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# On a conjecture of Turán

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# ON A CONJECTURE OF TURÁN

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The main result concerns a conjecture of Turán about minimal representations of $k$-tuples on an $n$-set by triples if $k$ is an odd number: The conjecture is not valid for $k > 3$ if $n = 2k - 1$.

## 1. INTRODUCTION

**Definition 1.1.** *Let $N = \{1, 2, \ldots, n\}$ be a set of $n$ elements, let $n > k > s$ be positive integers, let $R(n, k, s) = \{B_1, B_2, \ldots, B_m\}$ be a system of $s$-tuples with the following properties:*

a) $B_i \subset N$, $|B_i| = s$, $i = 1, 2, \ldots, m$; $m \leqq \binom{n}{s}$;

b) *if any $k$-subset $K \subset N$ is given then there exists at least one subset $B_j \in$ $\in R(n, k, s)$ such that $B_j \subset K$.*

*The system $R(n, k, s)$ is called a representation of $k$-tuples by $s$-tuples on an $n$-set. A representation is called minimal if it has the minimal number of $s$-tuples for given parameters $n, k, s$.*

Example 1.1. Let $n = 4$, $k = 3$, $s = 2$. Denote by $N, Q, S$ the following sets: $N = \{1, 2, 3, 4\}$, $Q = \{123, 124, 134, 234\}$, $S = \{12, 13, 14, 23, 24, 34\}$. It is clear that the minimal representation $R(4, 3, 2)$ is formed by 2 pairs: $R_{\min}(4, 3, 2) =$ $= \{12, 34\}$.

**Conjecture 1.1** (Turán's conjeture [2]). *Let $n > k > 3$ be positive integers where $k$ is an odd number. Then a minimal representation $R(n, k, 3)$ is formed by all triples on subsets $N_i$, where $i = 1, 2, \ldots, (k - 1)/2$, $N = \bigcup N_i$, $N_i \cap N_j = \emptyset$, $||N_i| - |N_j|| \leqq 1$.*

We refer to such representations as to *T-representations*.

Example 1.2. Let $n = 7$, $k = 5$, $s = 3$. The disjoint subsets are: $N_1 = \{1, 2, 3, 4\}$, $N_2 = \{5, 6, 7\}$. The T-representation $R(7, 5, 3) = \{123, 124, 134, 234, 567\}$. This representation is minimal.

**Theorem 1.1.** *Denote by $f(n. k, s)$ the number of s-tuples in a minimal represetation $R(n, k, s)$. Then the following relation holds:*

$$f(n, k, s) \geqq (n/(n-3))f(n-1, k, s). \tag{1}$$

The proof is given in [1].

Turán's conjecture is not valid in the case $R(9, 5, 3)$. The T-representation has 14 triples, but a minimal one has only 12 triples. These triples form the Steiner triple system of order 9. We will show in the following that the same situation occurs for infinitely many numbers $n$.

Further, we shall give results of the investigation of minimal representations $R(n, k, 3)$ for $k = 7, 9$ for some numbers $n$.

J. Suranyi proved [3] the validity of Turán's conjecture for representations $R(10, 5, 3)$.

## 2. MINIMAL REPRESENTATIONS $R(n, 9, 3)$ FOR $n < 18$

**Theorem 2.1.** *The following relations hold for $n = 10$ or $n = 11$: $f(10, 9, 3) = 2$ or $f(11, 9, 3) = 3$, respectively. There is exactly one representation formed by two or three triples, respectively. It is in both cases a T-representation.*

Proof. It holds $f(9, 9, 3) = 1$. According to (1) it follows that $f(10, 9, 3) \geqq \geqq ]10/7[ = 2$. A T-representation for $n = 10$ is formed by two disjoint triples, therefore it is minimal. If we consider two triples which are not disjoint then there always exists a 9-tuple which is not represented by such triples.

According to (1) we have for $n = 11$ the relation $f(11, 9, 3) \geqq ](11/8) . 2[ = 3$. A T-representation is formed by three disjoint triples, therefore it is minimal. A 9-tuple not represented by any three non-disjoint triples always exists.

For three triples we have various possibilities of their mutual relations. We say that we obtain various structures of three triples. If a triangle corresponds to each triple we obtain 12 different structures of three triples. (Fig. 1.)

