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HEAT SOURCES AND HEAT POTENTIALS

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We shall deal with potentials in R^{m+1} corresponding to the well-known kernel

$$\mathscr{E}(x, t) = \begin{cases} (4\pi t)^{-m/2} \exp\left(-\dfrac{|x|^2}{4t}\right), & x \in R^m, \quad t > 0, \\ 0, \quad x \in R^m, & t \leqq 0, \end{cases} \tag{1}$$

which represents a fundamental solution of the heat conduction operator

$$\frown = \frac{\partial}{\partial t} - \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$$

(cf. [1]). The term measure will always mean a finite positive Borel measure with a compact support in a Euclidean space. Let ν be a measure in R^m (describing a space distribution of heat sources) and let ϱ be a measure in R^1. Then the heat potential of $\mu = \nu \otimes \varrho$ defined by

$$\mathscr{E}\mu(x, t) = \int_{R^{m+1}} \mathscr{E}(x - \xi, t - \tau) \, d\mu(\xi, \tau) \tag{2}$$

may be interpreted as the temperature resulting at the time t and the point $x \in R^m$ under the action of time-variable heat sources which are so distributed that the quantity of heat emanating from a Borel set $M \subset R^m$ during the time interval $I \subset R^1$ is given by $\mu(M \times I) = \nu(M) \, \varrho(I)$. We shall adopt the following

Definition. Let $\alpha \geqq 0$ be a real number and suppose that ν is a measure in R^m. We shall say that ν is *α-admissible* if there is a non-trivial measure ϱ in R^1 such that the heat potential $u = \mathscr{E}\mu$ corresponding to $\mu = \nu \otimes \varrho$ satisfies the condition

$$u(x, t) - u(y, v) = o(|x - y|^\alpha + |t - v|^{\alpha/2}) \quad \text{as} \quad |x - y| + |t - v| \to 0+ . \tag{3}$$

Any ϱ with the above properties will be called an *α-admissible factor of ν.*

Let

$$\Omega(r, x) = \{\xi \in R^m;\ |\xi - x| < r\}$$

denote the open ball with center x and radius r. We are going to prove the following result characterizing all α-admissible measures in R^m for $\alpha \in \langle 0, 1)$.

Theorem. *If $\alpha \in \langle 0, 1)$, then a measure ν in R^m is α-admissible if and only if*

$$\sup_{x \in R^m} \int_0^\delta r^{1-m} \nu(\Omega(r, x))\, \mathrm{d}r = o(\delta^\alpha) \quad \text{as} \quad \delta \to 0+\ ; \tag{4}$$

for $\alpha \in (0, 1)$ the condition (4) may be replaced equivalently by (14).

Remark 1. Let ν be a non-trivial measure in R^m and denote by ε_{t_0} the Dirac measure (= unit point-mass) concentrated at a point t_0 in R^1. It is known that ε_{t_0} is never a 0-admissible factor of $\nu \not\equiv 0$ (compare [2]).

Remark 2. If $M \subset R^1$ and $\tau \in R^1$ we put

$$M - \tau = \{t - \tau;\ t \in M\}\,.$$

Given a measure ϱ in R^1 we may define the translated measure ϱ_τ by

$$\varrho_\tau(M) = \varrho(M - \tau)$$

on Borel sets $M \subset R^1$. Further we put for any $h > 0$

$$\varrho^h(.) = \frac{1}{h} \int_{-h}^0 \varrho_\tau(.)\, \mathrm{d}\tau\,.$$

The measure ϱ^h is absolutely continuous with respect to the Lebesgue measure λ in R^1 and the corresponding Radon-Nikodym derivative is given by the function

$$t \to \lim_{\varepsilon \to 0+} \varepsilon^{-1} \varrho^h(\langle t - \varepsilon, t))$$

which is everywhere defined and finite. Besides that, $\varrho^h(R^1) = \varrho(R^1)$. If ϱ is an α-admissible factor of ν, $\mu = \nu \otimes \varrho$ and $u = \mathscr{E}\mu$ is defined by (2), then Fubini's theorem yields

$$\mathscr{E}(\nu \otimes \varrho^h)(x, t) = \frac{1}{h} \int_0^h u(x, t + \tau)\, \mathrm{d}\tau\,.$$

Hence it follows that (3) is again satisfied with u replaced by $u^h = \mathscr{E}(\nu \otimes \varrho^h)$. In other words, ϱ^h is also an α-admissible factor of ν.

