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# RANGES OF $a$-HOMOGENEOUS OPERATORS AND THEIR PERTURBATIONS

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**1. Introduction.** Let us consider the equation

$$Jx - \mu Sx^+ + \nu Sx^- + Gx = f$$

where $\mu$ and $\nu$ are real parameters. The properties of the maps $J, S, G, x \mapsto x^+$, $x \mapsto x^-$ will be specified in Section 2. This paper continues the investigation in [6] and offers a generalization of the results contained in [5], [3] and [4]. We also complete some results from [7], Appendix V. In the proofs of the assertions contained in this paper we use the theory of Leray-Schauder degree. The properties of the degree used here are taken from [7].

Section 2 is a summary of the main results contained in the paper [6]. In Section 3 we give some applications of the second part of this paper to the boundary value problems for differential equations, particularly for the nonlinear Sturm-Liouville equation of the second order and for a certain type of partial differential equations. Section 4 is devoted to the study of the nonlinear Sturm-Liouville equation of the second order with constant coefficients. We discuss the existence of weak solutions of the homogeneous boundary value problem in dependence on the parameters $\mu$ and $\nu$. In the case of nonexistence of the weak solution we give some sufficient conditions on the right hand side of the equation in order that the boundary value problem may have at least one weak solution. The methods of the proofs are based on the properties of the Leray-Schauder degree and on the methods of classical analysis.

**2. Ranges of positive $a$-homogeneous nonlinear operators-Summary.** Let $X$, $Y$ and $Z$ be Banach spaces with zero elements $O_X$, $O_Y$, $O_Z$ and with norms $\|x\|_X$, $\|y\|_Y$, $\|z\|_Z$, respectively. A subset $C$ of $Z$ is called *a cone* if it is closed, convex, invariant under multiplication by nonnegative real numbers, and if $C \cap (-C) = \{O_Z\}$. We suppose that a given fixed cone $C$ in $Z$ has the following properties:

(Z 1) If $z \in Z$ then there exists a uniquely determined couple $z^+, z^- \in C$ such that $z = z^+ - z^-$ and $z_1^+ - z^+ \in C$, $z_1^- - z^- \in C$ for each $z_1^+, z_1^- \in C$, $z = z_1^+ - z_1^-$. For each $t \geqq 0$ it is

$$(tz)^+ = tz^+ \quad \text{and} \quad (tz)^- = tz^- , \quad (-z)^+ = z^- .$$

(Z 2) The mapping $z \to z^+$ is continuous.

(Z 3) $X \subset Z$ and the identity mapping $X \to Z$ is continuous.

Let $a > 0$ be fixed and let $J$ be a mapping defined on $X$ with values in the space $Y$, and suppose that the following assumptions are fulfilled:

(J 1) $J$ is positively $a$-homogeneous.

(J 2) $J$ is one-to-one, $J$ is continuous in $O_X$ and $J^{-1}$ is continuous.

(J 3) $J$ is odd.

Let $S$ be an operator defined on $Z$, acting into $Y$ and satisfying

(S 1) $S$ is positively $a$-homogeneous.

(S 2) $S$ is continuous.

(S 3) The mappings $x \to Sx^+$, $x \to Sx^-$ are completely continuous operators from $X$ into $Y$.

Suppose that $G : X \to Y$ is a completely continuous operator. Denote $\mathscr{R}_{[\mu,\nu]}(J, S, G) = \{f \in Y; \exists x_0 \in X : Jx_0 - \mu Sx_0^+ + \nu Sx_0^- + Gx_0 = f\}$ and

$\boldsymbol{A}_{-1} = \{[\mu, \nu] \in \mathbf{R}^2; \exists x_0 \neq O_X : Jx_0 - \mu Sx_0^+ + \nu Sx_0^- = O_Y\}$,

$\boldsymbol{A}_0 = \mathbf{R}^2 \setminus \boldsymbol{A}_{-1}$,

$\boldsymbol{A}_1 = \{[\mu, \nu] \in \boldsymbol{A}_0; \mathrm{d}[\tilde{F}; K_Y(1), O_Y] \neq 0\}$,

where $\tilde{F} : y \to y - \mu S(J^{-1}y)^+ + \nu S(J^{-1}y)^-$, $y \in Y$,

$\boldsymbol{A}_2 = \{[\mu, \nu] \in \boldsymbol{A}_0; \mathscr{R}_{[\mu,\nu]}(J, S, O) \neq Y\}$,

$\boldsymbol{A}_3 = \{[\mu, \nu] \in \mathbf{R}^2; \mathscr{R}_{[\mu,\nu]}(J, S, O) = Y\}$.

Then the sets $\boldsymbol{A}_i$, $i = -1, 0, 1, 2, 3$ are symmetric subsets of $\mathbf{R}^2$ and the following assertions are valid:

(i) $\boldsymbol{A}_0$ is open in $\mathbf{R}^2$ and moreover, if $[\alpha, \beta] \in \mathbf{R}^2$, $|\alpha| + |\beta| < c_2(\mu, \nu)/s$, $[\mu, \nu] \in \boldsymbol{A}_0$, then $[\mu + \alpha, \nu + \beta] \in \boldsymbol{A}_0$ where

$$c_2(\mu, \nu) = \inf_{\|x\|_X = 1} \|Jx - \mu Sx^+ + \nu Sx^-\|_Y > 0$$

and

$$s = \max\{\sup_{\|x\|_X = 1} \|Sx^+\|_Y \sup_{\|x\|_X = 1} \|Sx^-\|_Y\} < +\infty .$$

(ii) For $[\mu, \nu] \in \boldsymbol{A}_0$ the set $\mathscr{R}_{[\mu,\nu]}(J, S, O)$ is closed in $Y$.

(iii) $\boldsymbol{A}_1 \subset \boldsymbol{A}_3$.

(iv) $\boldsymbol{A}_1$ is an open subset of $\mathbf{R}^2$.

(v) $\boldsymbol{A}_1$ is a union of some components of $\boldsymbol{A}_0$.

(vi) Let $\boldsymbol{T}$ be a component of $\boldsymbol{A}_0$ containing a point $[\lambda, \lambda]$ for some real number $\lambda$. Then $\boldsymbol{T} \subset \boldsymbol{A}_1$.

(vii) Let $[\mu, \nu] \in \boldsymbol{A}_1$ and suppose that

$$\lim_{\|x\|_X \to \infty} \sup \frac{\|Gx\|_Y}{\|x\|_X^a} < c_2(\mu, \nu).$$

Then $\mathscr{R}_{[\mu,\nu]}(J, S, G) = Y$.

