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K. H. Kulkarni

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SUFFICIENT CONDITIONS FOR EDGE-LOCALLY CONNECTED AND n-CONNECTED GRAPHS

K. H. KULKARNI, Bombay

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INTRODUCTION

Let G be a graph. For an edge $x = uv$ of G, let $N(x)$ denote the set of all vertices which are different from u and v, and are adjacent with u or v. The subgraph of G induced by $N(x)$ is called the *neighbourhood* of the edge x and is denoted by $G(x)$. G is said to be *edge-locally connected* if the neighbourhood of every edge is connected. G is said to be *edge-locally n-connected* if the neighbourhood of every edge is n-connected. The *edge-local connectivity* of G is the maximum n such that G is edge-locally n-connected and is denoted by $\mathrm{ek}(G)$.

The concept of edge-local connectivity is edge-analogue of local connectivity of Chartrand and Pippert [2]. In [2] sufficient conditions for graphs to be locally connected and n-connected are given in terms of sum of degrees of pairs of vertices. VanderJagt [4] improved this sufficient condition for a graph to be locally connected. In this paper sufficient conditions for graphs to be edge-locally connected, and n-connected are given; further, it is shown that the sufficient condition for a graph to be edge-locally connected cannot be improved when the order of the graphs is congruent to 1(mod 3) or 2(mod 3).

The number of vertices a graph G contains will be denoted by $p(G)$. Other terminology is in conformity with that of [1].

SUFFICIENT CONDITIONS FOR GRAPHS TO BE EDGE-LOCALLY CONNECTED

Theorem 1. *If G is a graph of order p such that for every pair u, v of vertices,* $\deg u + \deg v > \frac{2}{3}(2p - 1)$, *then G is edge locally connected.*

Proof. Assume that G satisfies the hypothesis of the theorem but is not edge-locally connected. Let $x = uv$ be an edge of G such that $G(x)$ is disconnected. Let G_1 be a component of $G(x)$, and $G_2 = G(x) - G_1$. Since for every pair u, v of vertices of G, $\deg u + \deg v > \frac{2}{3}(2p - 1)$, it follows that $p(G(x)) \geqq \frac{1}{3}(2p - 1)$. Consider a vertex w_i of G_i, $i = 1, 2$. Then $\deg(w_1, G) \leqq p(G) - p(G(x)) + p(G_1) - 1$, and $\deg(w_4, G) \leqq p(G) - p(G_1) - 1$. Hence

$$\deg(w_1, G) + \deg(w_2, G) \leqq 2\,p(G) - p(G(x)) - 2 \leqq$$

$$\leqq 2p - \tfrac{1}{3}(2p - 1) - 2 = \tfrac{2}{3}(2p - 1) - 1\,.$$

This contradicts the fact that $\deg w_1 + \deg w_2 > \frac{2}{3}(2p - 1)$. Hence the theorem follows.

The following corollary is immediate.

Corollary. *If for a graph G of order p,* $\min\deg G > \frac{1}{3}(2p \text{ W } 1)$, *then G is edge-locally connected.*

We now show that Theorem 1 can be improved when the order of a graph is congruent to 1(mod 3) or 2(mod 3).

Theorem 2. *Let G be a graph of order $p = 3k + 1$ containing up to k vertices of degree $2k$ and remaining vertices of degree exceedtng $2k$. Then G is edge-locally connected.*

Proof. Suppose a graph G satisfies the hypothesis of the theorem but is not edge-locally connected. Then $\min\deg G = 2k$, for otherwise by the corollary to Theorem 1, G is edge-locally connected. Let $x = uv$ be an edge of G such that $G(x)$ is disconnected. Let G_1 be a component of $G(x)$ of minimum order, and $G_4 = G(x) - G_1$. Clearly, $p(G(x)) \geqq 2k - 1$. Supposz $p(G(x)) = 2k + r \text{ W } 1$, where r is 1 non-negative integer. Then $p(G_1) \leqq k + [\frac{1}{2}(r - 1)]$, where as usual $[x]$ denotes the greatest integer not greater than x. Now for a vertex w of G_1,

$$\text{(1)} \qquad \deg(w, G) \leqq p(G) - p(G(x)) + p(G_1) - 1 \leqq$$

$$\leqq 3k + 1 - (2k + r - 1) + k + [\tfrac{1}{2}(k - 1)] - 1 \leqq 2k - r + [\tfrac{1}{2}(k - 1)] + 1\,.$$