Proof of the theorem. Suppose first that ν is an α-admissible measure in R^m. Let ϱ be an α-admissible factor of ν. According to Remark 2 we may suppose that ϱ

is absolutely continuous (λ) and $\lim \varepsilon^{-1}\varrho(\langle t-\varepsilon, t\rangle)$ $(\varepsilon \to 0+)$ is everywhere defined and finite in R^1. Let us fix a $\tau \in R^1$ such that

$$\lim_{h\to 0+} \frac{\varrho(\langle \tau - h, \tau\rangle)}{h} = q > 0\,.$$

We have then for suitable $\delta > 0$ the implication

$$0 < h \leqq \delta \Rightarrow \tfrac{1}{2}qh \leqq \varrho(\langle \tau - h, \tau\rangle) \leqq 2qh\,. \tag{5}$$

Let $c > 0$ and consider the set

$$A(x, \tau, c) = \{[\xi, u] \in R^{m+1};\ \mathscr{E}(x-\xi, \tau-u) > c\} =$$
$$= \left\{[\xi, u] \in R^{m+1};\ u \in \left(\tau - \frac{1}{4\pi}\,c^{-2/m}, \tau\right),\ |x-\xi|^2 < r(u)\right\},$$

where

$$r(u) = 4(\tau - u)\log\left[c(4\pi(\tau-u))^{m/2}\right]^{-1}.$$

If $\xi \in R^m$ is fixed in such a way that

$$|x-\xi| = p\sqrt{\left(\frac{m}{2\pi e}\right)}\,c^{-1/m} \tag{6}$$

with $p \in \langle 0, 1)$, then

$$\{\xi\} \times \left\langle \tau - \frac{1}{4\pi e}\,c^{-2/m},\ \tau - \frac{p}{4\pi e}\,c^{-2/m}\right\rangle \subset A(x, \tau, c)\,. \tag{7}$$

This may be verified by a simple calculation; note that $A(x, \tau, c)$ is convex and

$$\frac{m}{2\pi e}\,c^{-2/m} = \max\left\{r(u);\ u \in \left(\tau - \frac{1}{4\pi}\,c^{-2/m}, \tau\right)\right\} = r\left(\tau - \frac{1}{4\pi e}\,c^{-2/m}\right).$$

According to (5) we obtain for c, p submitted to

$$\frac{1}{4\pi e}\,c^{-2/m} \leqq \delta\,, \quad p \in \langle 0, \tfrac{1}{8}\rangle \tag{8}$$

the estimate

$$\varrho\left(\left\langle \tau - \frac{1}{4\pi e}\,c^{-2/m},\ \tau - \frac{p}{4\pi e}\,c^{-2/m}\right\rangle\right) =$$
$$= \varrho\left(\left\langle \tau - \frac{1}{4\pi e}\,c^{-2/m}, \tau\right\rangle\right) - \varrho\left(\left\langle \tau - \frac{p}{4\pi e}\,c^{-2/m}, \tau\right\rangle\right) \geqq$$
$$\geqq \tfrac{1}{2}q\,\frac{1}{4\pi e}\,c^{-2/m} - 2q\,\frac{p}{4\pi e}\,c^{-2/m} = \frac{q}{4\pi e}\,(\tfrac{1}{2} - 2p)\,c^{-2/m} \geqq$$
$$\geqq \frac{q}{16\pi e}\,c^{-2/m}\,.$$

