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VARIATIONS OF DISCRETE ANALOGUES OF WIRTINGER'S INEQUALITY

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Discrete analogues of Wirtinger's inequality have been already studied by different methods of proofs. The basic theorem of the topic dealt with in our article is Theorem 1. Its first proof was published 1950 by I. J. SCHOENBERG (see [5]). The author uses the complex finite Fourier series and proves Theorem 1 for complex numbers.

In [3], published 1955, K. FAN, O. TAUSSKY and J. TODD discuss discrete analogues of several integral inequalities. The main tool they use to prove them are the properties of Hermitian matrices which are known from the calculus of variations (see [3], p. 77). In this way the authors prove the first three theorems of those which will be dealt with in this article (Theorems 1, 2 and 3). In [3], each theorem is proved separately.

In 1957, H. D. BLOCK in [2] proved the complex case of Theorem 1 using the properties of operators in the n-dimensional unitary space.

O. SHISHA published 1973 another proof of Theorem 1 (see [6]). He uses geometrical tools based on Fenchel's theorem for a spherical curve.

In our paper we prove the basic Theorem 1 using the real trigonometric polynomials (see [1], pp. 13–20). The method is analogous to that used by I. J. Schoenberg. As compared with the results achieved as far, we obtain also a sharpening of Theorem 1 (Theorem 5). We show that Theorems 2 and 3 follow immediately from Theorem 1. Theorem 4 is a discrete analogue of the integral inequality as proved in [4], p. 595. We derive its sharpening (Theorem 6). Theorems 4, 5, 6 are mentioned neither in [2] nor in [3], [5], [6].

First we give Theorems 1 through 6. Then we derive Theorems 2, 3 and 4 from the basic one – Theorem 1 – and Theorem 6 from Theorem 5. The proofs of Theorems 1 and 5 are given afterwards.

In the last part of the paper, a geometrical application – the proof of the isoperimetric inequality for some polygons – via Theorems 1 and 5 is given.

1. LIST OF THEOREMS

Theorem 1. *Let $x_1, \ldots, x_n$ be n real numbers such that*

$$\sum_{i=1}^{n} x_i = 0 . \tag{1.1}$$

Let us define $x_{n+1} = x_1$. Then

$$\sum_{i=1}^{n} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 . \tag{1.2}$$

The equality in (1.2) holds if and only if

$$x_i = A \cos \frac{2\pi i}{n} + B \sin \frac{2\pi i}{n}, \quad i = 1, \ldots, n, \quad A, B = \text{const.} \tag{1.3}$$

Theorem 2. *If $x_1, \ldots, x_n$ are n real numbers and $x_1 = 0$, then*

$$\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{i=2}^{n} x_i^2 . \tag{1.4}$$

The equality in (1.4) holds if and only if

$$x_i = A \sin \frac{(i-1)\pi}{2n-1}, \quad i = 1, \ldots, n, \quad A = \text{const.} \tag{1.5}$$

Theorem 3. *If $x_1, \ldots, x_n$ are n real numbers, then*

$$\sum_{i=0}^{n} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{2(n+1)} \sum_{i=0}^{n} x_i^2 , \tag{1.6}$$

where $x_0 = x_{n+1} = 0$. The equality in (1.6) holds if and only if

$$x_i = A \sin \frac{i\pi}{n+1}, \quad i = 1, \ldots, n, \quad A = \text{const.} \tag{1.7}$$

Theorem 4. *Let $x_1, \ldots, x_n$ be n real numbers satisfying (1.1). Then*

$$\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{2n} \sum_{i=1}^{n} x_i^2 . \tag{1.8}$$

The equality in (1.8) holds if and only if

$$x_i = A \cos \frac{(2i-1)\pi}{2n}, \quad i = 1, \ldots, n, \quad A = \text{const.} \tag{1.9}$$