Since r is a non-negative integer, $-r + [\frac{1}{2}(r - 1)] + 1$ is nonpositive. Also, since $\min\deg G = 2k$, it follows that $-r + [\frac{1}{2}(r - 1)] + 1 = 0$ and equality occurs in (1). Now $-r + [\frac{1}{2}(r - 1)] + 1 = 0$ if $r = 0$ or $r = 1$. Also, $p(G_1) = k + [\frac{1}{2}(r - 1)]$ when equality occurs in (1).

Case 1. Suppose $r = 0$. Then $p(G(x)) = 2k - 1$ and $p(G_1) = k - 1$. Now $p(G(x)) = 2k - 1$ implies $\deg(u, G) = 2k$ and $\deg(v, G) = 2k$. Also, the degree of every vertex of G_1 in G is $2k$. Thus G contains $k + 1$ vertices of degree $2k$ which contradicts the hypothesis.

Case 2. Suppose $r = 1$. Then $p(G(x)) = 2k$ and $p(G_1) = k$. Hence G_2 is also a component of $G(x)$ of minimum order k. Hence the degree of every vertex of G_2 in G is $2k$. Since the degree of every vertex of G_1 in G is $2k$, G contains $2k$ vertices of degree $2k$ which is again a contradiction. Hence the theorem follows.

On the same lines the following theorem can be proved.

Theorem 3. *Let G be a graph of order $p = 3k + 2$ containing up to $2k + 1$ vertices of degree $2k + 1$ and all others of degree exceeding $2k + 1$. Then G is edge-locally connected.*

By exhibiting examples, it will now be shown that for graphs of orders more than 9 Theorems 2 and 3 cannot be improved. Let G_1 and G_2 be disjoint graphs. The *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is the union of vertex sets of G_1 and G_2 and edge set is the union of edge sets of G_1 and G_2. The *sum* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph whose vertex set is the union of the vertex sets of G_1 and G_2, and two vertices are adjacent if and only if they are adjacent vertices of G_1, or adjacent vertices of G_2, or one is a vertex of G_1 and the other is a vertex of G_2.

Example 1. Let $p = 3k + 1$, $k > 2$. Let G_1, G_2, G_3 and G_4 be pair-wise disjoint graphs, where $G_1 = K_2$, $G_2 = K_k$, $G_3 = K_{k-1}$, and $G_4 = K_k$. Let $G = (G_1 \cup G_2) + (G_3 \cup G_4)$. Then in the graph G, the degree of every vertex belonging to $V(G_1)$ or $V(G_3)$ is $2k$; the degree of every vertex belonging to $V(G_2)$ or $V(G_4)$ exceeds $2k$. Thus G contains $k + 1$ vertices of degree $2k$ and remaining vertices of degree more than $2k$. But G is not edge-locally connected as the neighbourhood of the edge in G_1 being $G_3 \cup G_4$, is disconnected. Hence Theorem 2 is best possible.

Example 2. Let $p = 3k + 2$, $k > 2$. Let G_1, G_2, G_3, and G_4 be pair-wise disjoint graphs, where $G_1 = K_2$, $G_2 = K_k$, $G_3 = K_k$, and $G_4 = K_k$. Let $G = (G_1 \cup G_2) + (G_3 \cup G_4)$. Then G contains $2k + 2$ vertices of degree $2k + 1$ and remaining vertices of degree more than $2k + 1$. But G is not edge-locally connected as the neighbourhood of the edge in G_1 being $G_3 \cup G_4$, is disconnected. Hence theorem 3 is best possible.

SUFFICIENT CONDITION FOR GRAPHS TO BE EDGE-LOCALLY n-CONNECTED

Theorem 4. *Let G be a graph of order p such that for every pair u, v of vertices, $\deg u + \deg v > \frac{2}{3}(2p + n) - 1$, where $1 \leqq n \leqq p - 3$. Then G is edge-locally n-connected.*

Proof. Suppose that G satisfies the hypothesis of the theorem but is not edge-locally n-connected. Let $x = uv$ be an edge for which $G(x)$ is not n-connected. Following two cases arise.