In view of (7), (6) we have the inclusion

$$\left\{[\xi, u];\ |\xi - x| \leqq \frac{1}{8}\sqrt{\left(\frac{m}{2\pi e}\right)}\, c^{-1/m},\ \tau - \frac{1}{4\pi e} c^{-2/m} \leqq u \leqq \right.$$
$$\left. \leqq \tau - \tfrac{1}{2}|x - \xi|\, c^{-1/m} \frac{1}{\sqrt{(2\pi e m)}}\right\} \subset A(x, \tau, c)$$

whence we get

$$(9) \qquad (\nu \otimes \varrho)\,(A(x, \tau, c)) \geqq \frac{q}{16\pi e} c^{-2/m}\, \nu\left(\Omega\left(\frac{1}{8}\sqrt{\left(\frac{m}{2\pi e}\right)}\, c^{-1/m}, x\right)\right).$$

Consider first the case $\alpha = 0$. If $\mu = \nu \otimes \varrho$ and

$$(10) \qquad \mathscr{E}\mu(x, t)\left(= \int_0^\infty \mu(A(x, t, c))\,\mathrm{d}c\right)$$

is a continuous function of the variables x, t, then

$$(11) \qquad \lim_{a\to\infty} \sup_{x,t} \int_a^\infty \mu(A(x, t, c))\,\mathrm{d}c = 0$$

(compare Proposition below). Employing (9) we obtain for

$$\frac{1}{4\pi e} a^{-2/m} \leqq \delta\,, \quad s = \frac{q}{16\pi e}\,, \quad z = \frac{1}{8}\sqrt{\left(\frac{m}{2\pi e}\right)}$$

the inequality

$$\int_a^\infty \mu(A(x, \tau, c))\,\mathrm{d}c \geqq s \int_a^\infty c^{-2/m} \nu(\Omega(zc^{-1/m}, x))\,\mathrm{d}c =$$
$$= smz^{m-2} \int_0^{za^{-1/m}} r^{1-m} \nu(\Omega(r, x))\,\mathrm{d}r$$

which combined with (11) yields (4) for $\alpha = 0$.

Conversely, suppose that (4) holds with $\alpha = 0$. Fix an arbitrary measure ϱ in R^1 satisfying for a suitable $K > 0$ the estimate

$$(12) \qquad \varrho(\langle \tau - \delta, \tau \rangle) \leqq K\delta \quad (\tau \in R^1,\ \delta > 0)$$

and put $\mu = \nu \otimes \varrho$. The inclusion

$$A(x, \tau, c) \subset \Omega\left(\sqrt{\left(\frac{m}{2\pi e}\right)}\, c^{-1/m}, x\right) \times \left(\tau - \frac{1}{4\pi} c^{-2/m}, \tau\right)$$

together with (12) gives

$$\mu(A(x, \tau, c)) \leqq \frac{K}{4\pi} c^{-2/m} \nu\left(\Omega\left(\sqrt{\left(\frac{m}{2\pi e}\right)}\, c^{-1/m}, x\right)\right),$$

whence (putting $\zeta = \sqrt{(m/2\pi e)}$)

$$\int_a^\infty \mu(A(x, \tau, c))\,\mathrm{d}c \leq \frac{K}{4\pi}\, m\zeta^{m-2} \int_0^{\zeta a^{-1/m}} r^{1-m} \nu(\Omega(r, x))\,\mathrm{d}r\,.$$

Using (4) with $\alpha = 0$ we arrive at

$$\limsup_{a\to\infty}\,\sup_{x,t} \int_a^\infty \mu(A(x, \tau, c))\,\mathrm{d}c = 0$$

which quarantees that the potential (10) is a uniformly continuous function of the variable $[x, t] \in R^{m+1}$ (compare Proposition below). Thus the theorem is proved for $\alpha = 0$.

