of these results to the general case.

(ii) In accordance with the usual terminology, we shall call the b.v.p. (A, V, Q) the *Dirichlet problem* (for the operator A from (0.2)) if

$$V = W_0^{E,p}(\Omega).$$

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*) For X, Y two Banach spaces, the symbol $X \hookrightarrow Y$ means that there exists a constant $c > 0$ such that $\|u\|_Y \leq c\|u\|_X$ for every $u \in X$.

It is not necessary to prescribe the functional g in this case since – in view of the density of $C_0^\infty(\Omega)$ in V – it is $\langle g, v\rangle = 0$ for *every* $v \in V$ and the term $\langle g, v\rangle$ does not occur in (1.9).

(iii) Let us mention that S. M. Nikol'skiĭ [4] has investigated the Dirichlet problem for the *linear* case of the operator A:

$$(Au)(x) = \sum_{\alpha,\beta\in E} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u(x)),\quad x \in \Omega,$$

and has proved existence theorems for weak solutions from the space

$$W^{E,2}(\Omega).$$

2. FURTHER ASSUMPTIONS AND AUXILIARY ASSERTIONS

In order to be able to prove the existence theorem for the b.v.p. (A, V, Q), it is necessary to know the structure of the spaces $W^{E,p}(\Omega)$ in more detail. These spaces are investigated in [6], [7] and we shall mention some of the results.

2.1. The domain Ω. We shall suppose that the domain $\Omega \subset \mathbb{R}^N$ is bounded and that its boundary $\partial\Omega$ can be locally described by functions satisfying the Lipschitz condition. We shall write this fact by

$$\Omega \in \mathscr{C}^{0,1}\,; \tag{2.1}$$

for a more detailed description see e.g. [1].

2.2. The set E. We shall suppose that

(i) E is convex, i.e.

$$\operatorname{ch} E \cap \mathbb{N}_0^N = E$$

(ch E denotes the *convex hull* of E);

(ii) if $\alpha \in E$ and $\beta \in \mathbb{N}_0^N$, $\beta \leqq \alpha$ (i.e., $\beta_i \leqq \alpha_i$ for $i = 1, \ldots, N$), then

$$\beta \in E\,.$$

A set $B \subset E$ is called a *complete basis of E* if

(i) $\operatorname{ch}(B \cup \{\theta\}) \cap \mathbb{N}_0^N = E$;

(ii) to every $\alpha \in E - B$, there exist multiindices $\beta^{(i)} \in B$ $(i = 1, \ldots, N)$ such that

$$\alpha \leqq \beta^{(i)} \quad\text{and}\quad \alpha_i < \beta_i^{(i)}\,.$$

In [6], the following assertions are proved:

2.3. Theorem. *Under the assumptions of Sections* 2.1 *and* 2.2, *the norm*

$$|||u|||_{E,p} = \sum_{\alpha\in B} \|D^\alpha u\|_p + \|u\|_p$$

is equivalent to the norm $\|u\|_{E,p}$ *from* (1.3).

2.4. Theorem. *Under the assumptions of Sections* 2.1 *and* 2.2, *it is*

$$W^{E,p}(\Omega) \subset\subset W^{F,p}(\Omega) \quad *) \tag{2.2}$$

where

$$F = (E - B) \cup \{\theta\} . \tag{2.3}$$

2.5. Using some known theorems on Nemyckiĭ operators (see e.g. [8]), we conclude from the growth conditions (1.5) that the operator H_α:

$$H_\alpha(u)(x) = a_\alpha(x; \delta_E u(x))$$

is a continuous mapping from $W^{E,p}(\Omega)$ into $L^q(\Omega)$, $q = p/(p - 1)$. Using Hölder's inequality, we derive easily the estimate

$$\left|\int_\Omega a_\alpha(x; \delta_E u(x)) D^\alpha v(x) \, dx\right| \leqq \tag{2.4}$$

$$\leqq \int_\Omega |g_\alpha(x) D^\alpha v(x)| \, dx + c_\alpha \sum_{\beta\in E} \int_\Omega |D^\beta u(x)|^{p-1} |D^\alpha v(x)| \, dx \leqq$$

$$\leqq \|g_\alpha\|_q \|D^\alpha v\|_p + c_\alpha \sum_{\beta\in E} \|D^\beta u\|_p^{p-1} \|D^\alpha v\|_p \leqq$$

$$\leqq (\|g_\alpha\|_q + \tilde{c}_\alpha \|u\|_{E,p}^{p-1}) \|v\|_{E,p} .$$

It follows that the operator $\mathscr{A}$ defined in (1.6) is a bounded continuous mapping from $W^{E,p}(\Omega)$ into $(W^{E,p}(\Omega))^*$.

