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THE LAPLACE TRANSFORM OF EXPONENTIALLY BOUNDED VECTOR-VALUED FUNCTIONS (REAL CONDITIONS)

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The purpose of this paper is to find characteristic properties of the Laplace transforms of exponentially bounded vector-valued functions in arbitrary, i.e. in particular nonreflexive, Banach spaces (the "representability" problem). The known solution of this problem in reflexive Banach spaces is a special case (Corollary 12) of our main Theorem 10 which gives a complete answer to the representability problem formulated above. This theorem is preceded by some auxiliary facts from functional analysis, partly reminded, partly proved. In section 14 related results are commented.

1. We shall use the following notations: (1) $\mathbb{R}$ – the real number field, (2) (ω, ∞) – the set of all real numbers greater than ω if $\omega \in \mathbb{R}$, (3) $M_1 \to M_2$ – the set of all mappings of the whole set M_1 into the set M_2.

2. In the whole paper, E will denote a Banach space over $\mathbb{R}$. The set of all continuous linear functionals on E is denoted E^*. The basic notions from functional analysis necessary in the sequel can be found in [2], chap. 1 and 2. The notions of measurability and integrability of vector-valued functions and their properties are used in the scope of section 3.1–3.7 of [2].

3. Proposition. *Let $f \in (0, \infty) \to E$. Then*

(α) *the function lf is measurable for every $l \in E^*$,*

(β) *there is a null set $N \subseteq (0, \infty)$ and a separable subset $X \subseteq E$ such that $f(t) \in X$ for every $t \in (0, \infty) \setminus N$,*

if and only if the function f is measurable.

Proof. See Theorem 3.5.3 in [2].

4. Proposition. *Let $f \in (0, \infty) \to E$. If the function f is measurable, then*

$$\frac{1}{h}\int_0^h \|f(t + \tau) + f(t - \tau) - 2f(t)\| \, d\tau \to_{h\to 0+} 0$$

for almost every $t > 0$.

Proof. It is possible to follow almost word by word the argumentation of [3], Theorem 18.5.

5. Proposition. *Let $X \subseteq E$. If the set X is separable, then there exists a countable subset $L \in E^*$ such that for every $x_1, x_2 \in X$, $x_1 \neq x_2$, there is an $l \in L$ for which $l(x_1) \neq l(x_2)$.*

Proof. Since X is separable we can find a subset $X_0 \subseteq E$ such that

(1) X_0 is countable,

(2) $X_0 \subseteq X$, $\bar{X}_0 \supseteq X$.

According to Hahn-Banach theorem we can fix for every $z_1, z_2 \in X_0$, $z_1 \neq z_2$, a linear functional $l_{z_1,z_2} \in E$ so that

(3) $\|l_{z_1,z_2}\| = 1$, $l_{z_1,z_2}(z_1 - z_2) = \|z_1 - z_2\|$ for every $z_1, z_2 \in X_0$, $z_1 \neq z_2$.

Let us now consider some $x_1, x_2 \in X$, $x_1 \neq x_2$. According to (2) we can choose $z_1, z_2 \in X_0$ so that

(4) $\|z_1 - x_1\| \leqq \frac{1}{8}\|x_1 - x_2\|$, $\|z_2 - x_2\| \leqq \frac{1}{8}\|x_1 - x_2\|$.

It follows from (4) that

(5) $z_1 \neq z_2$,

(6) $\|(z_1 - x_1) - (z_2 - x_2)\| \leqq \frac{1}{4}\|x_1 - x_2\|$.

Further we get from (6) that

(7) $\|z_1 - z_2\| = \|(z_1 - x_1) - (z_2 - x_2) + (x_1 - x_2)\| \geqq$
$\geqq \|x_1 - x_2\| - \|(z_1 - x_1) - (z_2 - x_2)\| \geqq \|x_1 - x_2\| - \frac{1}{4}\|x_1 - x_2\| =$
$= \frac{3}{4}\|x_1 - x_2\|$.

Now it would suffice to prove that $l_{z_1,z_2}(x_1 - x_2) \neq 0$. Assume the contrary, i.e.

(8) $l_{z_1,z_2}(x_1 - x_2) = 0$.