(viii) For a given $[\mu, \nu] \in \boldsymbol{A}_2$ there exists $c_3(\mu, \nu) > 0$ such that if

$$\lim_{\|x\|_X \to \infty} \sup \frac{\|Gx\|_Y}{\|x\|_X^a} \leqq c_3(\mu, \nu)$$

then $\mathscr{R}_{[\mu,\nu]}(J, S, G) \neq Y$.

(ix) $\boldsymbol{A}_2$ is an open set in $\mathbf{R}^2$.

For the proofs of these assertions see [6].

**3. Applications to differential equations.** In this section we study the question of the existence of weak solutions of the boundary value problem for the nonlinear Sturm-Liouville equation of the second order and for partial differential equations of a certain type.

Let $L_p(\Omega)$, $C^k(\bar{\Omega})$ denote the usual function spaces on a bounded domain $\Omega$ in the real Euclidean $N$-space $\mathbf{R}^N$ (the boundary $\partial\Omega$ is sufficiently smooth if $N > 1$) with norms defined as usual, where $p \in \langle 1, \infty)$ is a real number and $k$ is a nonnegative integer. Let $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$, respectively, denote the Sobolev spaces (see e.g. [8], [9]) with the norms

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leqq k} \left( \int_\Omega |D^\alpha f(x)|^p \, dx \right)^{1/p}$$

and

$$\|f\|_{W_0^{k,p}(\Omega)} = \sum_{|\alpha| = k} \left( \int_\Omega |D^\alpha f(x)|^p \, dx \right)^{1/p},$$

respectively. It is possible to prove that the norms $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{W_0^{k,p}(\Omega)}$ are equivalent on the space $W_{0^{k,p}}(\Omega)$.

Let $V$ be a subspace of the Sobolev space $W^{1,p}(0, \pi)$ which fulfils one and only one from the following conditions:

(i) $V = W^{1,p}(0, \pi)$;

(ii) $V = \{u \in W^{1,p}(0, \pi);\ u(0) = 0\}$ or
$V = \{u \in W^{1,p}(0, \pi);\ u(\pi) = 0\}$;

(iii) $V = W_0^{1,p}(0, \pi)$.

In the case (i) the norm $\|\cdot\|_V$ is supposed to be equal to $\|\cdot\|_{W^{1,p}(0,\pi)}$, in the cases (ii) and (iii) the norm $\|\cdot\|_V$ is equal to $\|\cdot\|_{W_0^{1,p}(0,\pi)}$.

Let us suppose in the sequel $p \geqq 2$. Put $X = Z = V$, $Y = X^*$ and $C = \{u \in X; u(t) \geqq 0$ for all $t \in \langle 0, \pi \rangle\}$. Let $a, b, c$ be real functions defined on $\langle 0, \pi \rangle$. Suppose that $a(t) > 0$ for all $t \in \langle 0, \pi \rangle$ and $a \in C^1(\langle 0, \pi \rangle)$, $b(t) \geqq 0$, $c(t) > 0$ for all $t \in \langle 0, \pi \rangle$ and $b, c \in C(\langle 0, \pi \rangle)$. The real numbers $A_0, A_1, B_0, B_1$ are supposed to satisfy the inequalities

$$A_0 \geqq 0, \quad A_1 \geqq 0, \quad B_0 \geqq 0, \quad B_1 \geqq 0.$$

In the cases (i) and (ii), we assume moreover $b(t) \neq 0$ for all $t \in \langle 0, \pi \rangle$ or $A_0 + A_1 > 0$. Let $\lambda_1 c(t) - b(t) > 0$ for all $t \in \langle 0, \pi \rangle$, where $\lambda_1 > 0$ is the least eigenvalue of the problem

$$Ju - \lambda Su = O_Y$$

(the operators $J$ and $S$ will be defined by the relation (3.1) and (3.2) below). The fact $\lambda_1 > 0$ is proved in [7], Appendix V. In the case (i) suppose that

$$B_0 = B_1 = 0 \quad \text{and} \quad A_0 + A_1 > 0$$

or

$$\begin{cases} \lambda_1 B_0 - A_0 \geqq 0, & \lambda_1 B_1 - A_1 \geqq 0 \\ \lambda_1 B_0 - A_0 + \lambda_1 B_1 - A_1 > 0. \end{cases}$$

Denote

$$(3.1) \qquad (Ju, v)_X = \int_0^\pi [a(t) |u'(t)|^{p-2} u'(t) v'(t) + b(t) |u(t)|^{p-2} u(t) v(t)] \, dt + A_0 |u(0)|^{p-2} u(0) v(0) + A_1 |u(\pi)|^{p-2} u(\pi) v(\pi),$$

$$(3.2) \qquad (Su, v)_X = \int_0^\pi c(t) |u(t)|^{p-2} u(t) v(t) \, dt + B_0 |u(0)|^{p-2} u(0) v(0) + B_1 |u(\pi)|^{p-2} u(\pi) v(\pi),$$

$$(3.3) \qquad (F, v)_X = \int_0^\pi f(t) v(t) \, dt,$$

where the symbol $(.,.)_X$ is used for the duality between $X^*$ and $X$; $f \in L_1(0, \pi)$.

**3.1. Definition.** Let $f \in L_1(0, \pi)$ and let

$$(3.4) \qquad (Ju, v)_X - \mu(Su^+, v)_X + \nu(Su^-, v)_X = (F, v)_X$$

hold for each $v \in V$. Then $u$ is called *the weak solution* of the nonlinear Sturm-Liouville equation of the second order with the right hand side $f$.

**3.2. Lemma.** *The operators $J$ and $S$ satisfy the conditions* (J 1)–(J 3) *and* (S 1)–(S 3), *respectively, from Section* 2.

Proof. The continuity of Němyckij's operator acting from $L_p(0, \pi)$ into $L_q(0, \pi)$ ($q = p/(p - 1)$) and the continuity of the imbedding from $W^{1,p}(0, \pi)$ into $C(\langle 0, \pi\rangle)$ imply that the operator $J$ is continuous. The conditions (J 1) and (J 3) can be verified. There exists $c > 0$ such that

$$(3.5) \qquad (Ju - Jv,\ u - v)_X \geqq c\|u - v\|_X^p$$

holds for each $u, v \in X$ (because the inequality

$$(|x|^{p-2}\, x - |y|^{p-2}\, y)\,(x - y) \geqq \tilde{c}|x - y|^p$$

holds for any real numbers $x$ and $y$ with a suitable constant $\tilde{c} > 0$). From the theorem of Minty-Browder (see e.g. [1]) we conclude the surjectivity of $J$. The inequality (3.5) implies the injectivity of $J$ and the continuity of $J^{-1}$. The condition (J 2) is verified. The operator $S$ is a strongly continuous mapping of $X$ into $Y$ because the imbedding from $W^{1,p}(0, \pi)$ into $C(\langle 0, \pi\rangle)$ is strongly continuous. Thus the conditions on the operator $S$ can be verified.