Theorem 5 (sharpening of Theorem 1 for n even). *Let* $n = 2m$, $n \geqq 4$, let $x_1, \ldots, x_n$ *be n real numbers satisfying* (1.1). *Let us define* $x_{n+i} = x_i$, $i = 1, \ldots, m$. *Then*

$$(1.10)\qquad \sum_{i=1}^{n} (x_i - x_{i+1})^2 \geqq$$

$$\geqq \left(\sin^2 \frac{2\pi}{n} - \sin^2 \frac{\pi}{n}\right) \sum_{i=1}^{n} (x_i + x_{i+m})^2 + 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 .$$

The equality in (1.10) *holds if and only if*

$$(1.11)\qquad x_i = A \cos \frac{2\pi i}{n} + B \sin \frac{2\pi i}{n} + C \cos \frac{4\pi i}{n} + D \sin \frac{4\pi i}{n} ,$$

$$i = 1, \ldots, n , \quad A, B, C, D = \text{const} .$$

Remark. 1. For $n \geqq 4$ the inequality $\sin^2 (2\pi/n) - \sin^2 (\pi/n) > 0$ holds.

2. Choosing a number μ, $0 < \mu < \sin^2 (2\pi/n) - \sin^2 (\pi/n)$, we can derive in the same way as in the proof of (1.10) (see the proof of (3.9)) that the following inequality holds:

$$(1.10')\qquad \sum_{i=1}^{n} (x_i - x_{i+1})^2 \geqq \mu \sum_{i=1}^{n} (x_i + x_{i+m})^2 + 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 ,$$

where the numbers $x_1, \ldots, x_n$ satisfy the assumptions of Theorem 5. The equality in (1.10′) holds if and only if x_i satisfy (1.3).

Theorem 6 (sharpening of Theorem 4). *Let* $x_1, \ldots, x_n$ *be n real numbers satisfying* (1.1), $n \geqq 2$. *Then*

$$(1.12)\qquad \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \geqq$$

$$\geqq \left(\sin^2 \frac{\pi}{n} - \sin^2 \frac{\pi}{2n}\right) \sum_{i=1}^{n} (x_i + x_{n+1-i})^2 + 4 \sin^2 \frac{\pi}{2n} \sum_{i=1}^{n} x_i^2 .$$

The equality in (1.12) *holds if and only if*

$$(1.13)\qquad x_i = A \cos \frac{(2i-1)\pi}{2n} + B \cos \frac{(2i-1)\pi}{n} , \quad i = 1, \ldots, n ,$$

$$A, B = \text{const} .$$

2. APPLICATION OF THE BASIC THEOREMS

Now it will be shown how to derive Theorem 2 from Theorem 1. Let $y_1, \ldots, y_{2(2n-1)}$ be $2(2n - 1)$ real numbers defined as follows:

$$(2.1)\qquad y_k = \begin{cases} x_k\,, & k = 1, \ldots, n\,, \\ x_{2n-k+1}\,, & k = n+1, \ldots, 2n-1\,, \\ -x_{k+1-2n}\,, & k = 2n, \ldots, 3n-1\,, \\ -x_{4n-k}\,, & k = 3n, \ldots, 2(2n-1)\,. \end{cases}$$

Since $\sum_{i=1}^{2(2n-1)} y_k = 0$, we can, putting $y_{4n-1} = y_1$, apply the results of Theorem 1 to (2.1). As $x_1 = 0$, the following equalities hold:

$$\sum_{k=1}^{2(2n-1)} y_k^2 = 4\sum_{i=2}^{n} x_i^2\,, \quad \sum_{k=1}^{2(2n-1)} (y_k - y_{k+1})^2 = 4\sum_{i=1}^{n-1} (x_i - x_{i+1})^2\,.$$

Hence, (1.4) holds. The equality will hold for (1.5), since $y_1 = 0$ in the new computation.