Then it follows from (3), (5) and (7) that

(9) $\frac{3}{4}\|x_1 - x_2\| \leqq \|z_1 - z_2\| = l_{z_1,z_2}(x_1 - x_2) =$

$= l_{z_1,z_2}((z_1 - x_1) - (z_2 - x_2) + (x_1 - x_2)) =$

$= l_{z_1,z_2}((z_1 - x_1) - (z_2 - x_2)) \leqq \|(z_1 - x_1) - (z_2 - x_2)\| \leqq \frac{1}{4}\|x_1 - x_2\|.$

Since $x_1 \neq x_2$ the inequality (9) is clearly contradictory and hence (8) cannot be true, i.e. $l_{z_1,z_2}(x_1 - x_2) \neq 0$. We conclude

(10) for every $x_1, x_2 \in X$, $x_1 \neq x_2$ there exist $z_1, z_2 \in X_0$, $z_1 \neq z_2$ so that $l_{z_1,z_2}(x_1) \neq l_{z_1,z_2}(x_2)$.

Let us now denote $L = \{l_{z_1,z_2} : z_1, z_2 \in X_0,\ z_1 \neq z_2\}$. It follows from (1) that L is a countable set and (10) can be written in the form of the assertion of our Proposition.

6. Lemma. *Let x_k, $k \in \{1, 2, \ldots\}$, be a sequence in E. If*

(α) *the set $\{x_k : k \in \{1, 2, \ldots\}\}$ is relatively weakly compact in E,*

(β) *the sequence $l(x_k)$, $k \in \{1, 2, \ldots\}$, is convergent for every $l \in E^*$,*

then the sequence x_k is weakly convergent in E.

Proof. Let us denote $A_k = \{x_j : j \in \{1, 2, \ldots\},\ j \geqq k\}$ for every $k \in \{1, 2, \ldots\}$. Further let $\bar{A}_k$ be weak closures of A_k, $k \in \{1, 2, \ldots\}$.

By assumption (α), $\bigcap_{k=1}^{\infty} \bar{A}_k$ is non-empty, i.e. we can fix an $x_0 \in E$ so that

(1) $x_0 \in \bar{A}_k$ for every $k \in \{1, 2, \ldots\}$.

On the other hand, let $\varepsilon > 0$ and $l \in E^*$.

It follows from (β) that there is a $k_0 \in \{1, 2, \ldots\}$ so that $|l(x) - l(y)| \leqq \varepsilon$ for every $x, y \in A_{k_0}$ which implies

(2) $|l(x) - l(y)| \leqq \varepsilon$ *for every* $x, y \in \bar{A}_{k_0}$.

By (1) and (2) we have $|l(x_k) - l(x_0)| \leqq \varepsilon$ for every $k \geqq k_0$.

Since $\varepsilon > 0$ and $l \in E^*$ were arbitrary we obtain the required result.

7. Proposition. *Let x_k, $k \in \{1, 2, \ldots\}$, be a sequence in E. If there exist subsets $X \subseteq E$ and $L \subseteq E^*$ such that*

(α) $x_k \in X$ *for every* $k \in \{1, 2, \ldots\}$,

(β) *the set X is relatively weakly compact in E,*

(γ) *for every, $x, y \in X$, $x \neq y$, there is an $l \in L$ so that $l(x) \neq l(y)$,*

(δ) *the sequence $l(x_k)$, $k \in \{1, 2 \ldots\}$, is convergent for every $l \in L$,* *then the sequence* x_k, $k \in \{1, 2, \dots\}$, *is weakly convergent.*

Proof. Let us denote $C(X)$ the algebra of all real continuous functions on X (the continuity is considered here and in the rest of the proof with respect to the weak topology induced on X).

Further, let $C_0(X)$ be the set of all functions of the form

$$\sum_{\substack{i_1, i_2, \dots, i_r \in \\ \{0, 1, \dots, r\}}} a_{i_1 i_2 \dots i_r} \, l_1(x)^{i_1} \, l_2(x)^{i_2} \dots l_r(x)^{i_r}, \quad x \in E,$$

where r runs through $\{0, 1, \dots\}$ and $a_{i_1 i_2 \dots i_r} \in \mathbb{R}$.

It is easy to see from the assumptions (γ) and (δ) that

(1) $C_0(X)$ is a subalgebra of $C(X)$ containing all constant functions and separating the points of X.

(2) the sequence $f(x_k)$, $k \in \{1, 2, \dots\}$, is convergent for every $f \in C_0(X)$,

Using the Weierstrass-Stone theorem we obtain from (1) that

(3) $\overline{C_0(X)} = C(X)$, i.e. every $f \in C(X)$ is uniform limit of a sequence f_k, $k \in \{1, 2, \dots\}$, from $C_0(X)$.

It follows from (2) and (3) that

(4) the sequence $f(x_k)$, $k \in \{1, 2, \dots\}$, is convergent for every $f \in C(X)$.

As a particular case of (4) we have

(5) the sequence $l(x_k)$, $k \in \{1, 2, \dots\}$, is convergent for every $l \in E^*$.