3. EXISTENCE THEOREM

The theorem on existence of a weak solution of the b.v.p. (A, V, Q) is based on the following theorem of Leray and Lions (see e.g. [2]):

3.1. Theorem. *Let* X *be a reflexive Banach space. Let* T *be an operator from* X *into* X^* *and let the following conditions be fulfilled*:

(i) *the operator* T *is bounded*;
(ii) *the operator* T *is demicontinuous*;
(iii) *the operator* T *is coercive*, i.e.

$$\lim_{\|u\|_X \to \infty} \frac{\langle Tu, u\rangle}{\|u\|_X} = +\infty ; \tag{3.1}$$

*) For X, Y two Banach spaces, the symbol $X \subset\subset Y$ means that the imbedding operator of $X \subset Y$ is *compact.*

(iv) *there exists a bounded mapping* Φ *from* $X \times X$ *into* X^* *such that*

$$\Phi(u, u) = Tu\ ; \tag{3.2}$$

(v) *for every* $u, w, h \in X$ *and any real sequence* $\{t_n\}$ *such that* $t_n \to 0$, *it is*

$$\Phi(w, u + t_n h) \rightharpoonup \Phi(w, u)\ ; \tag{3.3}$$

(vi) *for every* $u, w \in X$, $u \neq w$ *it is*

$$\langle \Phi(u, u) - \Phi(u, w), u - w \rangle > 0\ ; \tag{3.4}$$

(vii) *if* $u_n \rightharpoonup u$ *and*

$$\langle \Phi(u_n, u_n) - \Phi(u_n, u), u_n - u \rangle \to 0 \tag{3.5}$$

for $n \to \infty$, *then for every* $w \in X$

$$\Phi(u_n, w) \rightharpoonup \Phi(u, w)\ ; \tag{3.6}$$

(viii) *if* $w \in X$, $u_n \rightharpoonup u$ *and*

$$\Phi(u_n, w) \rightharpoonup z \tag{3.7}$$

for $n \to \infty$, *then*

$$\langle \Phi(u_n, w), u_n \rangle \to \langle z, u \rangle\ . \tag{3.8}$$

Then $T(X) = X^*$, *i.e. the equation*

$$Tu = f$$

has for every $f \in X^*$ *at least one solution* $u \in X$.

3.2. For $E \subset \mathbb{N}_0^N$, B a complete basis of E, we introduce the following notation:

$$\delta_E u = (\delta_{E-B} u, \delta_B u)$$

and – in accordance with this notation – we write for $\xi \in \mathbb{R}^M$

$$\xi = (\zeta, \eta)\,, \quad \zeta = \{\zeta_\beta;\ \beta \in E - B\}\,, \quad \eta = \{\eta_\gamma;\ \gamma \in B\}\ . \tag{3.9}$$

Now, we are able to formulate the main theorem:

3.3. Theorem. *Let* $\Omega \subset \mathbb{R}^N$ *and* $E \subset \mathbb{N}_0^N$ *fulfil the assumptions of Sections* 2.1 *and* 2.2, *respectively. Let* A *be the differential operator from* (0.2) *and suppose that the functions* $a_\alpha(x; \xi) = a_\alpha(x; \zeta, \eta)$ *fulfil the growth conditions* (1.5) *and satisfy the following conditions*:

(I) *for every* $\xi = (\zeta, \eta)$, $\hat{\xi} = (\zeta, \hat{\eta}) \in \mathbb{R}^M$; $\eta \neq \hat{\eta}$, *and for a.e.* $x \in \Omega$, *it is*

$$\sum_{\gamma \in B} [a_\gamma(x; \zeta, \eta) - a_\gamma(x; \zeta, \hat{\eta})]\,(\eta_\gamma - \hat{\eta}_\gamma) > 0\ ; \tag{3.10}$$

(II) *there exist constants* $c_1 > 0$, $c_2 > 0$, $c_3 \geqq 0$ *such that for every* $\xi \in \mathbb{R}^M$ *and a.e.* $x \in \Omega$ *it is*

$$(3.11) \qquad \sum_{\alpha \in E} a_\alpha(x; \xi)\, \xi_\alpha \geqq c_1 \sum_{\gamma \in B} |\xi_\gamma|^p + c_2 |\xi_\theta|^p - c_3 ;$$

(III) *for a.e.* $x \in \Omega$, *it is*

$$(3.12) \qquad \lim_{|\gamma| \to \infty} \frac{\sum_{\gamma \in B} a_\gamma(x; \zeta, \eta)\, \eta_\gamma}{|\eta| + |\eta|^{p-1}} = \infty$$

uniformly with respect to bounded subsets of ζ.