Let us present some regularity properties of the weak solution.

**3.3. Theorem.** *Let $u$ be a weak solution of the boundary value problem* (3.4) *with $f \in L_1(0, \pi)$. Then $u \in C^1(\langle 0, \pi\rangle)$. Moreover, if $f \in C(\langle 0, \pi\rangle)$ then $a(t)\,|u'(t)|^{p-2}\,.$ $.\,u'(t) \in C^1(\langle 0, \pi\rangle)$.*

Proof. Using (3.1), (3.2), (3.3), (3.4) and integration by parts we obtain

$$(3.6) \qquad \int_0^\pi M(t)\, v'(t)\, \mathrm{d}t = 0\,,$$

where

$$M(t) = a(t)\,|u'(t)|^{p-2}\, u'(t) -$$
$$- \int_0^t \{b(\tau)\,|u(\tau)|^{p-2}\, u(\tau) - \mu\, c(\tau)\,|u^+(\tau)|^{p-2}\, u^+(\tau) +$$
$$+ \nu\, c(\tau)\,|u^-(\tau)|^{p-2}\, u^-(\tau) - f(\tau)\}\, \mathrm{d}\tau\,.$$

The function $M(t)$ is an element of $L_q(0, \pi)$ ($q = p/(p - 1)$) and the identity (3.6) holds for each $v \in \mathscr{D}(0, \pi)$ (where $\mathscr{D}(0, \pi)$ is the set of all infinitely differentiable functions with compact supports in $(0, \pi)$). It is

$$(3.7) \qquad \int_0^\pi \frac{\mathrm{d}M}{\mathrm{d}t}\, v(t)\, \mathrm{d}t = 0$$

for each $v \in \mathscr{D}(0, \pi)$, where $\mathrm{d}M/\mathrm{d}t$ denotes the derivative of $M(t)$ in the sense of distributions. The expression (3.7) implies $M(t) = \bar{c}$ almost everywhere in $\langle 0, \pi \rangle$, where $\bar{c}$ is a constant. Let us denote

$$F(t, z) = a(t)\,|z|^{p-2}\,z -$$
$$- \int_0^t \{b(\tau)\,|u(\tau)|^{p-2}\,u(\tau) - \mu\,c(\tau)\,|u^+(\tau)|^{p-2}\,u^+(\tau) +$$
$$+ \nu\,c(\tau)\,|u^-(\tau)|^{p-2}\,u^-(\tau) - f(\tau)\}\,\mathrm{d}\tau - \bar{c}\,, \quad t \in \langle 0, \pi \rangle\,, \quad z \in \mathbf{R}^1\,.$$

By the same argument as in the proof of Lemma 3.2 there exists a constant $c_1 > 0$ such that

$$(F(t, z_1) - F(t, z_2))\,(z_1 - z_2) \geqq c_1 |z_1 - z_2|^p$$

for each $t \in \langle 0, \pi \rangle$, $z_1, z_2 \in \mathbf{R}^1$. This inequality implies for each $t \in \langle 0, \pi \rangle$ the existence of $z(t)$ which is determined uniquely and

$$F(t, z(t)) = 0\,. \tag{3.8}$$

Moreover, the function $z(t)$ is continuous on $\langle 0, \pi \rangle$. However, from (3.8) we obtain $z(t) = u'(t)$ almost everywhere in $\langle 0, \pi \rangle$. The proof of the second part of this theorem is similar to the first one.

**3.4. Remark.** Let us remark that many other interesting properties can be proved for the weak solution of the boundary value problem (3.4). Let us mention for instance that if the function $u$ is a weak solution of the boundary value problem (3.4) and $f \equiv 0$ then $u$ and its derivative $u'$ have only a finite number of zeros in $\langle 0, \pi \rangle$. For the proof see [7], Appendix V.

**3.5. Theorem.** *Let* $[\mu, \nu] = [\lambda + \alpha,\ \lambda + \beta]$, *where* $|\alpha| + |\beta| < c_2(\lambda, \lambda)/s$ *(for* $c_2(\lambda, \lambda)$ *and* $s$ *see Section* 2*). Then the boundary value problem* (3.4) *has at least one weak solution* $u \in V$ *for an arbitrary right hand side* $f \in L_1(0, \pi)$. *If* $f \in C(\langle 0, \pi \rangle)$ *then the boundary value problem* (3.4) *has at least one classical solution in the sense of* 3.3.

**3.6. Theorem.** *Let* $[\mu, \nu] \in A_1$ *and let* $g(t, z)$ *be a real function defined on* $\langle 0, \pi \rangle \times \mathbf{R}^1$. *Let the function* $g(t, z)$ *satisfy Carathéodory's conditions and, moreover, let there exist a function* $r(t) \in L_q(0, \pi)$ *so that*

$$|g(t, z)| \leqq r(t) + c_2(\mu, \nu)\,|z|^{p-1}$$

*holds for each* $z \in \mathbf{R}^1$ *and for almost all* $t \in (0, \pi)$ $(q = p/(p-1))$.

*Then the boundary value problem*

$$(Ju, v)_X - \mu(Su^+, v)_X + \nu(Su^-, v)_X + (Gu, v)_X = (F, v)_X \tag{3.9}$$

*has at least one weak solution for an arbitrary right hand side* $f \in L_1(0, \pi)$. *If we* *suppose* $g(t, z) \in C(\langle 0, \pi\rangle \times \mathbf{R}^1)$ *and* $f \in C(\langle 0, \pi\rangle)$, *the boundary value problem* (3.9) *has at least one classical solution in the sense of* 3.3.

**3.7. Remark.** The last expression on the left hand side of (3.9) is defined as follows:

$$(Gu, v)_X = \int_0^\pi g(t, u(t))\, v(t)\, \mathrm{d}t\,, \quad u, v \in X\,.$$

To prove Theorems 3.5 and 3.6 means nothing else than to verify the assumptions from Section 2.

**3.8. Remark.** Similar theorems about the existence of weak solutions of the boundary value problem (3.4) or (3.9), may be formulated for the nonlinear Sturm-Liouville equation of the fourth order (for operators $J$ and $S$ see for instance [7], Appendix V).