The required weak convergence of the sequence x_k, $k \in \{1, 2, \dots\}$, follows from (α), (β) and (5) by means of Lemma 7.

8. Proposition. *Let* $\omega \in \mathbb{R}$ *and* $\varphi \in (0, \infty) \to \mathbb{R}$, *If*

(α) *the function* φ *is measurable,*

(β) *there is a constant* M *such that* $|\varphi(t)| \leqq M e^{\omega t}$ *for almost every* $t > 0$,

then

$$\frac{1}{p!} \left(\frac{p}{t}\right)^{p+1} \int_0^\infty e^{-p\tau/t} \tau^p \, \varphi(\tau) \, d\tau \to_{p \to \infty, p > \omega t} \varphi(t)$$

for every $t > 0$ *such that*

$$\frac{1}{h} \int_0^h |\varphi(t + \tau) + \varphi(t - \tau) - 2\,\varphi(t)| \, d\tau \to_{h \to 0+} 0 \, .$$

Proof. Immediate consequence of Theorem 6a in [1], Chap. VII.

9. Proposition. *Let* $\omega \geqq 0$, $M \geqq 0$ *and* $\Phi \in (\omega\ \ \infty) \to \mathbb{R}$. *Then*

(A_1) *the function* Φ *is infinitely differentiable on* (ω, ∞),

(A_2) $|(\mathrm{d}^p/\mathrm{d}\lambda^p)\, \Phi(\lambda)| \leqq Mp!/(\lambda - \omega)^{p+1}$ *for every* $\lambda > \omega$ *and* $p \in \{0\ \ 1\ \ldots\}$,

if and only if there exists a function $\varphi \in (0, \infty) \to \mathbb{R}$ *such that*

(B_1) φ *is measurable on* $(0, \infty)$,

(B_2) $|\varphi(t)| \leqq M\mathrm{e}^{\omega t}$ *for almost every* $t > 0$,

(B_3) $\int_0^\infty \mathrm{e}^{-\lambda\tau}\, \varphi(\tau)\, \mathrm{d}\tau = \Phi(\lambda)$ *for every* $\lambda > \omega$.

Proof. An easy extension of Theorem 16a in [1], Chap. VII.

10. Theorem. *Let* $\omega \geqq 0$, $M \geqq 0$ *and* $F \in (\Phi, \infty) \to E$. *Then*

(A_1) *the function* F *is infinitely differentiable on* (ω, ∞),

(A_2) $\|(\mathrm{d}^p/\mathrm{d}\lambda^p)\, F(\lambda)\| \leqq Mp!/(\lambda - \omega)^{p+1}$ *for every* $\lambda > \omega$ *and* $p \in \{0, 1, \ldots\}$,

(A_3) *for almost every* $t > 0$, *the set*

$$\left\{\frac{1}{p!}\left(\frac{p}{t}\right)^{p+1} F^{(p)}\left(\frac{p}{t}\right) : p \in \{0, 1, \ldots\}\,, \quad p > \omega t\right\}$$

is relatively weakly compact in E,

if and only if there exists a function $f \in (0, \infty) \to E$ *such that*

(B_1) f *is measurable on* $(0, \infty)$,

(B_2) $\|f(t)\| \leqq M\mathrm{e}^{\omega t}$ *for almost every* $t > 0$,

(B_3) $\int_0^\infty \mathrm{e}^{-\lambda\tau} f(\tau)\, \mathrm{d}\tau = F(\lambda)$ *for every* $\lambda > \omega$.

Proof. "Only if". For the sake of simplicity, let us denote

(1) $f_p(t) = \dfrac{(-1)^p}{p!}\left(\dfrac{p}{t}\right)^{p+1} F^{(p)}\left(\dfrac{p}{t}\right)$ for $t > 0$ and $p \in \{0, 1, \ldots\}$, such that $p > \omega t$.

It is easy to see from (A_1) that there is a subspace E_0 of E such that

(2) E_0 is closed and separable,

(3) $F(\lambda) \in E_0$ for every $\lambda > \omega$.

It follows from (2) and (3) that

(4) $F^{(p)}(\lambda) \in E_0$ for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

Now (1) and (4) imply

(5) $f_p(t) \in E_0$ for every $t > 0$ and $p \in \{0, 1, \ldots\}$ such that $p > \omega t$.

In view of Proposition 5 we obtain from (2) that there is a set $L \subseteq E^*$ such that

(6) L is a countable

(7) for every $x \in E_0$, $x \neq 0$, there is an $l \in L$ such that $l(x) \neq 0$.