We denote the vertices of triangles by 1, 2, ..., 9. For 11 structures ninetuples are given not represented by the triples of the structures.

If any three triples on the set $N = \{1, 2, \ldots, 11\}$ are given then they must have one of the structures in Fig. 1. So we obtain a permutation which transforms these given triples into one of the 12 cases. If we obtain the case a) then the given triples form a representation $R(11, 9, 3)$ isomorphic to the triples of the case a). If we obtain another case than a) then the triples do not form a representation. Proof. Suppose

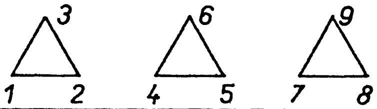
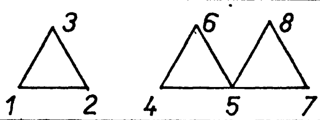
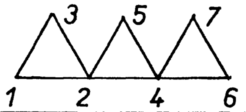
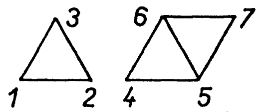
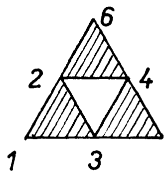
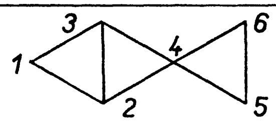
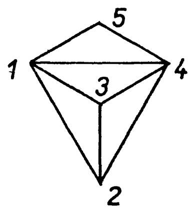
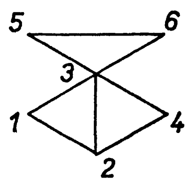
| CASE | STRUCTURE | A 9-TUPLE NOT REPRESENTED BY THE STRUCTURE |
|---|---|---|
| a) |  | NONE |
| b) |  | 1 2 4 6 7 8 9 10 11 |
| c) |  | 1 3 4 5 6 8 9 10 11 |
| d) |  | 1 2 4 5 7 8 9 10 11 |
| e) |  | 1 2 4 5 7 8 9 10 11 |
| f) |  | 1 2 4 5 7 8 9 10 11 |
| g) |  | 1 2 4 6 7 8 9 10 11 |
| h) |  | 1 2 4 5 7 8 9 10 11 |

Fig. 1.

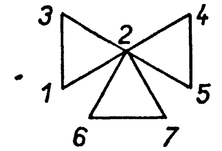
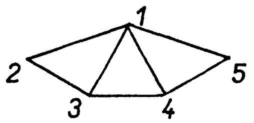
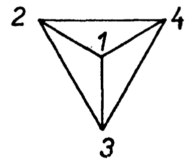
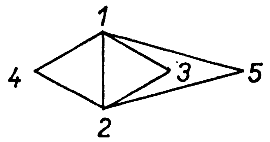
| | | |
|---|---|---|
| *i)* |  | *1 3 4 5 6 7 8 9 10* |
| *j)* |  | *2 3 4 5 6 7 8 9 10* |
| *k)* |  | *2 3 4 5 6 7 8 9 10* |
| *l)* |  | *1 3 4 5 6 7 8 9 10* |

Fig. 1 - cont.

that the given triples represent the set $K$ of all 9-tuples on 11-set. The mentioned permutation transforms the given triples into a certain case different from a) and the set $K$ into itself. Thus the triples of that case different from a) represent the set $K$. We obtain a contradiction because these triples do not represent the set $K$.

Thus the T-representation is the minimal representation for $R(11, 9, 3)$.

**Theorem 2.2.** *There is only one minimal representation* $R(12, 9, 3)$. *It is the T-representation which is formed by four disjoint triples.*

Proof. According to (1) the following relation holds: $f(12, 9, 3) \geqq (12/9) . 3 = 4$. The $T$-representation for these parameters is formed by 4 disjoint triples, therefore it is minimal.

Suppose that there exists a minimal representation $R(12, 9, 3)$ which is not formed by 4 disjoint triples. Any element of the set $N = \{1, 2, \ldots, 12\}$ occurs in the representation not more than once. Suppose that an element $a$ occurs more than once. Let us delete the triples with the element $a$ from the representation. We obtain at most two triples which must form a representation $R(11, 9, 3)$. Thus we have a contradiction with Theorem 2.1. Therefore the element $a$ occurs at most once and the triples must be disjoint. We obtain a contradiction with the assumption.