Let $k$ be a positive integer, $\Omega \subset \mathbf{R}^N$ a bounded domain ($N \geqq 1$) with a lipschitzian boundary $\partial\Omega$ if $N > 1$. Let $a_{ij} \in L_1(\Omega)$, $a_{ij} = a_{ji}$ ($i$ and $j$ are multiindices). Suppose there exists a constant $\gamma > 0$ such that

$$\sum_{|i|=|j|=k} a_{ij}(t)\, \eta_i \eta_j \geqq \gamma \sum_{|i|=k} \eta_i^2 \tag{3.10}$$

for all $\eta_i \in \mathbf{R}^1$, $|i| = k$, and almost all $t \in \Omega$. Put $X = W_0^{k,2}(\Omega)$, $Y = X^*$, $Z = L_2(\Omega)$, $C = \{f \in L_2(\Omega);\ f(t) \geqq 0$ for almost all $t \in \Omega\}$. For $c \in L_\infty(\Omega)$ define the operators $J$ and $S$:

$$(Ju, v)_X = \sum_{|i|=|j|=k} \int_\Omega a_{ij}(t)\, D^i u(t)\, D^j v(t)\, \mathrm{d}t \tag{3.11}$$

and

$$(Sz, v)_X = \int_\Omega c(t)\, z(t)\, v(t)\, \mathrm{d}t\,, \tag{3.12}$$

for all $u \in X$, $v \in X^*$, $z \in Z$.

**3.9. Definition.** Let $f \in L_2(\Omega)$ and let $g(t, z)$ be acting from $\Omega \times \mathbf{R}^1$ into $\mathbf{R}^1$ and satisfy Carathèodory's conditions. Suppose there exists such a function $r(t) \in L_2(\Omega)$ and a constant $c_2 > 0$ that

$$|g(t, z)| < r(t) + c_2|z|$$

holds for each $z \in \mathbf{R}^1$ and almost all $t \in \Omega$. The function $u \in W_0^{k,2}(\Omega)$ is said to be the weak solution of the Dirichlet problem

$$\begin{gathered} \sum_{|i|=|j|=k} (-1)^j\, D^j(a_{ij}(t)\, D^i u(t)) - \mu\, c(t)\, u^+(t) + \nu\, c(t)\, u^-(t) + \\ + g(t, u(t)) = f(t)\,, \quad t \in \Omega\,, \\ u(t) = 0\,, \quad t \in \partial\Omega \end{gathered} \tag{3.13}$$

if the identity

$$(3.14)\qquad (Ju, v)_X - \mu(Su^+, v)_X + \nu(Su^-, v)_X + (Gu, v)_X = (F, v)_X$$

holds for all $v \in W_0^{k,2}(\Omega)$ (the operators $G$ and $F$ are the same as in the special case $\Omega = (0, \pi)$).

**3.10. Lemma.** *The operators $J$ and $S$ satisfy the conditions* (J 1)–(J 3) *and* (S 1)–(S 3), *respectively, from Section* 2.

For the proof we use the same arguments as in 3.2.

Denote by $\sigma(S(J^{-1}))$ the spectrum of the completely continuous operator $S(J^{-1})$.

**3.11. Theorem.** *Let the couple of parameters $\mu$ and $\nu$ satisfy $[\mu, \nu] = [\lambda + \alpha, \lambda + \beta]$, where $\alpha, \beta, \lambda$ are real numbers such that*

$$|\alpha| + |\beta| < \frac{\gamma}{s} \operatorname{dist}(\lambda, \sigma(S(J^{-1}))) .$$

*Then the Dirichlet problem* (3.13) (*with $g \equiv 0$*) *has at least one weak solution for every $f \in L_2(\Omega)$.*

Proof. The space $Y$ is a Hilbert space and that is why

$$c_2(\lambda, \lambda) = \inf_{\|u\|_X = 1} \|Ju - \lambda Su\|_Y = \inf_{\|u\|_X = 1} \|Ju - \lambda S(J^{-1}(Ju))\|_Y \geqq$$
$$\geqq \operatorname{dist}(\lambda, \sigma(S(J^{-1}))) \inf_{\|u\|_X = 1} \|Ju\|_Y \geqq \gamma \operatorname{dist}(\lambda, \sigma(S(J^{-1})))$$

(see e.g. [10]). Now it is sufficient to apply the assertions from Section 2.

**4. Nonlinear Sturm-Liouville equation of the second order with constant coefficients.** This section deals with the solvability of the homogeneous Dricihlet problem for the Sturm-Liouville equation of the second order with constant coefficients. The results from Sections 2 and 3 are used in the proofs of the assertions of this part and the the sets $A_i$, $i = -1, 0, 1, 2, 3$ are investigated.

We are concerned first with the initial value problem

$$(4.1)\qquad -(|u'(t)|^{p-2} u'(t))' - \mu|u^+(t)|^{p-2} u^+(t) + \nu|u^-(t)|^{p-2} u^-(t) = f(t),$$
$$u(t_0) = \alpha_1, \quad u'(t_0) = \alpha_2, \quad t \in \mathbf{R}^1$$

where $\alpha_1, \alpha_2, t_0$ are real numbers and $f \in L_{1,loc}(\mathbf{R}^1)$ (the space of locally Lebesgue integrable functions on a real line $\mathbf{R}^1$).

**4.1. Definition.** Let $u$ be a real function of the real variable, suppose $u'$ to be continuous and $|u'|^{p-2} u'$ absolutely continuous on each compact interval in $\mathbf{R}^1$. If the function $u$ fulfils the initial conditions in (4.1) and the equation (4.1) holds almost everywhere in $\mathbf{R}^1$ then $u$ is called *a solution of the initial value problem* (4.1).

**4.2. Remark.** If $f \in C(I)$ for an interval $I \subset \mathbf{R}^1$ then $|u'|^{p-2} u' \in C^1(I)$ and the equation (4.1) holds for each $t \in I$ (see 3.3).

**4.3. Remark.** Suppose that $\mu > 0$ and $\nu > 0$. It is possible to prove that the condition $f \in L_{1,loc}(\mathbf{R}^1)$ guarantees the existence of a solution of the initial value problem (4.1) and the solution is determined uniquely. The method of the proof of these assertions is similar to that used in the theory of ordinary differential equations of the type $y' = f(x, y)$ (see e.g. [2]).

Elementary properties of the equation

$$-(|u'|^{p-2} u')' - \mu|u^+|^{p-2} u^+ + \nu|u^-|^{p-2} u^- = k \tag{4.2}$$

where $k$ is a constant, yield the following assertions.

If the function $u$ satisfies (4.2) and the initial conditions

$$u(0) = 0\,, \quad u'(0) = \alpha_2 > 0$$

then

$$t_0 = \inf\{t > 0;\ u'(t) = 0\} \tag{4.3}$$

is a finite number and

$$u^+(t_0 + t) = u^+(t_0 - t)$$

for all $t \in \langle 0, t_0 \rangle$.

If $\alpha_2 < 0$, it is possible to prove that $t_0$ defined by (4.3) is a finite number and

$$u^-(t_0 + t) = u^-(t_0 - t)$$

holds for each $t \in \langle 0, t_0 \rangle$.