Then there exists at least one weak solution $u \in W^{E,p}(\Omega)$ *of the b.v.p.* (A, V, Q) *from Sections* 1.6, 1.5.

Proof. Let V be the space from 1.5(i) and let us define an operator T on V by the formula

$$(3.13) \qquad \langle Tw, v \rangle = \sum_{\alpha \in E} \int_\Omega a_\alpha(x; \delta_E(w(x) + \varphi(x)))\, D^\alpha v(x)\, dx ,$$

where φ is the function from 1.5(iii). The growth conditions (1.5) guarantee that T maps V into V^*; a comparison with (1.6) implies immediately that

$$Tw = \mathscr{A}(w + \varphi) .$$

Hence, if $\hat{u} \in V$ is such that

$$\langle T\hat{u}, v \rangle = \langle f, v \rangle + \langle g, v \rangle \quad \text{for every} \quad v \in V$$

(f, g being the functionals from 1.5(iv), (v)), then the element

$$u = \hat{u} + \varphi$$

is a weak solution of the b.v.p. (A, V, Q).

Consequently, it remains to show that the operator T from (3.13) satisfies the assumptions (i)–(viii) of Theorem 3.1 with $X = V$.

Conditions (i) and (ii) are direct consequences of the results of Section 2.5.

Ad (iii): Taking $\xi = \delta_E u(x)$ in (3.11) and then integrating over Ω, we obtain the inequality

$$\langle Tu, u \rangle \geqq c_4 |||u|||^p_{E,p} - c_3 \operatorname{meas} \Omega ;$$

hence the coerciveness of T follows by Theorem 2.3.

Ad (iv): For $u, v, w \in V$, we take

$$\langle \Phi_1(u, w), v \rangle = \sum_{\gamma \in B} \int_\Omega a_\gamma(x; \delta_{E-B}(u(x) + \varphi(x)), \delta_B(w(x) + \varphi(x)))\, D^\gamma v(x)\, \mathrm{d}x\,,$$

$$\langle \Phi_2(u), v \rangle = \sum_{\beta \in E-B} \int_\Omega a_\beta(x; \delta_E(u(x) + \varphi(x)))\, D^\beta v(x)\, \mathrm{d}x$$

and define

$$\Phi(u, w) = \Phi_1(u, w) + \Phi_2(u)\,.$$

Obviously, $\Phi(u, u) = Tu$; the boundedness of the mapping Φ follows from the estimate (2.4).

Condition (v) is again a direct consequence of the results of Section 2.5.

Ad (vi): Taking $\zeta = \delta_{E-B}(u(x) + \varphi(x))$, $\eta = \delta_B(u(x) + \varphi(x))$, $\hat{\eta} = \delta_B(w(x) + \varphi(x))$ in (3.10), we have

$$\langle \Phi_1(u, u) - \Phi_1(u, w), u - w \rangle > 0\,,$$

and this implies that (3.4) is fulfilled since $\langle \Phi_2(u) - \Phi_2(u), u - w \rangle = 0$.

Ad (vii): Let us denote, for $u_n \rightharpoonup u$ in V,

$$(3.14) \qquad F_n(x) = \sum_{\gamma \in B} [a_\gamma(x; \delta_{E-B}(u_n(x) + \varphi(x)), \delta_B(u_n(x) + \varphi(x)) - a_\gamma(x; \delta_{E-B}(u_n(x) + \varphi(x)), \delta_B(u(x) + \varphi(x)))]\, D^\gamma(u_n(x) - u(x))\,.$$

In view of the condition (3.10) it is $F_n(x) \geqq 0$ for a.e. $x \in \Omega$. Since $\langle \Phi_2(u_n) - \Phi_2(u_n), u_n - u \rangle = 0$, condition (3.5) means that

$$(3.15) \qquad \int_\Omega F_n(x)\, \mathrm{d}x \to 0 \quad \text{for} \quad n \to \infty\,.$$

Since $u_n \rightharpoonup u$ in V, it follows from Theorem 2.4 and from the reflexivity of V that

$$(3.16) \qquad u_n \to u \quad \text{in} \quad W^{F,p}(\Omega)$$

with F given by (2.3).