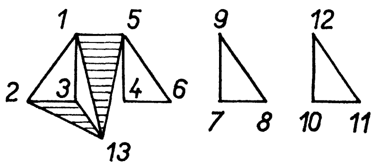
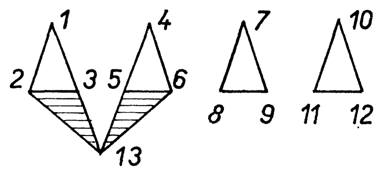
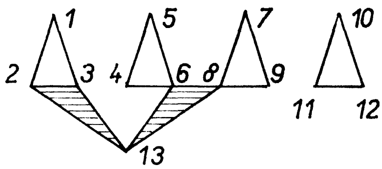
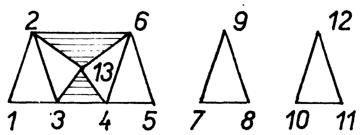
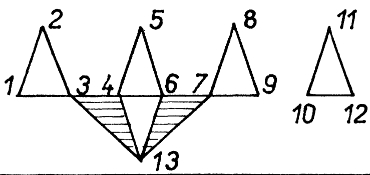
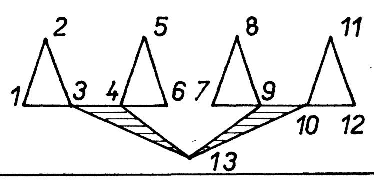
| CASE | STRUCTURE | A 9-TUPLE NOT REPRESENTED BY THE STRUCTURE |
|---|---|---|
| (a) |  | 1 2 5 6 7 8 10 11 13 |
| (b) |  | 1 2 4 5 7 8 10 11 13 |
| (c) |  | —— " —— |
| (d) |  | 1 2 4 6 7 8 10 11 13 |
| (e) |  | 1 2 4 5 7 8 10 11 13 |
| (f) |  | —— " —— |

Fig. 2.

**Theorem 2.3.** *A minimal representation* $R(13, 9, 3)$ *contains 7 triples. There exist at least 4 nonisomorphic minimal representations and just one of them has the* *following property: any element of its triples occurs in at most two triples, i.e. the degree of any element equals at most 2.*

Proof. Let us suppose that $f(13, 9, 3) = 6$. In the corresponding representation $R(13, 9, 3)$ there must exist an element of the set $N = \{1, 2, \ldots, 13\}$ whose degree is equal to 2. No element can have a greater degree than 2. These assertions can be proved analogously as in the proof of Theorem 2.2.

We can suppose that the element of the set $N$ denoted by *13* has the degree equal to 2. Let us omit both triples with the element *13* from the representation considered. We obtain 4 triples which must form a representation $R(12, 9, 3)$. According to Theorem 2.2 these triples must be disjoint and therefore we obtain the representation $R(13, 9, 3)$ by adding to them two triples with the element *13*. Let the disjoint triples be *123, 456, 789, 10 11 12*. Then we have the following possibilities for two triples with the element *13* (Fig. 2):

| | |
|---|---|
| a) *2 3 13, 1 5 13* | d) *3 4 13, 2 6 13* |
| b) *2 3 13, 5 6 13* | e) *3 4 13, 6 7 13* |
| c) *2 3 13, 6 8 13* | f) *3 4 13, 9 10 13* |

Two triples with the element *13* and four disjoint triples form a structure of six triples. We obtain six structures which are mutually nonisomorphic. This fact follows immediately from the graphical representation of the triples by triangles.

To each obtained structure we can easily find at least one 9-tuple which is not represented by the triples of the same structure. The following 9-tuples fullfil this condition:

ad a): *1 2 5 6 7 8 10 11 13*,
ad b), c), f): *1 2 4 5 7 8 10 11 13*,
ad d): *1 2 4 6 7 8 10 11 13*,
ad e): *1 2 4 5 8 9 10 11 13*.

It follows that the representation satisfying our assumptions does not exist and therefore $f(13, 9, 3) = 7$. The T-representation for these parameters contains seven triples, therefore it is minimal:

(2) = {*1 2 3, 1 2 4, 1 3 4, 2 3 4, 5 6 7, 8 9 10, 11 12 13*}.