Now consider the case $\alpha \in (0, 1)$. Let μ be a measure in R^{m+1} and denote by $u = \mathscr{E}\mu$ its heat potential. Then the equation

$$\frown u = \mu$$

holds in R^{m+1} in the sense of the distribution theory. Suppose now that for all $[x, t]$, $[y, t']$ in

$$\overline{\Omega(2r, \xi)} \times \langle \tau - (2r)^2, \tau + (2r)^2 \rangle$$

the estimate

$$|u(x, t) - u(y, t')| \leq Q(r)\,(|x - y|^\alpha + |t - t'|^{\alpha/2})$$

holds.

There is an infinitely differentiable function $\varphi(x, t)$ vanishing outside

$$\overline{\Omega(2r, \xi)} \times \langle \tau - (2r)^2, \tau + (2r)^2 \rangle$$

such that $\varphi = 1$ on $\overline{\Omega(r, \xi)} \times \langle \tau - r^2, \tau \rangle$, $0 \leq \varphi \leq 1$ and

$$\left|\frac{\partial \varphi}{\partial t}\right| + \sum_{i=1}^m \left|\frac{\partial^2 \varphi}{\partial x_i^2}\right| \leq 2(m + 1)\, r^{-2}\,.$$

Then

$$\mu(\overline{\Omega(r, \xi)} \times \langle \tau - r^2, \tau \rangle) \leq \int_{R^{m+1}} \varphi\,\mathrm{d}\mu =$$

$$= -\int_{R^{m+1}} \left(\frac{\partial \varphi(x, t)}{\partial t} + \sum_{i=1}^m \frac{\partial^2 \varphi(x, t)}{\partial x_i^2}\right) [u(x, t) - u(\xi, \tau)]\,\mathrm{d}x\,\mathrm{d}t\,.$$

Hence we conclude that

(13) $$\mu(\overline{\Omega(r, \xi)} \times \langle \tau - r^2, \tau \rangle) \leq k\, Q(r)\, r^{m+\alpha}$$

with an absolute constant k (independent of r, μ). Assuming $\mu = \nu \otimes \varrho$ with ϱ

absolutely continuous (λ) and having an everywhere defined finite density $\lim_{\varepsilon \to 0+} \varepsilon^{-1} \varrho(\langle t - \varepsilon, t \rangle)$, we may again choose $\tau \in R^1$ and $q, \delta > 0$ such that (5) holds. Combining (13) and (5) we get for $r^2 \leqq \delta$

$$\nu(\Omega(r, \xi)) \leqq 2k\, Q(r)\, q^{-1} r^{m-2+\alpha} .$$

If (3) holds, then $\lim_{r \to 0+} Q(r) = 0$ and we obtain

$$\sup_x \nu(\Omega(r, x)) = o(r^{m-2+\alpha}) \quad \text{as} \quad r \to 0+ . \tag{14}$$

Conversely, assume (14) and fix an arbitrary measure ϱ in R^1 satisfying (12). Then $\mu = \nu \otimes \varrho$ satisfies

$$\sup_{x,\tau} \mu(\Omega(r, x) \times \langle \tau - r^2, \tau \rangle) = o(r^{m+\alpha}) \quad \text{as} \quad r \to 0+ ,$$

which implies that $u = \mathscr{E}\mu$ fulfils (3) (compare Remark 5 and Lemma 4 in [3] and note that the derivatives of u have zero limits at infinity). To make the proof complete it remains to observe that (4) and (14) are equivalent for $\alpha \in (0, 1)$.

Remark 3. The assertion of the theorem (but not that of Remark 1) remains valid if o is replaced by O simultaneously in (4) and in the relation (3) occurring in the definition of α-admissibility (compare also [4]), provided $\alpha > 0$.

We shall now complete the detailed proof of the condition for continuity of the heat potential that has been useful in the course of the proof of the theorem.