Now, there exist a set $N \subset \Omega$, meas $N = 0$, and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$(3.17) \quad F_{n_k}(x) \to 0\,, \quad D^\beta u_{n_k}(x) \to D^\beta u(x)\,, \quad |g_\alpha(x)| < \infty \quad \text{for} \quad x \in \Omega - N$$

($\alpha \in E$, $\beta \in F$) as a consequence of (3.15), (3.16) and of the fact that $g_\alpha(x) \in L^q(\Omega)$ (see (1.5)). Using the second and the third relation in (3.17) for a fixed $x \in \Omega - N$, we derive the estimate

$$(3.18) \quad F_{n_k}(x) \geqq \sum_{\gamma \in B} a_\gamma(x; \delta_{E-B}(u_{n_k}(x) + \varphi(x)), \delta_B(u_{n_k}(x) + \varphi(x)))\, D^\gamma u_{n_k}(x) - \sum_{\gamma \in B} |a_\gamma(x; \delta_{E-B}(u_{n_k}(x) + \varphi(x)), \delta_B(u_{n_k}(x) + \varphi(x)))| \,.\, |D^\gamma u(x)| -$$

$$- \sum_{\gamma \in B} |a_\gamma(x; \delta_{E-B}(u_{n_k}(x) + \varphi(x)), \delta_B(u(x) + \varphi(x)))| \cdot |D^\gamma(u_{n_k}(x) - u(x))| \geqq$$
$$\geqq \sum_{\gamma \in B} a_\alpha(x; \delta_{E-B}(u_{n_k}(x) + \varphi(x)), \delta_B(u_{n_k}(x) + \varphi(x))) D^\gamma u_{n_k}(x) -$$
$$- c(1 + \sum_{\gamma \in B} |D^\gamma u_{n_k}(x)|^{p-1} + \sum_{\gamma \in B} |D^\gamma u_{n_k}(x)|).$$

Assuming that $|\delta_B u_{n_k}(x)| \to \infty$, we conclude from (3.18) and from the condition (3.12) that $F_{n_k}(x) \to \infty$, which contradicts the first relation in (3.17). Consequently, if $\tilde{\eta}$ is a limit point of the sequence $\{\delta_B u_{n_k}(x)\}$, then necessarily $|\tilde{\eta}| < \infty$. Letting $n_k \to \infty$ in $F_{n_k}(x)$, we have – in view of (3.17) and of the continuity of $a_\alpha(x; \xi)$ with respect to ξ – the identity

$$\sum_{\gamma \in B} [a_\gamma(x; \delta_{E-B}(u(x) + \varphi(x)), \tilde{\eta} + \delta_B \varphi(x)) -$$
$$- a_\gamma(x; \delta_{E-B}(u(x) + \varphi(x)), \delta_B(u(x) + \varphi(x)))] \cdot [\tilde{\eta}_\gamma - D^\gamma u(x)] = 0.$$

However, this means that

$$\tilde{\eta} = \delta_B u(x)$$

as a consequence of (3.10) and thus we have

$$D^\gamma u_{n_k}(x) \to D^\gamma u(x), \quad \gamma \in B, \quad x \in \Omega - N.$$

This together with the second relation in (3.17) implies that

$$\delta_E u_{u_k}(x) \to \delta_E u(x) \quad \text{for a.e.} \quad x \in \Omega$$

and consequently,

$$(3.19) \qquad a_\alpha(x; \delta_E(u_{n_k}(x) + \varphi(x))) \to a_\alpha(x; \delta_E(u(x) + \varphi(x)))$$

for a.e. $x \in \Omega$.

Since $u_n \rightharpoonup u$ in V, the sequence $\{u_n\}$ is bounded and consequently, the sequence $\{a_\alpha(x; \delta_E(u_n(x) + \varphi(x)))\}$ is bounded in $L^q(\Omega)$. Then it follows from the Lebesgue Dominated Convergence Theorem that

$$(3.20) \qquad a_\alpha(x; \delta_E(u_{n_k} + \varphi)) \rightharpoonup a_\alpha(x; \delta_E(u + \varphi)) \quad \text{in} \quad L^q(\Omega).$$

Now, one can show by the usual procedure that (3.20) holds not only for $\{u_{n_k}\}$ but for the whole sequence $\{u_n\}$, and this implies the relation (3.6).