By means of Proposition 9 we obtain from (A_1) and (A_2) that for every $l \in E^*$ we can fix a function $\varphi_l \in (0, \infty) \to R$ such that

(8) φ_l is measurable on $(0, \infty)$ for every $l \in E^*$,

(9) for every $l \in E^*$, $|\varphi_l(t)| \leqq M e^{\omega t}$ for almost all $t > 0$,

(10) $\int_0^\infty e^{-\lambda\tau} \varphi_l(\tau) \, d\tau = l(F(\lambda))$ for every $\lambda > \omega$ and $l \in E^*$.

By use of Proposition 8 we get from (1), (8), (9) and (10) that

(11) for every $l \in E^*$, $l(f_p(t)) \to_{p \to \infty, p > \omega t} \varphi_l(t)$ for almost every $t > 0$.

It follows from (6) and (11) that there is a set $S_1 \subseteq (0, \infty)$ such that

(12) the set $(0, \infty) \setminus S_1$ is measurable of measure zero,

(13) $l(f_p(t)) \to \varphi_l(t)$ for every $t \in S_1$ and $l \in L$.

On the other hand, it follows from (A_3) that there is a set $S_2 \subseteq (0, \infty)$ such that

(14) the set $(0, \infty) \setminus S_2$ is measurable of measure zero,

(15) the set $\{f_p(t) : p \in \{0, 1, \ldots\}, p > \omega t\}$ is relatively weakly compact in E for every $t \in S_2$.

Let now

(16) $S = S_1 \cap S_2$.

It follows from (12)–(16) that

(17) the set $(0, \infty) \setminus S$ is measurable of measure zero,

(18) the sequence $l(f_p(t))$, $p \in \{0, 1, \ldots\}$, $p > \omega t$, is convergent for every $t \in S$ and $l \in L$,

(19) the set $\{f_p(t) : p \in \{0, 1, \ldots\}, p > \omega t\}$ is relatively weakly compact in E for every $t \in S$.

Now we shall prove that

(20) the sequence $f_p(t)$, $p \in \{0, 1, \ldots\}$, $p > \omega t$ is weakly convergent in E for every $t \in S$.

Indeed, let us first fix a $t \in S$ and a $p_0 \in \{0, 1, \ldots\}$, so that $p_0 > \omega t$. Let us write $x_k = f_{p_0+k}(t)$, $k \in \{1, 2. \ldots\}$, and let X be the weak closure of the set $\{f_p(t) : p \in \in \{0, 1, \ldots\}, p > \omega t\}$. Then the condition ($\alpha$) of Proposition 7 is evidently fulfilled, the condition (β) follows from (19), the condition (γ) from (2), (5) and (7) and the condition (δ) from (18). Now the conclusion of Proposition 7 gives (20).

Let us now define in view of (20)

(21) $f(t) = \text{weak}-\lim_{p\to\infty, p>\omega t} f_p(t)$ for every $t \in S$, $f(t) = 0$ for $t \in (0, \infty) \setminus S$.

It follows from (2), (5) and (21) that

(22) $f(t) \in E_0$ for every $t > 0$.

Further, it follows from (13), (16), (17), (20) and (21) that

(23) for every $l \in E^*$, $l(f(t)) = \varphi_l(t)$ for almost every $t > 0$.

Now (8) and (23) imply the condition (α) of Proposition 3 and (2) and (22) the condition (β). Hence the conclusion of this Proposition shows that

(24) the statement ($\mathbf{B}_1$) holds.

It follows from (1), (21) and (A_2) that

(25) $$|l(f(t))| = |\lim_{p\to\infty, p>\omega t} l(f_p(t))| \leqq \lim_{p\to\infty, p>\omega t} |l(f_p(t))| \leqq \lim_{p\to\infty, p>\omega t} \|l\| \, \|f_p(t)\| \leqq$$
$$\leqq \|l\| \lim_{p\to\infty, p>\omega t} \left\| \frac{(-1)^p}{p!} \left(\frac{p}{t}\right)^{p+1} F^{(p)}\left(\frac{p}{t}\right) \right\| \leqq$$
$$\leqq \|l\| \lim_{p\to\infty, p>\omega t} \left(\frac{1}{p!} \left(\frac{p}{t}\right)^{p+1} \frac{Mp!}{\left(\frac{p}{t} - \omega\right)^{p+1}} \right) = M\|l\| \lim_{p\to\infty, p>\omega t} \frac{1}{\left(1 - \frac{\omega t}{p}\right)^{p+1}} =$$
$$= Me^{\omega t}\|l\| \text{ for every } t \in S \text{ and } l \in E^*.$$

Immediate consequence of (25) is

(26) $\|f(t)\| \leqq Me^{\omega t}$ for every $t \in S$.