If the function $u$ is a solution of (4.2) with $k = 0$ and

$$u(0) = 0\,, \quad u'(0) = \alpha_2 \neq 0$$

then $u$ is a periodic function with the period $((\lambda_1/\mu)^{1/p} - (\lambda_1/\nu)^{1/p})\,\pi$ where $\lambda_1$ is the least eigenvalue of the boundary value problem

$$\begin{gathered} -(|u'|^{p-2} u')' - \lambda|u|^{p-2} u = 0\,, \\ u(0) = u(\pi) = 0\,. \end{gathered} \tag{4.4}$$

These assertions based only on the elementary properties of the equation (4.2) enable us to prove the following theorem.

**4.4. Theorem.** *All eigenvalues of the boundary value problem* (4.4) *form a sequence* $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$ *with*

$$\lim_{n\to\infty} \lambda_n = +\infty .$$

*To the least eigenvalue* $\lambda_1$ *there corresponds one and only one eigenfunction* $u$ (*we suppose that* $v'(0) = 1$ *for each eigenfunction* $v$) *Moreover,* $u(t) > 0$ *for all* $t \in (0, \pi)$. *If* $\lambda_n (n \geqq 2)$ *is an eigenvalue of* (4.4) *and* $v_n$ *is the corresponding eigenfunction then there exists* $t_0 \in (0, \pi)$ *such that* $v_n(t_0) = 0$. *To each* $\lambda_n$ *there corresponds one and only one eigenfunction* $v_n$.

Proof. Let $\lambda \in \mathbf{R}^1$ be an eigenvalue of (4.4) and let $v$ be the corresponding eigenfunction. Assume $v(t) > 0$ in some right reduced neighbourhood of zero $P_+(0)$. For $\lambda < 0$ we obtain from the equation (4.4) that

$$\{t \in (0, \pi\rangle;\ v(t) = 0\} = \emptyset .$$

For $\lambda = 0$ we obtain $v \equiv 0$ in $\langle 0, \pi\rangle$. This yields the inequality $\lambda > 0$ for each eigenvalue of the problem (4.4). Denote $\lambda_1 = \inf\{\lambda > 0; \lambda \text{ is an eigenvalue of (4.4)}\}$. Using Remark 4.3 it is possible to prove that the set of eigenvalues of the problem (4.4) is nonempty. Assume that $\lambda_1 > 0$. There exists a sequence of eigenvalues $\{\tau_m\}_{m=1}^{\infty}$, a sequence of the corresponding eigenfunctions $\{w_m\}_{m=1}^{\infty}$ and a sequence of real numbers $\{t_m\}_{m=1}^{\infty}$ so that $\lim_{m\to\infty} \tau_m = \lambda_1$ and $\|t_m w_m\|_X = 1$, $m = 1, 2, \ldots$ There exists a subsequence $\{t_{m_k} w_{m_k}\}_{k=1}^{\infty}$ and $w_0 \in X$ such that $t_{m_k} w_{m_k} \rightharpoonup^X w_0$ (i.e. $\{t_{m_k} w_{m_k}\}_{k=1}^{\infty}$ converges weakly to $w_0$ in the space $X$). The operator $S$ is strongly continuous and so $S(t_{m_k} w_{m_k}) \to^{X^*} S w_0$ and $\tau_{m_k} S(t_{m_k} w_{m_k}) \to^{X^*} \lambda_1 S w_0$. Thus we have $J(t_{m_k} w_{k_k}) \to^{X^*}$ $\to^{X^*} \lambda_1 S w_0$. In virtue of the continuity of the operator $J^{-1}$, it is $t_{m_k} w_{m_k} \to^X w_0$ and so

$$J w_0 - \lambda_1 S w_0 = 0 .$$

It is proved that $\lambda_1$ is an eigenvalue. For $\lambda_1 = 0$ it is $J(t_{m_k} w_{m_k}) \to^{X^*} O_X$ and so $t_{m_k} w_{m_k} \to^X O_X$ which is a contradiction with $\|t_{m_k} w_{m_k}\|_X = 1$. So we have $\lambda_1 > 0$.

Let $u_1$ be the eigenfunction corresponding to $\lambda_1$. Suppose there exists such $t \in$ $\in (0, \pi)$ that $u(t) = 0$. Choose $t_0 \in (0, \pi)$ so that $t_0 = \min\{t \in (0, \pi);\ u(t) = 0\}$ (this step is senseful according to 3.4). Define

$$\tilde{u}(t) = u\left(\frac{t_0}{\pi}\, t\right), \quad t \in \langle 0, \pi\rangle .$$

Then

$$-(|\tilde{u}'|^{p-2}\, \tilde{u}')' - \left(\frac{t_0}{\pi}\right)^p \lambda_1(|\tilde{u}|^{p-2}\, \tilde{u}) = 0 ,$$

$$\tilde{u}(0) = \tilde{u}(\pi) = 0$$

which is a contradiction with the fact that $\lambda_1$ is the least eigenvalue. We have proved that no eigenfunction corresponding to $\lambda_1$ changes its sign.

Directly from the equation (4.4) it is possible to prove that to each eigenvalue $\lambda$ there corresponds one and only one eigenfunction $v$. Denote $\lambda_k = k^p \lambda_1$, $k \geqq 2$, $k$ an integer. Define a function $v_k$ in this way:

$$v_k : t \mapsto \begin{cases} \frac{1}{k} u_{\lambda_1}(kt), & t \in \left\langle 2l \frac{\pi}{k}, (2l+1)\frac{\pi}{k} \right), \\ -\frac{1}{k} u_{\lambda_1}(kt), & t \in \langle 0, \pi \rangle \setminus \left\langle 2l \frac{\pi}{k}, (2l+1)\frac{\pi}{k} \right) \end{cases}$$

where $l = 1, 2, \ldots, \frac{1}{2}k$ if $k$ is even, $l = 1, 2, \ldots, [\frac{1}{2}k] + 1$ if $k$ is an odd number; the symbol $[t]$ denotes the integer part of the real number $t$ and $u_{\lambda_1}$ is an eigenfunction corresponding to the least eigenvalue $\lambda_1$. In this way we obtain eigenfunctions $v_k$ which correspond to the eigenvalues $\lambda_k$ for all $k \geqq 2$. On the other hand, if $v$ is an eigenfunction corresponding to $\lambda_k$ for some $k \geqq 2$ then according to 4.3 we have $v = v_k$ in $\langle 0, \pi \rangle$. Finally, if $\lambda \neq \lambda_1$ is an eigenvalue of (4.4) and $v$ is the corresponding eigenfunction then there exists $t \in (0, \pi)$ such that $v(t) = 0$. Put $t_0 = \inf \{t \in (0, \pi \rangle;\ v(t) = 0\}$. According to 4.3 it is

$$v(t) = \frac{t_0}{\pi} u\left(\frac{\pi}{t_0} t\right), \quad t \in \langle 0, t_0 \rangle .$$

Similarly, if $t_1 = \inf \{t \in (t_0, \pi \rangle;\ v(t) = 0\}$ then

$$v(t) = -\frac{t_1 - t_0}{\pi} u\left(\frac{\pi}{t_1 - t_0}(t - t_0)\right), \quad t \in \langle t_0, t_1 \rangle .$$

This fact implies the existence of $k \geqq 2$ such that $\lambda = k^p \lambda_1 = \lambda_k$.