Ad (viii): It is

$$(3.21) \qquad \langle \Phi(u_n, w), u_n \rangle = \langle \Phi(u_n, w) - \Phi(u, w), u_n - u \rangle +$$
$$+ \langle \Phi(u, w), u_n - u \rangle + \langle \Phi(u_n, w), u \rangle.$$

Since $u_n \rightharpoonup u$, we have

$$\langle \Phi(u, w), u_n - u \rangle \to 0$$

and in virtue of the assumption (3.7)

$$\langle \Phi(u_n, w), u \rangle \to \langle z, u \rangle .$$

Further, (3.16) holds and hence in view of the results of Section 2.5,

$$a_\alpha(x; \delta_{E-B}(u_n + \varphi), \delta_B(w + \varphi)) \to a_\alpha(x; \delta_{E-B}(u + \varphi), \delta_B(w + \varphi))$$

in $L^q(\Omega)$, i.e.

$$\Phi(u_n, w) \to \Phi(u, w) .$$

Since the sequence $\{u_n - u\}$ is bounded, we have

$$\langle \Phi(u_n, w) - \Phi(u, w),\ u_n - u \rangle \to 0$$

and finally, (3.21) yields

$$\langle \Phi(u_n, w), u_n \rangle \to \langle z, u \rangle ,$$

which is (3.8).

Thus the assumptions of Theorem 3.1 are verified and consequently, Theorem 3.3 is proved.

3.4. Remark. Under the notation of Section 1.5, let us define a functional Φ on V by the formula

$$\Phi(v) = \int_0^1 \left(\sum_{\alpha \in E} \int_\Omega a_\alpha(x; t\delta_E\, v(x) + \delta_E\, \varphi(x))\, D^\alpha v(x)\, \mathrm{d}x \right) \mathrm{d}t - \langle f, v \rangle - \langle g, v \rangle .$$

Let us assume that the assumptions of Theorem 3.3 are fulfilled and that, moreover,

(i) the derivatives $\partial a_\alpha / \partial \xi_\beta$ exist for all $\alpha, \beta \in E$ and fulfil the symmetry conditions

$$\frac{\partial a_\alpha}{\partial \xi_\beta} = \frac{\partial a_\beta}{\partial \xi_\alpha}, \quad \alpha, \beta \in E ,$$

(ii) the functions

$$b_{\alpha\beta}(x; \xi) = \xi_\alpha \frac{\partial a_\alpha}{\partial \xi_\beta}(x; \xi) , \quad \alpha, \beta \in E ,$$

fulfil again the growth conditions (1.5).

Then one can prove that *there exists an element $u_0 \in V$ which realizes the minimum of the functional Φ. The element*

$$u_0 + \varphi$$

is then a weak solution of the b.v.p. (A, V, Q).

In applications, differential operators occur which not only need not involve some of the derivatives (which is expressed by the presence of D^α with $\alpha \in E$ only), but various derivatives are also allowed to have *various degrees of growth* (which can be expressed by the fact that we consider not a single exponent $p > 1$ but an M-tuple

$$\bar{p} = \{p_\alpha;\ \alpha \in E\}\,, \quad p_\alpha > 1)\,. \tag{4.1}$$

We shall now deal with b.v.p.'s with such more general differential operators.

4.1. The space $W^{E,\bar{p}}(\Omega)$. Let $E \subset \mathbb{N}_0^N$ be the set from Section 2.1, $\bar{p}$ given by (4.1), and assume

$$p_\alpha \leqq p_\beta \quad \text{for} \quad \alpha \geqq \beta\,. \tag{4.2}$$

The set

$$W^{E,\bar{p}}(\Omega) = \{u = u(x);\ D^\alpha u \in L^{p_\alpha}(\Omega) \quad \text{for} \quad \alpha \in E\} \tag{4.3}$$

is a *reflexive Banach space* if equipped with the norm

$$\|u\|_{E,\bar{p}} = \sum_{\alpha \in E} \|D^\alpha u\|_{p_\alpha}\,. \tag{4.4}$$

Again we set

$$W_0^{E,\bar{p}}(\Omega) = \overline{C_0^\infty(\Omega)}\,,$$

the closure being taken in the norm (4.4).

4.2. The domain Ω. We shall say that

$$\Omega \in \mathscr{D}(H, \delta)$$

with $H > 0$, $\delta > 0$, if Ω is a bounded domain in $\mathbb{R}^N$ such that for every $x \in \Omega$ there exists a closed cube C with edges of the length H parallel to the coordinate axes and with one vertex at the origin, such that for $y \in \Omega$, $|y - x| < \delta$, we have

$$y + C \subset \Omega\,.$$

It can be shown (see [7]) that

$$\mathscr{D}(H, \delta) \subset \mathscr{C}^{0,1}\,.$$

Now, we shall mention some results which are proved in [6] and [7]:

4.3. Theorem. *Let* $\Omega \in \mathscr{D}(H, \delta)$, $E_1 = \{\alpha \in \mathbb{N}_0^N;\ |\alpha| \leqq 1\}$, $\bar{p} = (p_0, p_1, \ldots, p_N)$, $p_0 \geqq p_i \geqq 1$, $i = 1, \ldots, N$. *Then*

$$W^{E_1,\bar{p}}(\Omega) \subset\subset L^q(\Omega)$$

with

$$\frac{1}{q} = \frac{1}{N}\left(\sum_{i=1}^{N} \frac{1}{p_i} - 1\right) \quad \textit{for} \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1\,,$$

$$q \geqq 1 \quad \textit{arbitrary} \quad \textit{for} \sum_{i=1}^{N} \frac{1}{p_i} \leqq 1\,.$$

For $1 \leqq r < q$ *it is*

$$W^{E_1,\bar{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)\,.$$

4.4. The space $C^{0,\bar{\mu}}(\bar{\Omega})$. Let $\bar{\mu} = (\mu_1, \ldots, \mu_N)$ with $0 \leqq \mu_i \leqq 1$. The set $C^{0,\bar{\mu}}(\bar{\Omega})$ of functions continuous on $\bar{\Omega}$ and such that

$$\|u\|_{0,\bar{\mu}} = \sup_{x\in\Omega} |u(x)| + \sup_{\substack{x,y\in\Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{\sum_{i=1}^{N} |x_i - y_i|^{\mu_i}} < \infty \tag{4.5}$$

is a Banach space under the norm (4.5).

4.5. Theorem. *Let* $\Omega \in \mathscr{D}(H, \delta)$, $E_1 = \{\alpha \in \mathbb{N}_0^N;\ |\alpha| \leqq 1\}$, $\bar{p} = (p_0, p_1, \ldots, p_N)$, $p_0 \geqq p_i > N$, $i = 1, \ldots, N$. *Then*

$$W^{E_1,\bar{p}}(\Omega) \hookrightarrow C^{0,\bar{\mu}}(\bar{\Omega})$$

with $\bar{\mu} = (\mu_1, \ldots, \mu_N)$,

$$\mu_i = 1 - \frac{N}{p_i}\left[1 - \sum_{j=1}^{N} \frac{1}{p_j} + \frac{N}{p_i}\right]^{-1}. \tag{4.6}$$

If $\bar{\nu} = (\nu_1, \ldots, \nu_N)$ *with* $0 \leqq \nu_i < \mu_i$, $i = 1, \ldots, N$, *then*

$$W^{E_1,\bar{p}}(\Omega) \hookrightarrow\hookrightarrow C^{0,\bar{\nu}}(\bar{\Omega})\,.$$

4.6. Notation. (i) The set E_1 considered in Sections 4.3 and 4.5 is very special. But using Theorems 4.3 and 4.5 repeatedly for $u \in W^{E,\bar{p}}(\Omega)$ (and for certain derivatives of u) with E a general subset of $\mathbb{N}_0^N$ satisfying the assumptions of Section 2.2, one can derive imbedding theorems also for general spaces $W^{E,\bar{p}}(\Omega)$. Let us denote by q_β (for $\beta \in E - B$, B being a complete basis of E, $\theta \notin B$) the exponent for which the operator

$$D^\beta : W^{E,\bar{p}}(\Omega) \to L^{q_\beta}(\Omega)$$

is continuous (in virtue of Theorem 4.3). Thus we have defined q_β for $\beta \in E - B$; for $\gamma \in B$ we take $r_\gamma = p_\gamma$.

(ii) If $r_\beta < q_\beta$ for *all* $\beta \in F = E - B$, then the second assertion of Theorem 4.3 implies compactness of the imbedding

$$W^{E,\bar{p}}(\Omega) \hookrightarrow\hookrightarrow W^{F,\bar{r}}(\Omega)\,, \tag{4.7}$$

where $\bar{r} = \{r_\beta;\ \beta \in F\}$.