In an analogous way Theorems 3 and 4 can be derived from Theorem 1. We shall show, only schematically, how to define the numbers $\{y_k\}$.
For Theorem 3:

$$(2.2)\qquad 0, x_1, x_2, \ldots, x_n\,,\ 0, -x_1, -x_2, \ldots, -x_n\,.$$

For Theorem 4:

$$(2.3)\qquad x_1, x_2, \ldots, x_n\,,\ x_n, \ldots, x_2, x_1\,.$$

Theorem 6, a sharpening of Theorem 4, can be derived from Theorem 5 via (2.3). Here, $n_1 = 2m$.

Remark. Theorem 2 can be derived from Theorem 3 when the numbers $\{y_k\}_{k=1}^{2(n-1)}$ are defined as follows (schematically written):

$$x_2, x_3, \ldots, x_n\,,\ x_n, \ldots, x_2\,,$$

$y_0 = y_{2n-1} = x_1 = 0.$

3. PROOFS OF THE BASIC THEOREMS

In [1], p. 13–20, W. Blaschke has defined trigonometric polynomials. Let $z_1, \ldots, z_n$ be n numbers. First we assume n odd, $n = 2m + 1$. In [1] it is shown that we can choose such numbers $\xi_0, \xi_1, \ldots, \xi_m, \xi_1^*, \ldots, \xi_m^*$ that the following equalities hold:

$$(3.1)\qquad z_p = \xi_0 + \sum_{k=1}^{m}\left(\xi_k \cos kp\frac{2\pi}{n} + \xi_k^* \sin kp\frac{2\pi}{n}\right), \quad p = 1, \ldots, n\,,$$

$$(3.2)\qquad \frac{1}{n}\sum_{p=1}^{n} z_p^2 = \xi_0^2 + \frac{1}{2}\sum_{k=1}^{m}(\xi_k^2 + \xi_k^{*2})\,,$$

$$(3.3)\qquad \frac{1}{n}\sum_{p=1}^{n}(z_p - z_{p+1})^2 = 2\sum_{k=1}^{m}(\xi_k^2 + \xi_k^{*2})\sin^2 k\frac{\pi}{n}\,.$$

Let now n be even, $n = 2m$. We can choose numbers $c_0, c_1, \ldots, c_m, c_1^*, \ldots, c_{m-1}^*$ in an analogous way, but now

(3.1′)

$$z_p = c_0 + \sum_{k=1}^{m-1} \left(c_k \cos kp \frac{2\pi}{n} + c_k^* \sin kp \frac{2\pi}{n} \right) + c_m \cos mp \frac{2\pi}{n}, \quad p = 1, \ldots, n.$$

Inserting

(3.4) $$\xi_0 = c_0, \quad \xi_k = c_k, \quad \xi_k^* = c_k^*, \quad k = 1, \ldots, m-1,$$
$$\xi_m = \sqrt{(2)}\, c_m, \quad \xi_m^* = 0,$$

the equalities (3.2) and (3.3) will hold, too.

It can be easily shown that if $\sum_{p=1}^{n} z_p = 0$, then

(3.5) $$\xi_0 = 0.$$

The proof of Theorem 1 is now very simple. Using (3.2), (3.3) and (3.5) for $x_1, \ldots, x_n$, we conclude that (1.2) will hold, provided

(3.6) $$\sin^2 \frac{k\pi}{n} \geqq \sin^2 \frac{\pi}{n}, \quad k = 1, \ldots, m,$$

is satisfied. (3.6) is true, since $0 \leqq k\pi/n \leqq \pi/2$, $k = 1, \ldots, m$. For $x \in \langle 0, \pi/2 \rangle$ the function $\sin x$ is growing. The equality in (1.2) holds if and only if $\xi_i = \xi_i^* = 0$, $i = 2, \ldots, m$, ξ_1, ξ_1^* are arbitrary, i.e. if and only if (1.3) is satisfied.

To prove Theorem 5 we shall use (3.2), (3.3) (with (3.4)) and (3.5).