Let us recall that in Section 3 we have defined the weak solution of the boundary value problem

$$(4.5) \qquad \begin{aligned} -(|u'|^{p-2} u')' - \mu |u^+|^{p+2} u^+ + \nu |u^-|^{p-2} u^- &= f, \\ u(0) = u(\pi) &= 0 . \end{aligned}$$

**4.5. Theorem.** *Boundary value problem* (4.5) *with $f \equiv 0$ has a nontrivial weak solution if and only if one of the following conditions holds:*

(i) $\mu = \lambda_1$, $\nu$ *arbitrary;*

(ii) $\mu$ *arbitrary,* $\nu = \lambda_1$;

(iii) $\mu > \lambda_1$, $\nu > \lambda_1$,

$$w_1(\mu, \nu) = \frac{(\mu)^{1/p} (\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p}) (\lambda_1)^{1/p}} \in \mathbf{N} ,$$

$$w_2(\mu, \nu) = \frac{((\mu)^{1/p} - (\lambda_1)^{1/p})(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} \in \mathbf{N},$$

$$w_3(\mu, \nu) = \frac{((\nu)^{1/p} - (\lambda_1)^{1/p})(\mu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} \in \mathbf{N},$$

*where* $\mathbf{N}$ *denotes the set of all positive integers.*

Proof. Let $u$ be a nontrivial weak solution of (4.5). Then $u \in C^1(\langle 0, \pi \rangle)$ according to 3.3 and according to 3.4 the function $u$ has only a finite number of zeros in $\langle 0, \pi \rangle$. If the function $u$ has no zero in $(0, \pi)$ then according to 4.3 we obtain (i) or (ii). In the opposite case it is possible to divide the interval $\langle 0, \pi \rangle$ into a finite number of subintervals so that on each of them it is either $u(t) \geqq 0$ or $u(t) \leqq 0$. In accordance with 4.3 it is

$$u(t) = K_1 u_{\lambda_1}\left(\left(\frac{\mu}{\lambda_1}\right)^{1/p} (t - \alpha)\right), \quad t \in \left(\alpha, \alpha + \left(\frac{\lambda_1}{\mu}\right)^{1/p} \pi\right)$$

if $u(t) > 0$ on $(\alpha, \alpha + (\lambda_1/\mu)^{1/p}\pi)$;

$$u(t) = -K_2 u_{\lambda_1}\left(\left(\frac{\nu}{\lambda_1}\right)^{1/p} (t - \beta)\right), \quad t \in \left(\beta, \beta + \left(\frac{\lambda_1}{\nu}\right)^{1/p} \pi\right)$$

if $u(t) < 0$ on $(\beta, \beta + (\lambda_1/\nu)^{1/p}\pi)$, where $K_1 > 0$, $K_2 > 0$ are suitable constants such that $u \in C^1(\langle 0, \pi \rangle)$ and $u_{\lambda_1}$ is an eigenfunction corresponding to the least eigenvalue $\lambda_1$. If $u \in C_0^1(\langle 0, \pi \rangle)$ (i.e. $u \in C^1(\langle 0, \pi \rangle)$ and $u(0) = 0$, $u(\pi) = 0$) then the condition (iii) is necessarily fulfilled. On the other hand, if one of the conditions (i), (ii) or (iii) is fulfilled then in the same way as in the first part of the proof it is possible to construct a nontrivial weak solution of (4.5) with $f \equiv 0$.

From Section 2 and from the previous theorem we obtain the existence result for weak solutions of (4.5). The reader is invited to see the figure in 4.10.

**4.6. Theorem.** *Let the parameters* $\mu$ *and* $\nu$ *fulfil one of the conditions*

(i) $\mu < \lambda_1$, $\nu < \lambda_1$;

(ii) $\mu > \lambda_1$, $\nu > \lambda_1$,

$$\frac{((\mu)^{1/p} - (\lambda_1)^{1/p})(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} < 1, \quad \frac{((\nu)^{1/p} - (\lambda_1)^{1/p})(\mu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} < 1;$$

$$k - 1 < \frac{((\mu)^{1/p} - (\lambda_1)^{1/p})(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} < k, \quad k - 1 < \frac{((\nu)^{1/p} - (\lambda_1)^{1/p})(\mu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} < k;$$

$k \in \mathbf{N}$, $k \geqq 2$. *Then the boundary value problem* (4.5) *has at least one weak solution for an arbitrary right hand side* $f \in L_1(0, \pi)$.

**4.7. Theorem.** *Let the assumptions of* (4.6) *be fulfilled. Moreover, let* $g : \langle 0, \pi\rangle \times \mathbf{R}^1 \to \mathbf{R}^1$ *satisfy the conditions from* 3.6. *Then the boundary value problem*

$$-(|u'(t)|^{p-2}\, u'(t))' - \mu|u^+(t)|^{p-2}\, u^+(t) + \nu|u^-(t)|^{p-2}\, u^-(t) + g(t, u(t)) = f(t),$$
$$u(0) = u(\pi) = 0$$

*has at least one weak solution.*

Denote by $\Phi_{+1}^{\{\mu,\nu\}}$ and $\Phi_{-1}^{\{\mu,\nu\}}$ (if there is no danger of misunderstanding, we write $\Phi_{+1}$ and $\Phi_{-1}$ only) the solutions of the initial value problem

$$(4.6)\qquad -(|u'(t)|^{p-2}\, u'(t))' - \mu|u^+(t)|^{p-2}\, u^+(t) + \nu|u^-(t)|^{p-2}\, u^-(t) = 0,$$
$$u(0) = 0,\quad u'(0) = 1 \quad\text{and}\quad u(0) = 0,\quad u'(0) = -1,$$

respectively. Using the elementary properties of the solution of (4.6) we can prove that

$$\{[\mu, \nu] \in \mathbf{R}^2;\ \mu \leqq 0,\ \nu > \lambda_1\} \cup \{[\mu, \nu] \in \mathbf{R};\ \mu > \lambda_1,\ \nu \leqq 0\} \subset \mathbf{A}_2 .$$

Theorem 4.6 implies that $[\mu, \nu]$ is an element of a component of $\mathbf{A}_0$ which does not contain the point $[\lambda, \lambda]$ for any $\lambda \in \mathbf{R}^1$ if and only if

$$(4.7)\qquad \Phi_{+1}^{\{\mu,\nu\}}(\pi) \,.\, \Phi_{-1}^{\{\mu,\nu\}}(\pi) > 0 .$$

In the sequel we shall prove that in the case (4.7) there exists no weak solution of (4.5) for a certain right hand side $f \in L_1(0, \pi)$.