Now we introduce three representations which are not isomorphic to the representation (2):

(3) = {*1 2 3, 4 5 6, 7 8 9, 10 11 12, 1 5 13, 2 6 13, 3 4 13*};
(4) = {*1 2 3, 4 5 6, 7 8 9, 10 11 12, 1 2 13, 3 6 13, 4 5 13*};
(5) = {*1 2 3, 4 5 6, 7 8 9, 7 9 12, 8 10 12, 2 11 13, 1 3 13*}.

The representation (2) has 4 elements of degree 3 and 9 elements of degree 1, the representation (3) has one element of degree 3, six elements of degree 2 and six elements of degree 1. The representation (4) has the same degrees of elements as the representation (3), but they are not isomorphic as we can see from the graphical representation by triangles in Fig. 3. The representation (5) has 5 elements of degree 2 and 8 elements of degree 1.

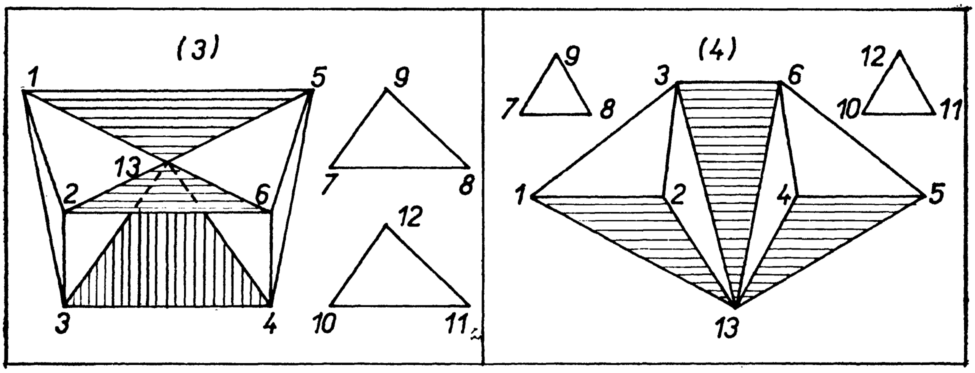



Fig. 3.

The proof of the last assertion of our Theorem will be omitted because it is too lengthy. It is necessary to consider about 70 structures of seven triples on a 13-set whose elements have the degree less than or equal to 2.

**Theorem 2.4.** *The T-representations* $R(n, 9, 3)$ *for* $n = 14, 15, 16$ *are minimal but* $R(17, 9, 3)$ *is not.*

Proof. For $n = 14$ we obtain by Theorems 1.1 and 2.3 the following relation: $f(14, 9, 3) \geqq ](14/11) . 7[ = 9$. The T-representation is formed by 10 triples on the subsets $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9, 10\}$, $\{11, 12, 13, 14\}$.

We shall prove that there is no representation $R(14, 9, 4)$ with 9 triples. Suppose that such a representation exists. Any element of the 14-set has in this representation degree at most 2 which can be proved analogously as in the proof of Theorem 2.2.

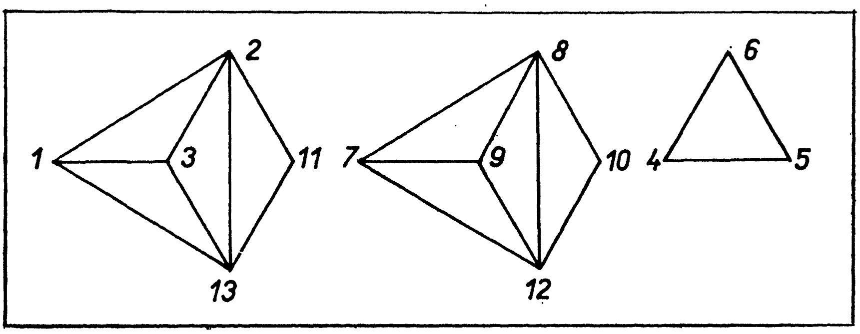



Fig. 4.

Let us omit 2 triples with the same element. We can suppose that the omitted element is denoted by *14*. We obtain 7 triples of the representation $R(13, 7, 3)$ in which the elements of the 13-set have degree equal to at most 2. According to Theorem 2.3 there exists just one representation with this property; it is the representation (5). Let us add to it 2 triples with the element *14* so that the degree of no element exceeds 2. Let us illustrate the representation (5) by triangles in Fig. 4. Both triples with the element *14* can clearly contain only the elements *10, 11, 4, 5, 6,* i.e. the elements with degree one. There are only 3 different cases:

(a) = {*10 11 14, 4 5 14*},
(b) = {*4 10 14, 5 11 14*},
(c) = {*4 5 14, 6 11 14*}.