The equality

(3.7) $$x_i + x_{i+m} = \sum_{k=1}^{m} \left\{ \xi_k \left[\cos ki \frac{2\pi}{n} + \cos k(i+m) \frac{2\pi}{n} \right] + \right.$$
$$\left. + \xi_k^* \left[\sin ki \frac{2\pi}{n} + \sin k(i+m) \frac{2\pi}{n} \right] \right\}$$

implies by virtue of (3.2) and (3.5) that

(3.8) $$\frac{1}{n} \sum_{i=1}^{n} (x_i + x_{i+m})^2 = \frac{1}{2} \sum_{k=1}^{m} (\xi_k^2 + \xi_k^{*2}) \left(1 + \cos km \frac{2\pi}{n} \right)^2 =$$
$$= \frac{1}{2} \sum_{k=1}^{m} (\xi_k^2 + \xi_k^{*2}) [1 + (-1)^k]^2.$$

(1.10) will hold if the inequality

$$\left(\sin^2 \frac{2\pi}{n} - \sin^2 \frac{\pi}{n} \right) \frac{n}{2} \sum_{k=1}^{m} (\xi_k^2 + \xi_k^{*2}) [1 + (-1)^k]^2 +$$

$$+ 2n \sin^2 \frac{\pi}{n} \sum_{k=1}^{m} (\xi_k^2 + \xi_k^{*2}) \leqq 2n \sum_{k=1}^{m} (\xi_k^2 + \xi_k^{*2}) \sin^2 k \frac{\pi}{n}$$

is fulfilled, i.e.

$$(3.9) \quad \sum_{k=1}^{m} (\xi_k^2 + \xi_k^{*2}) \left\{ \sin^2 k \frac{\pi}{n} - \sin^2 \frac{\pi}{n} - \frac{1}{4} \left(\sin^2 \frac{2\pi}{n} - \sin^2 \frac{\pi}{n} \right) [1 + (-1)^k]^2 \right\} \geqq 0 .$$

Let us denote

$$\varrho_k = \sin^2 k \frac{\pi}{n} - \sin^2 \frac{\pi}{n} - \frac{1}{4} \left(\sin^2 \frac{2\pi}{n} - \sin^2 \frac{\pi}{n} \right) [1 + (-1)^k]^2 .$$

In case of k odd,

$$\varrho_k = \sin^2 k \frac{\pi}{n} - \sin^2 \frac{\pi}{n} \geqq 0$$

(see (3.6)) with the equality holding only for $k = 1$. In case of k even,

$$\varrho_k = \sin^2 k \frac{\pi}{n} - \sin^2 \frac{2\pi}{n}$$

and (3.6) implies again $\varrho_k \geqq 0$. Here the equality holds only for $k = 2$. The inequality (1.10) with the equality condition (1.11) is proved.

Remark. The inequality (1.10′) follows immediately from the proof of (1.10) given above. The form of the numbers ϱ_k in this case is

$$\varrho_k = \sin^2 k \frac{\pi}{n} - \sin^2 \frac{\pi}{n} \geqq 0 \quad \text{for} \quad k \quad \text{odd} ,$$

$$\varrho_k = \sin^2 k \frac{\pi}{n} - \sin^2 \frac{\pi}{n} - \mu > 0 \quad \text{for} \quad k \quad \text{even} .$$

Now $\varrho_k > 0$ for $k > 1$, $\varrho_1 = 0$. The equality condition (1.3) for (1.10′) is an immediate consequence of this fact.

4. GEOMETRICAL APPLICATION

Let $P = A_1 \ldots A_n$ denote an equilateral closed n-gon in E_2 of area F and perimeter L. In [1], p. 13–20, the inequality

$$(4.1) \qquad L^2 \geqq 4n \operatorname{tg} \frac{\pi}{n} F$$

is proved on the basis of trigonometric polynomials. The equality in (4.1) holds if and only if P is a regular n-gon.