(iii) Let us denote by G the subset of all multiindices $\beta \in F$ such that the operator

$$D^\beta : W^{E,\bar{p}}(\Omega) \to C^0(\bar{\Omega})$$

is — in virtue of Theorem 4.5 — *continuous.*

4.7. Growth conditions. The direct analogue of the growth conditions (1.5) has the following form: there exist constants $c_\alpha > 0$ and functions $g_\alpha \in L^{p_\alpha^*}(\Omega)$ with $p_\alpha^* = p_\alpha/(p_\alpha - 1)$ such that for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^M$, it is

$$|a_\alpha(x; \xi)| \leqq g_\alpha(x) + c_\alpha \sum_{\beta \in E} |\xi_\beta|^{p_\beta/p_\alpha^*}. \tag{4.8}$$

These conditions can be further generalized using the imbeddings mentioned above: Let us take

$$s_\alpha = 1 \qquad \text{for} \quad \alpha \in G, \tag{4.9}$$

$$s_\alpha = \frac{r_\alpha}{r_\alpha - 1} \quad \text{for} \quad \alpha \in E - G$$

(for G, r_α see 4.6 (i)–(iii)); then the generalized growth conditions have the following form: for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^M$, it is

$$|a_\alpha(x; \xi)| \leqq c_\alpha\Big(\sum_{\beta \in G} |\xi_\beta|\Big) \Big[g_\alpha(x) + \sum_{\beta \in E - G} |\xi_\beta|^{r_\beta/s_\alpha}\Big] \tag{4.10}$$

where $g_\alpha \in L^{s_\alpha}(\Omega)$ and c_α is a continuous nonnegative function defined on $[0, \infty)$.

4.8. Analogously as in Section 2.5, it follows from the growth conditions (4.8) or (4.10) that the operator H_α:

$$H_\alpha(u)(x) = a_\alpha(x; \delta_E u(x))$$

is a continuous mapping from $W^{E,\bar{p}}(\Omega)$ into $L^{p_\alpha^*}(\Omega)$ or $L^{s_\alpha}(\Omega)$, respectively, and estimates analogous to (2.4) can be derived. So we have — for the case of conditions (4.10) —

$$\left|\int_\Omega a_\alpha(x; \delta_E u(x)) D^\alpha v(x) \,\mathrm{d}x\right| \leqq$$

$$\leqq \begin{cases} \sup\limits_{x \in \Omega} c_\alpha\Big(\sum\limits_{\beta \in G} |D^\beta u(x)|\Big) \Big(\|g_\alpha\|_1 + \sum\limits_{\beta \in E - G} \|D^\beta u\|_{r_\beta}^{r_\beta}\Big) & \text{for} \quad \alpha \in G, \\ \sup\limits_{x \in \Omega} c_\alpha\Big(\sum\limits_{\beta \in G} |D^\beta u(x)|\Big) \Big(\|g_\alpha\|_{s_\alpha} + \sum\limits_{\beta \in E - G} \|D^\beta u\|_{r_\beta}^{r_\beta/s_\alpha}\Big) \|D^\alpha v\|_{r_\alpha} & \text{for} \quad \alpha \in E - G, \end{cases}$$

and it follows that the operator $\mathscr{A}$ defined in (1.6) is a bounded continuous mapping from $W^{E,\bar{p}}(\Omega)$ into $(W^{E,\bar{p}}(\Omega))^*$.

Consequently, using the spaces $W^{E,\bar{p}}(\Omega)$ and the growth conditions (4.8) or (4.10), we are able to define the weak solution of the b.v.p. (A, V, Q) analogously as in Section 1.6, writing always $\bar{p}$ instead of p. The analogue of Theorem 3.3 reads then as follows:

4.9. Theorem. *Let $\Omega \in \mathscr{D}(H, \delta)$, let $E \subset \mathbb{N}_0^N$ fulfil the assumptions of Section 2.2. Let A be the differential operator from (0.2) and suppose that the functions $a_\alpha(x; \xi) = a_\alpha(x; \zeta, \eta)$ fulfil the conditions 1.3(i), (ii), the growth conditions (4.8) or (4.10), the condition (3.10), the condition (3.12) with $p = \max_{\gamma \in B} p_\gamma$ and the coerciveness condition*

$$(4.11) \qquad \lim_{\|u\|_{E,\bar{p}} \to \infty} \frac{1}{\|u\|_{E,\bar{p}}} \sum_{\beta \in E} \int_\Omega a_\alpha(x; \delta_E(u(x) + \varphi(x))\, D^\alpha u(x) = \infty\,.$$

Finally, let the compact imbedding (4.7) hold.
Then there exists at least one weak solution $u \in W^{E,\bar{p}}(\Omega)$ of the b.v.p. (A, V, Q).