**4.8. Lemma.** *Suppose there exists such a* $t_0 \in (0, \pi)$ *that*

$$\Phi_{\pm 1}(t) > 0,\quad \Phi'_{\pm 1}(t) < 0 \quad\text{for all}\quad t \in \langle t_0, \pi\rangle .$$

*Then there exists a right hand side* $f \in L_1(0, \pi)$ *such that the boundary value problem* (4.5) *has no weak solution.*

Proof. Let $f : \mathbf{R}^1 \to \mathbf{R}^1$ be such a function that $f \in L_1(\mathbf{R}^1)$, $f(t) = 0$ for all $t \in (-\infty, t_0\rangle \cup (\pi, +\infty)$ and $f(t) < 0$ for $t \in (t_0, \pi\rangle$. We have $f \in L_1(0, \pi)$. Let $\Phi_\alpha$ be the weak solution of the boundary value problem (4.5) with the right hand side $f$ and suppose $\Phi_\alpha(0) = 0$, $\Phi_\alpha(0) = \alpha$. For $\alpha > 0$ according to 4.3 it is

$$\Phi_\alpha(t) = \alpha\, \Phi_{+1}(t),\quad t \in \langle 0, t_0\rangle .$$

Put $t_1 = \inf\{t \in (t_0, \pi\rangle;\ \Phi_\alpha(t) = 0\}$. The interval $(t_0, t_1)$ contains a point $\tau_1$ with the property

$$(4.8)\qquad \left(\frac{\Phi_\alpha}{\Phi_{+1}}\right)'(\tau_1) < 0 .$$

In the opposite case

$$\frac{\Phi_\alpha(\tau)}{\Phi_{+1}(\tau)} \geqq \frac{\Phi_\alpha(t_0)}{\Phi_{+1}(t_0)} = \alpha > 0,\quad \tau \in (t_0, t_1)$$

which is impossible. From (4.8) we obtain

$$(4.9)\qquad (\Phi'_\alpha \Phi_{+1} - \Phi_\alpha \Phi'_{+1})(\tau_1) < 0$$

which is the same as $F(\tau_1) < 0$, where

$$F : \tau \mapsto (|\Phi'_\alpha|^{p-2}\, \Phi'_\alpha (\Phi_{+1})^{p-1} - (\Phi_\alpha)^{p-1}\, |\Phi'_{+1}|^{p-2}\, \Phi'_{+1})(\tau),$$

for the function $z \to |z|^{p-2}\, z$ is increasing on $\mathbf{R}^1$. It is possible to prove the existence of a set $\mathscr{A} \subset (t_0, \tau_1)$ with *meas* $\mathscr{A} > 0$ such that the following conditions are fulfilled:

(i) $\Phi'_\alpha(t) < 0$,
(ii) $F(t) < 0$,
(iii) $F'(t) < 0$
for all $t \in \mathscr{A}$.

Really, if $\Phi'_\alpha(t) < 0$ for all $t \in (t_0, \tau_1)$ then (ii) and (iii) are fulfilled because $F(t_0) = 0$, $F(\tau_1) < 0$ and $F$ is absolutely continuous on $(t_0, \tau_1)$. In the opposite case denote $\tau_2 = \sup\{\tau < \tau_1;\ \Phi'_\alpha(\tau) = 0\}$. It is $\tau_2 < \tau_1$ (see 3.4) and according to (4.9) we have $\Phi'_\alpha(\tau_1) < 0$. We conclude $\Phi'_\alpha(t) < 0$, $t \in (\tau_2, \tau_1)$. Since $F(\tau_2) > 0$, $F(\tau_1) < 0$, the conditions (i)–(iii) aree fulfilled. We have

$$(4.10)\qquad F'(t) = F_1(t) + F_2(t) < 0 \quad \text{for all} \quad t \equiv \mathscr{A},$$

where

$$F_1(t) = [(|\Phi'_\alpha|^{p-2}\, \Phi'_\alpha)' (\Phi_{+1})^{p-1} - (\Phi_\alpha)^{p-1} (|\Phi'_{+1}|^{p-2}\, \Phi'_{+1})'](t),$$
$$F_2(t) = [(|\Phi'_\alpha|^{p-2}\, \Phi'_\alpha)(\Phi_{+1}^{p-1})' - (\Phi_\alpha^{p-1})' (|\Phi'_{+1}|^{p-2}\, \Phi'_{+1})](t) =$$
$$= (p-1)\, \Phi'_\alpha \Phi'_{+1} [|\Phi'_\alpha|^{p-2}\, \Phi_{+1}^{p-2} - \Phi_\alpha^{p-2} |\Phi'_{+1}|^{p-2}](t).$$

The condition (ii) implies $(|\Phi'_\alpha|^{p-2}\, \Phi_{+1}^{p-2} - \Phi_\alpha^{p-2} |\Phi'_{+1}|^{p-2})(t) > 0$ for all $t \in \mathscr{A}$, So we have

$$(4.11)\qquad F_2(t) > 0, \quad t \in \mathscr{A}.$$

From the relations (4.10), (4.11) we conclude

$$(4.12)\qquad F_1(t) < 0, \quad t \in \mathscr{A}.$$

On the other hand, the equation (4.6) implies

$$F_1(t) = -f(t) \cdot (\Phi_{+1})^{p-1}(t) > 0, \quad t \in \mathscr{T},$$

where $\mathscr{T} \subset \mathscr{A}$, *meas* $\mathscr{T} > 0$. This fact contradicts (4.12). For $\alpha = 0$ we have $\Phi_0(t) = 0$ for all $t \in \langle 0, t_0 \rangle$. Denoting

$$t_1 = \inf\{t \in (t_0, \pi);\ \Phi_0(t) = 0\},$$

we obtain the existence of $z_0 \in (t_0, t_1)$ such that $\Phi_0'(z_0) = 0$. Suppose that $z_0$ is chosen as follows:

$$z_0 = \sup \{z \in (t_0, t_1);\ \Phi_0'(z) = 0\} .$$

There exists a point $\tau_1 \in (z_0, t_1)$ such that the conditions (i)–(iii) are fulfilled but we write $\Phi_0$ instead of $\Phi_\alpha$, $\alpha > 0$. The rest of the proof is similar to that for $\alpha > 0$. For $\alpha < 0$ it is