(4.1) can be derived from Theorem 1. Let us choose a cartesian coordinate system $S = \{O, x, y\}$ in E_2 with O being the centroid of P. Let $A_i = [x_i, y_i]$, $i = 1, \ldots, n$, in S. Let us denote $A_{n+1} = A_1$. Then the equalities

$$\sum_{i=1}^{n} x_i = 0\,, \quad \sum_{i=1}^{n} y_i = 0\,, \quad x_{n+1} = x_1\,, \quad y_{n+1} = y_1$$

hold and the assumptions of Theorem 1 for the numbers $\{x_i\}$, $\{y_i\}$ are fulfilled.

For P the following relations hold:

$$\frac{L^2}{n} = \sum_{i=1}^{n} [(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2]\,, \tag{4.2}$$

$$F = \left|\tfrac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - y_i x_{i+1})\right| = \left|\tfrac{1}{4} \sum_{i=1}^{n} [(x_i + x_{i+1})(y_i - y_{i+1}) + (y_i + y_{i+1})(x_{i+1} - x_i)]\right|\,. \tag{4.3}$$

Using (4.3) we can write

$$\begin{aligned} 8 \operatorname{tg} \frac{\pi}{n} F &= \\ &= 2 \operatorname{tg} \frac{\pi}{n} \sum_{i=1}^{n} [(x_i + x_{i+1})(\pm y_i \mp y_{i+1}) + (y_i + y_{i+1})(\pm x_{i+1} \mp x_i)] \leqq \\ &\leqq \sum_{i=1}^{n} (\pm y_i \mp y_{i+1})^2 + \operatorname{tg}^2 \frac{\pi}{n} \sum_{i=1}^{n} (x_i + x_{i+1})^2 + \\ &+ \sum_{i=1}^{n} (\pm x_{i+1} \mp x_i)^2 + \operatorname{tg}^2 \frac{\pi}{n} \sum_{i=1}^{n} (y_i + y_{i+1})^2 = \\ &= \sum_{i=1}^{n} [(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2] + \\ &+ \operatorname{tg}^2 \frac{\pi}{n} \sum_{i=1}^{n} \{[4x_i^2 - (x_i + x_{i+1})^2] + [4y_i^2 - (y_i - y_{i+1})^2]\} = \\ &= \sum_{i=1}^{n} \left\{\left(1 - \operatorname{tg}^2 \frac{\pi}{n}\right) [(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2]\right\} + \\ &+ 4 \operatorname{tg}^2 \frac{\pi}{n} \sum_{i=1}^{n} (x_i^2 + y_i^2)\,. \end{aligned} \tag{4.4}$$

Now, using (1.2), (4.4) and (4.2), we derive the following inequality:

$$\begin{aligned} 8 \operatorname{tg} \frac{\pi}{n} F &\leqq \\ &\leqq \sum_{i=1}^{n} \left(1 - \operatorname{tg}^2 \frac{\pi}{n} + \frac{1}{\cos^2 \frac{\pi}{n}}\right) [(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2] = 2 \frac{L^2}{n}\,. \end{aligned} \tag{4.5}$$

(4.5) is the inequality (4.1). The equality condition (the regularity of P) follows from (1.3) and the equality conditions in (4.4).

Using (1.10′), we can derive a sharpening of (4.1) for n even. Let $n = 2m$, $n \geqq 4$. In the coordinate system S let u_i be defined as follows:

$$u_i^2 = (x_i + x_{i+m})^2 + (y_i + y_{i+m})^2 ,$$

where $x_{n+i} = x_i$, $y_{n+i} = y_i$, $i = 1, \ldots, m$. Then using (1.10′) for

$$\mu = \frac{1}{4} \sin^2 \frac{\pi}{n}$$

and (4.4) we obtain the inequality

$$4n \operatorname{tg} \frac{\pi}{n} F + \frac{n}{8} \operatorname{tg}^2 \frac{\pi}{n} \sum_{i=1}^{n} u_i^2 \leqq L^2 \tag{4.6}$$

with the equality holding only for the case of a regular n-gon.

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