But the following 9-tuple (aa) is not represented by 7 triples of the representation (5) and by two triples of (a). The same holds analogously for (bb) and (cc).

(aa) = *1 2 4 6 7 8 10 13 14*,
(bb) = *1 2 4 6 7 8 12 13 14*,
(cc) = *1 2 5 6 7 8 10 13 14*.

This fact implies that we obtain no representation $R(14, 9, 3)$ with 9 triples. Thus a minimal representation $R(14, 9, 3)$ is formed by 10 triples.

According to this result, we have for $n = 15$ the estimate $f(15, 9, 4) \geqq ](15/12) . 10[ = 13$. The T-representation on the subsets {*1, 2, 3*}, {*4, 5, 6, 7*}, {*8, 9, 10, 11*}, {*12, 13, 14, 15*} has just 13 triples, therefore it is minimal.

According to this result, we obtain for $n = 16$ the estimate $f(16, 9, 4) \geqq ] (16/3) . 13[ = 16$. The T-representation on the subsets {*1, 2, 3, 4*}, {*5, 6, 7, 8*}, {*9, 10, 11, 12*}, {*13, 14, 15, 16*} has just 16 triples, therefore it is minimal.

According to the last result, we obtain for $n = 17$ the estimate $f(17, 9, 4) \geqq \geqq ](17/14) . 16[ = 20$. The T-representation on the subsets {*1, 2, 3, 4*}, {*5, 6, 7, 8*}, {*9, 10, 11, 12*}, {*13, 14, 15, 16, 17*} has 22 triples. We shall show in the following (Theorem 4.1) that the minimal representation actually contains only 20 triples.

## 3. MINIMAL REPRESENTATIONS $R(n, 7, 3)$ FOR $n < 14$

**Theorem 3.1** *If* $n = 8, 9, \ldots, 12$ *then Turán's conjecture holds for representations* $R(n, 7, 3)$.

Proof. It is clear that $f(7, 7, 3) = 1$. Thus $f(8, 7, 3) \geqq ]8/5[ = 2$. Further, we have the relation $f(9, 7, 3) \geqq ](9/6) . 2[ = 3$. In these cases the T-representations contain just $f(n, 7, 3)$ triples.

For $n = 10$ we obtain the relation $f(10, 7, 3) \geqq ](10/7) . 3[ = 5$. But the corresponding T-representation is formed by 6 triples on the subsets {*1, 2, 3*}, {*4, 5, 6*},

$\{7, 8, 9, 10\}$. We will show that the T-representation serves as a minimal representation $R(10, 7, 3)$.

Suppose that a minimal representation $R(10, 7, 3)$ is formed by 5 triples. There must exist an element of degree 2, but no element of degree 3 or greater in such a representation. These assertions can be proved analogously as in the proof of Theorem 2.2.

We can suppose that the element of degree 2 is denoted by *10* and omit two triples with the element *10*. We obtain 3 triples which must represent all 7-tuples on a 9-set.

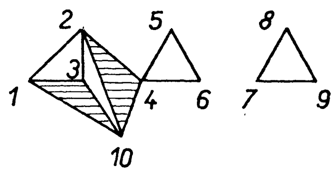
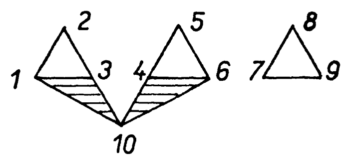
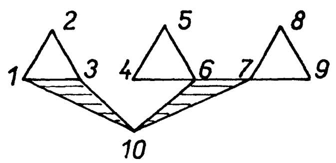
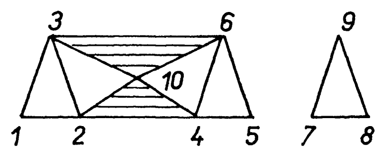
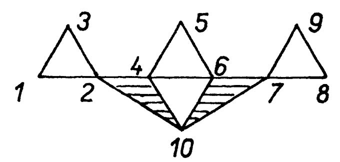
| CASE | STRUCTURE | A 7-TUPLE NOT REPRESENTED BY THE STRUCTURE |
|---|---|---|
| (a) |   | 1 2 5 6 7 8 10 |
| (b) |   | 1 2 4 5 7 8 10 |
| (c) |   | 1 2 4 5 7 8 10 |
| (d) |   | 1 3 4 5 7 8 10 |
| (e) |   | 1 3 4 5 8 9 10 |

Fig. 5.