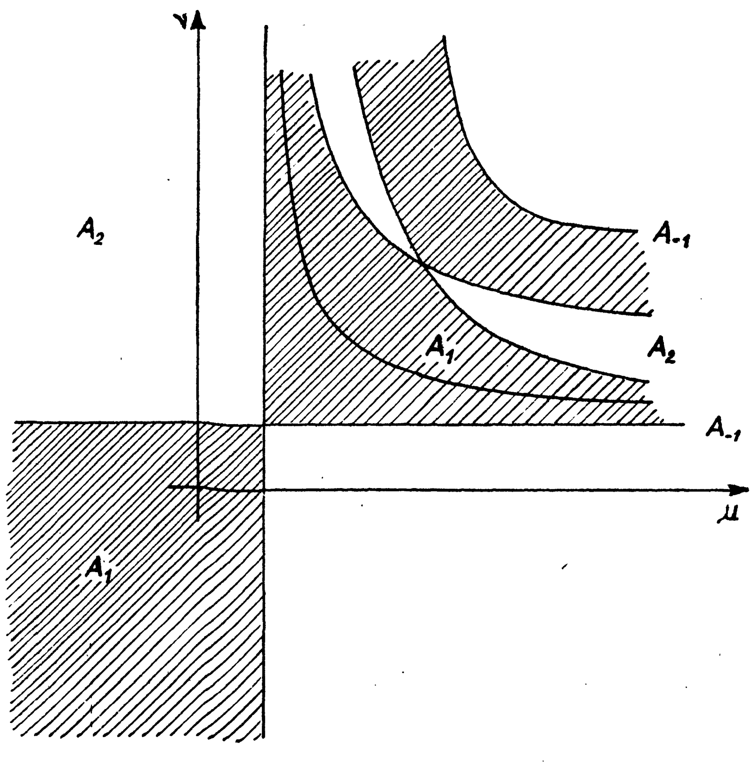
$$\Phi_\alpha(t) = |\alpha|\ \Phi_{-1}(t)\,, \quad t \in \langle 0, t_0 \rangle$$

and the proof is quite analogous to that for $\alpha > 0$. It means that for the right hand side $f$ defined above there exists no weak solution of the boundary value problem (4.5).

The other cases can be proved by modifying the proof of Lemma 4.8. Thus we obtain the following theorem.

**4.9. Theorem.** *If the condition (4.7) is fulfilled then there exists a right hand side $f \in L_1(0, \pi)$ such that the boundary value problem (4.5) has no weak solution.*

**4.10. Remark.** Theorems 4.5, 4.6 and 4.9 give us the classification of parameters $[\mu, \nu]$ in the sense of Section 2.



**4.11. Remark.** The proofs of the previous assertions imply the existence of a right hand side $f \in L_\infty(0, \pi)$ (i.e. the space of almost everywhere bounded functions) such that the boundary value problem (4.5) has no weak solution. The abstract part of this paper implies the existence of a function $f \in C^\infty(\langle 0, \pi \rangle)$ the support of which is situated "near the point $\pi$", with the same property.

It is possible to state some sufficient conditions on the right hand side $f$ in order that the boundary value problem (4.5) may have a weak solution. Consider the initial value problem

$$-(|u'|^{p-2} u')' - \mu|u^+|^{p-2} u^+ + \nu|u^-|^{p-2} u^- = \varepsilon f \tag{4.13}$$
$$u(0) = 0, \quad u'(0) = 1,$$

where $f \in L_1(\mathbf{R}^1)$. Let $u_\varepsilon$ be a solution of this initial value problem. For $\mu > 0$, $\nu > 0$, the function $u$ is determined uniquely.

**4.12. Theorem.** *Let $\varepsilon_0$ be a real number. Then*

$$\lim_{\varepsilon \to \varepsilon_0} \|u_\varepsilon - u_{\varepsilon_0}\|_{C(\langle 0,\pi \rangle)} = 0. \tag{4.14}$$

Proof. Fix $\delta > 0$ so that $0 \notin P_\delta(\varepsilon_0)$ and consider $\varepsilon \in P_\delta(\varepsilon_0)$ only. Denote

$$Q(z, t, \varepsilon) = -\mu|z^+|^{p-2} z^+ + \nu|z^-|^{p-2} z^- - \varepsilon f(t), \quad z, t \in \mathbf{R}^1;$$
$$q(\tau) = |\tau|^{p-2} \tau, \quad \tau \in \mathbf{R}^1.$$

It is possible to rewrite (4.13) into an equivalent form

$$[u'(t), v'(t)] = [q^{-1}(v(t)), Q(u(t), t, \varepsilon)] \tag{4.13$'$}$$
$$[u(0), v(0)] = [0, 1].$$

It is possible to show that the vector function $[q^{-1}(z_2), Q(z_1, t, \varepsilon)]$ satisfies the assumptions stated in [2], Theorem 4.2, Chapter 2. This fact implies (4.14).

The idea of the sufficient conditions upon the right hand sides $f$ is based on the following theorem.

**4.13. Theorem.** *Let $[\mu, \nu] \in A_2$, i.e.*

$$\Phi_{+1}(\pi) > 0, \quad \Phi_{-1}(\pi) > 0 \quad \text{or} \quad \Phi_{+1}(\pi) < 0, \quad \Phi_{-1}(\pi) < 0.$$

*If there exists a solution $u_\alpha$ of the initial value problem*

$$-(|u'|^{p-2} u')' - \mu|u^+|^{p-2} u^+ + \nu|u^-|^{p-2} u^- = f,$$
$$u(0) = 0, \quad u'(0) = \alpha$$

*such that $u_\alpha(\pi) \leqq 0$ or $u_\alpha(\pi) \geqq 0$, respectively, then for the right hand side $f \in L_1(0, \pi)$ in question there exists a weak solution of the boundary value problem (4.5).*

Proof of this theorem is based on Theorem 4.12.

Let us mention for the illustration that if $\Phi_{+1}(\pi) < 0$ and $\Phi_{-1}(\pi) < 0$ and if the function $f \in L_1(\mathbf{R}^1)$ is such that $f(t) = 0$ for $t \in (-\infty, t_0\rangle$, $f(t) < 0$ for $t \in (t_0, \pi)$ and $f(t) = 0$ for $t \in \langle \pi, +\infty)$, where $t_0$ is an arbitrary point of the interval $(\pi - \frac{1}{2}\pi(\lambda_1/\mu)^{1/p}, \pi)$, then there exists a weak solution of the boundary value problem (4.5). To prove this assertion it is sufficient to apply Theorem 4.13 and a slightly modified proof of Lemma 4.8.

**4.14. Remark.** It is interesting to see that if

$$\Phi_{+1}^{\{\mu;\nu\}}(\pi) \,.\, \Phi_{-1}^{\{\mu;\nu\}}(\pi) < 0$$

then applying Theorem 4.12 we can prove that there exists a weak solution of the boundary value problem (4.5) for any admissible right hand side. However, the same result was proved using the abstract part of this paper in Theorem 4.6, based on the Leray-Schauder degree.

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