Case 1. Suppose $G(x) = K_j$, for some $j \leqq n$. Assume that G contains a vertex w which is not adjacent with u or v. Then $\deg u \leqq j + 1 \leqq n + 1$ and $\deg w \leqq p - 3$, so that $\deg u + \deg v \leqq p + n - 2$. By hypothesis, $p + n - 2 > \frac{2}{3}(2p + n) - 1$. This implies that $n > p$ which is not possible. Hence every vertex of G is adjacent with u or v. Suppose now that there exists a vertex w which is adjacent with only one of u and v, say w is adjacent with u. Then $\deg w = j$ and $\deg u \leqq j + 1$. Hence $\deg u + \deg w \leqq j + j + 1 \leqq 2n + 1$. Again, by hypothesis $2n + 1 > \frac{2}{3}(2p + n) - 1$. This implies that $n > p - 1$ which again, is not possible. This proves that $G = K_p$; and hence G is edge-locally n-connected for every n, $1 \leqq n \leqq p - 3$, which is contrary to the initial hypothesis.

Case 2. Suppose $G(x)$ contains a set T of t ($\leqq n - 1$) vertices whose removal from $G(x)$ results into a disconnected graph. Let G_1 be a component of $G(x) - T$ of minimum order, $G_2 = (G(x) - T) - G_1$, and $G_3 = G - (N(x) \cup \{u, v\})$. Let $p(G_i) = m_i$, $i = 1, 2, 3$ and w be a vertex of G_1. Then $\deg u \leqq m_1 + m_2 + t + 1$ and $\deg(w, G) \leqq p - m_2 - 1$. By hypothesis, $\deg u + \deg w > a$, where $a = \frac{2}{3}(2p + n) - 1$. Hence $\deg w > a - \deg u$, so that $p - m_2 - 1 > a - m_1 - m_2 - t - 1$. Hence

$$(2) \qquad m_1 > a - p - t\,.$$

$$(3) \qquad m_2 > a - p - t\,, \quad \text{since} \quad m_2 \geqq m_1\,.$$

Now, $m_3 = p - m_1 - m_2 - t - 2$. Hence from (2) and (3)

$$(4) \qquad m_3 < p - 2a + 2p + 2t - t - 2 = 3p - 2a + t - 2\,.$$

Let w' be a vertex of G_2. Then $\deg(w', G) \leqq m_2 + m_3 + t + 1$. Hence

$$(5) \qquad \deg(w, G) + \deg(w', G) \leqq (m_1 + m_3 + t + 1) + (m_2 + m_3 + t + 1) =$$
$$= (m_1 + m_2 + m_3 + t + 2) + t + m_3 = p + t + m_3 <$$
$$< p + t + 3p - 2a + t - 2\,, \text{ from (4)} =$$
$$= 4p - 2a - 2 + 2t \leqq (4p + 2n - 3) - 2a - 1\,, \quad \text{since} \quad t \leqq n - 1\,.$$

Now $a = \frac{2}{3}(2p + n) - 1$. Hence $3a = 4p + 2n - 3$. Substituting this in (5), $\deg(w, G) + \deg(w', G) < 3a - 2a - 1 = a - 1 = \frac{2}{3}(2p + n) - 2$. This is again a contradiction to the initial hypothesis. Hence the theorem follows.

Corollary. *If G is a graph of order p for which* $\min \deg G > \frac{2}{3}(2p + n) - 1$, *where $1 \leqq n \leqq p - 3$, then G is edge-locally n-connected.*

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References

[1] *M. Behzad* and *G. Chartrand:* Introduction to the theory of Graphs. Allyn and Bacon, Boston (1971).

[2] *G. Chartrand* and *R. E. Pippert:* Locally connected graphs. Časopis Pěst. Mat. *99* (1974), 158—163.

[3] *K. H. Kulkarni:* Some topics in graph theory, Ph. D. Thesis, Karnatak Univ. Dharwar (1976).

[4] *D. W. VanderJagt:* Sufficient conditions for locally connected graphs. Časopis Pěst. Mat. *99* (1974), 400—404.

Author's address: Department of Mathematics, Victoria Jubilee Technical Institute, Bombay 400 019, India.