The proof of Theorem 4.9 is a more complicated analogue of the proof of Theorem 3.3. Let us mention that in the case of the growth conditions (4.8), the coerciveness condition (4.11) is satisfied if there exist constants $c_1 > 0$, $c_2 > 0$, $c_3 \geqq 0$ such that for every $\xi \in \mathbb{R}^M$ and a.e. $x \in \Omega$ it is

$$\sum_{\alpha \in E} a_\alpha(x; \xi)\, \xi_\alpha \geqq c_1 \sum_{\gamma \in B} |\xi_\gamma|^{p_\gamma} + c_2 |\xi_\theta|^{p_\theta} - c_3\,.$$

5. AN INTERPRETATION OF THE WEAK SOLUTION

5.1. The "usual" Sobolev spaces $W_0^{k,p}(\Omega)$ can be characterized by the conditions

$$D^\beta u|_{\partial\Omega} = 0 \quad \text{for} \quad |\beta| \leqq k - 1\,,$$

where $w|_{\partial\Omega}$ is the *trace* of the function w on the boundary $\partial\Omega$ of Ω. The anisotropic spaces $W_0^{E,\bar{p}}(\Omega)$ can be characterized again in a similar manner: the difference consists in the fact that *traces of only certain specially chosen derivatives in special coordinate directions* can be considered (for details see [5], [7]). For example, if $\Omega \subset \mathbb{R}^2$ has the form indicated in Fig. 1 with $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and if

$$(5.1) \qquad E = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1)\}\,,$$

then $W_0^{E,\bar{p}}(\Omega)$ may be characterized as follows:

$$(5.2) \qquad W_0^{E,\bar{p}}(\Omega) = \left\{u \in W^{E,\bar{p}}(\Omega);\ u|_{\partial\Omega} = \frac{\partial u}{\partial x}\bigg|_{\partial\Omega} = \frac{\partial u}{\partial y}\bigg|_{\Gamma_1 \cup \Gamma_3} = 0\right\},$$

i.e. *without any condition on* $\left.\dfrac{\partial u}{\partial y}\right|_{\Gamma_2}$.

5.2. For $\Omega \subset \mathbb{R}^2$ from Fig. 1, let us consider the differential operator A from (0.2) with the set E given by (5.1) (e.g., one can consider the operator

$$(Au)(x, y) = \frac{\partial^2}{\partial x^2}\left(\left|\frac{\partial^2 u}{\partial x^2}\right|^r \operatorname{sgn}\frac{\partial^2 u}{\partial x^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\left(\left|\frac{\partial^2 u}{\partial x\,\partial y}\right|^s \operatorname{sgn}\frac{\partial^2 u}{\partial x\,\partial y}\right) - \Delta u\,,$$

for which $\bar{p} = (2, 2, 2, r, s)$). Let $u \in W^{E,\bar{p}}(\Omega)$ be the weak solution of the Dirichlet problem with $\varphi \equiv 0$ and f given by

$$\langle f, v\rangle = \int_\Omega f(x, y)\, v(x, y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

Then $u \in V = W_0^{E,\bar{p}}(\Omega)$; if this weak solution u is smooth enough, one can show that u solves the "classical" b.v.p.

$$\begin{cases} Au = f \quad \text{on} \quad \Omega\,, \\ u\big|_{\partial\Omega} = 0\,, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_1 \cup \Gamma_3} = 0\,. \end{cases}$$

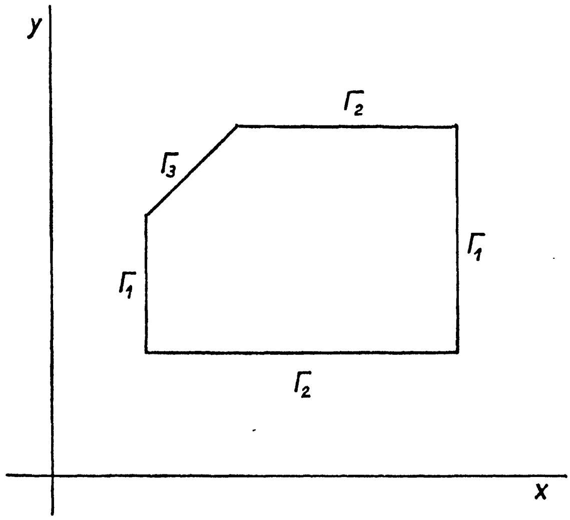

Fig. 1

Here, the difference between the isotropic and anisotropic case is demonstrated: if we considered the set $\tilde{E} = \{\alpha \in \mathbb{N}_0^2;\ |\alpha| \leqq 2\}$ which differs from our set E only by the multiindex $(0, 2)$ and for which $W^{\tilde{E},\bar{p}}(\Omega)$ is the "usual" Sobolev space $W^{2,\bar{p}}(\Omega)$, the boundary conditions would assume the form

$$u\big|_{\partial\Omega} = 0\,, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\,,$$

i.e. *including conditions for the normal derivative on* Γ_2.

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