If we consider all structures of 3 triples in Fig. 1 we find easily that only the structure of three disjoint triples can serve as representation $R(9, 7, 3)$.

The other structures do not represent the 7-tuples which are formed by the first 7 elements of the 9-tuples given in Fig. 1.

Thus we obtain the required representation $R(10, 7, 3)$ by adding two triples with the element *10* to 3 disjoint triples. No element can have a greater degree than 2. There are 5 possibilities of adding two triples: see Fig. 5, where the added triples are marked by shading. The 7-tuples given in Fig. 5 are not represented by the triples of the 5 structures. Therefore the supposition was false and thus the minimal representation $R(10, 7, 3)$ has 6 triples. It is e.g. the T-representation for this case.

According to this result, we have the following estimate for $n = 11 : f(11, 7, 3) \geqq \geqq ](11/8) . 6[ = 9$. The T-representation on the subsets $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, $\{9, 10, 11\}$ is formed by 9 triples and therefore it is minimal.

According to the last result, we have the following estimate for $n = 12$ : $f(12, 7, 3) \geqq ](12/9) . 9[ = 12$. Therefore the T-representation formed by 12 triples on the subsets $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, $\{9, 10, 11, 12\}$ is minimal.

So we have proved the assertions of Theorem 3.1.

**Theorem 3.2.** *A minimal representation* $R(13, 7, 3)$ *is formed by* 16 *triples. Turán's conjecture does not hold in this case.*

Proof. The assertion is a special case of Theorem 4.1.

## 4. THE MAIN THEOREM

**Theorem 4.1.** *Let* $k \geqq 5$ *be an odd number. Then the T-representation* $R(2k - 1, k, 3)$ *is not minimal. For each* $k > 3$ *there exists a representation* $R(2k - 1, k, 3)$ *with* $2k + 2$ *triples.*

Proof. We will use the induction by $k$. It is well known that the theorem holds for $k = 5$. Let us suppose that it holds for an odd integer $k > 3$ and let us prove it for the next odd integer $k + 2$.

The T-representation $R(2k - 1, k, 3)$ has $2k + 4$ triples. We split the set $N = \{1, 2, \ldots, 2k - 1\}$ into $(k - 1)/2$ disjoint classes. The cardinality of each class is equal to 4 except one class with cardinality 5. Evidently, the following relation holds: $4 . ((k - 1)/2 - 1) + 5 = 2k - 1$. The number of triples in a T-representation is therefore equal to

$$((k - 1)/2 - 1) . \binom{4}{3} + \binom{5}{3} = 2k + 4 .$$

By the assumption there exists a representation $R(2k - 1, k, 3)$ with $2k + 2$ triples.

We shall search for a representation $R(2k + 3, k + 2, 3)$. We decompose the set

$N' = \{1, 2, \ldots, 2k - 1, 2k, 2k + 1, 2k + 2, 2k + 3\}$ into two disjoint classes:

$$A = \{1, 2, \ldots, 2k - 1\}, \quad B = \{2k, 2k + 1, 2k + 2, 2k + 3\}.$$

Let us consider any $(k + 2)$-tuple of the set $N'$. If this $(k + 2)$-tuple has at least $k$ elements in the class $A$ then it is represented by a certain triple of the representation $R(2k - 1, k, 3)$ whose triples represent all $k$-tuples of the class $A$. If the $(k + 2)$-tuple has 3 or 4 elements in the class $B$ then it is represented by a certain triple on the class $B$.

Therefore all $(k + 2)$-tuples on the set $N'$ are represented either by the triples of the representation $R(2k - 1, k, 3)$ or by the triples on the class $B$. The number of the triples is therefore equal to $2k + 2 + 4 = 2k + 6$.

However, the T-representation $R(2k + 3, k + 2, 3)$ has $2(k + 2) + 4 = 2k + 8$ triples. Thus the theorem is proved.

## *References*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