Proposition. *The heat potential $\mathscr{E}\mu$ corresponding to a measure μ in R^{m+1} is finite and continuous on R^{m+1} if and only if*

$$\lim_{a \to \infty} \sup_{x,t} \int_a^\infty \mu(A(x, t, c))\, dc = 0 . \tag{15}$$

Proof. Put for $a \geqq 0$

$$\mathscr{E}_a = \min(a, \mathscr{E}) , \quad \mathscr{E}_a\mu(x, t) = \int_{R^{m+1}} \mathscr{E}_a(x - \xi, t - \tau)\, d\mu(\xi, \tau) .$$

For any $x_0 \in R^m$ and $t > t_0$ the estimate

$$\mathscr{E}\mu(x_0, t) \geqq [4\pi(t - t_0)]^{-1/m} \mu(\{[x_0, t_0]\}) \tag{16}$$

shows that $\mu(\{[x_0, t_0]\}) = 0$ whenever $\mathscr{E}\mu$ is locally bounded. Suppose now that $\mathscr{E}\mu$ is finite and continuous. Then $\mathscr{E}_a(x - \xi, t - \tau) \to \mathscr{E}_a(x_0 - \xi, t_0 - \tau)$ for μ-almost every $[\xi, \tau] \in R^{m+1}$ (i.e. for every $[\xi, \tau] \neq [x_0, t_0]$) as $[x, t] \to [x_0, t_0]$, so that $\mathscr{E}_a\mu$ is continuous on R^{m+1}. Since $\mathscr{E}_a\mu \nearrow \mathscr{E}\mu$ as $a \nearrow \infty$ we conclude from Dini's theorem (which may be applied to the Aleksandrov compactification of R^{m+1}, because all the functions in question tend to zero at infinity) that

$$\lim_{a \to \infty} \sup_{x,t} [\mathscr{E}\mu(x, t) - \mathscr{E}_a\mu(x, t)] = 0 . \tag{17}$$

Noting that, for fixed $[x, t] \in R^{m+1}$, $\mathscr{E}(x - \xi, t - \tau) - \mathscr{E}_a(x - \xi, t - \tau)$ vanishes outside $A(x, t, a)$ and equals $\mathscr{E}(x - \xi, t - \tau) - a$ for $[\xi, \tau] \in A(x, t, a)$ we get

$$\mathscr{E}\chi(x, t) - \mathscr{E}_a\mu(x, t) = \int_{A(x,t,a)} [\mathscr{E}(x - \xi, t - \tau) - a] \, \mathrm{d}\mu(\xi, \tau) =$$

$$= \int_0^\infty \mu(\{[\xi, \tau] \in A(x, t, a); \mathscr{E}(x - \xi, t - \tau) > a + c\}) \, \mathrm{d}c = \int_a^\infty \mu(A(x, t, c)) \, \mathrm{d}c .$$

The equality

$$\mathscr{E}\mu(x, t) - \mathscr{E}_a\mu(x, t) = \int_a^\infty \mu(A(x, t, c)) \, \mathrm{d}c \tag{18}$$

together with (17) yields (15). Conversely, assume (15). In view of (18), $\mathscr{E}_a\mu \nearrow \mathscr{E}\mu$ uniformly as $a \nearrow \infty$. Since the functions $\mathscr{E}_a\mu$ are bounded, the same holds of $\mathscr{E}\mu$ and (16) shows that μ does not charge points. As we have seen above, this implies the uniform continuity of $\mathscr{E}_a\mu$ and, consequently, of $\mathscr{E}\mu$ as well.

Remark 4. If ν is a measure in R^m and $m \geqq 2$, then we denote by

$$U \nu(x) = \int_{R^m} p(x - \xi) \, \mathrm{d}\nu(\xi)$$

its Newtonian (in the case $m > 2$) or logarithmic (in the case $m = 2$) potential corresponding to the kernel

$$p(x) = \begin{cases} |x|^{2-m} & \text{if } m > 2 , \\ \log \dfrac{1}{|x|} & \text{if } m = 2 . \end{cases}$$

If $\alpha \in \langle 0, 1)$, then ν satisfies (4) if and only if

$$U \nu(x) - U \nu(y) = o(|x - y|^\alpha) \quad \text{as} \quad |x - y| \to 0+ . \tag{19}$$

This assertion remains valid for $\alpha > 0$ if o is replaced by O in (19) and (4) simultaneously (compare [5]–[9]).

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