Now we see from (17) and (26) that

(27) the statement ($\mathbf{B}_2$) holds.

Finally we obtain from (8)–(10), (23), (24) and (26) that

$$(28)\quad l\left(\int_0^\infty e^{-\lambda\tau} f(\tau)\,d\tau\right) = \int_0^\infty e^{-\lambda\tau}\, l(f(\tau))\,d\tau =$$
$$= \int_0^\infty e^{-\lambda\tau}\, \varphi_l(\tau)\,d\tau = l(F(\lambda)) \text{ for every } \lambda > \omega \text{ and } l \in E^*.$$

An immediate consequence of (28) is

(29) the statement (B_3) holds.

The proof of the "only if" part is executed by (24), (26) and (29).

"If" Let us first fix an $f \in (0, \infty) \to E$ such that (B_1), (B_2) and (B_3) hold.

In particular, by (B_3) we have $F(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\,d\tau$ for every $\lambda > \omega$ and we easily obtain from (B_1), (B_2) by procedures usual in the classical Laplace transform that

(1) the statements (A_1) and (A_2) hold.

Let us now denote

$$(2)\quad S = \left\{t : t > 0,\ \frac{1}{h}\int_0^h \|f(t+\tau) + f(t-\tau) - 2f(t)\|\,d\tau \to_{h\to 0+} 0\right\}.$$

According to Proposition 4 we obtain from (B_1) and (2) that

(3) the set $(0, \infty) \setminus S$ is measurable of measure zero.

Taking $\varphi(t) = l(f(t))$ for $t > 0$ and $l \in E^*$, we obtain easily from (B_1), (B_2) and (B_3) and from (2) by means of Proposition 8 that

$$(4)\quad \frac{(-1)^p}{p!}\left(\frac{p}{t}\right)^{p+1} l\left(F^{(p)}\left(\frac{p}{t}\right)\right) \to_{p\to\infty, p>\omega t} l(f(t)) \text{ for every } t \in S \text{ and } l \in E^*.$$

But (4) implies

(5) the set $\left\{\frac{1}{p!}\left(\frac{p}{t}\right)^{p+1} F^{(p)}\left(\frac{p}{t}\right) : p \in \{0, 1, \ldots\},\ p > \omega t\right\}$ is relatively weakly compact in E for every $t \in S$.

Combining (2) and (5) we have

(6) the statement (A_3) holds.

The proof of "if" part is given by (1) and (6).

11. Remark. The condition (A_3) in the preceding Theorem 10 can be replaced by a formally more general one, namely

(A_3') *for almost every* $t > 0$, *there exists a sequence* $t_p > 0$, $p \in \{0, 1, \ldots\}$, *such that*

$$t_p \to_{p \to \infty} t\,,$$

the set $\left\{\frac{1}{p!}\left(\frac{p}{t_p}\right)^{p+1} F^{(p)}\left(\frac{p}{t_p}\right) : p \in \{0, 1 \ldots\},\ p > \omega t_p\right\}$ *is relatively weakly compact in* E.

The proof of corresponding version of Theorem 10, with the condition (A_3') above, remains almost unchanged and the necessary little adaptations can be left to the reader, but instead of Proposition 8 we need the following

8′. Proposition. *Let* $\omega \in R$ *and* $\varphi \in (0, \infty) \to \mathbb{R}$. *If* (α), (β) *as in Proposition* 8, *then*

$$\frac{1}{p!}\left(\frac{p}{t_p}\right)^{p+1} \int_0^\infty \mathrm{e}^{-p\tau/t_p}\, \tau^p\, \varphi(\tau)\, \mathrm{d}\tau \to_{p\to\infty, p>\omega t_p} \varphi(t)$$

for every $t > 0$ *and every sequence* $t_p > 0$, $p \in \{0, 1, \ldots\}$, *such that* $t_p \to_{p\to\infty} t$ *and*

$$\frac{1}{h}\int_0^h \left|\varphi(t+\tau) + \varphi(t-\tau) - 2\,\varphi(t)\right| \mathrm{d}\tau \to_{h\to 0+} 0.$$

Proof. See [6], Theorem 1.1. Pollard uses a little different record of the assertion, but the reader easily shows that both formulations are equivalent.

The requirement $t_p \to_{p\to\infty} t$ cannot be weakened as seen on the special case $\varphi(t) = t,\ t > 0$.

12. Theorem. (Miyadera) *Let* $\omega \geqq 0$, $M \geqq 0$ *and* $F \in (\omega, \infty) \to E$. *If the space* E *is reflexive, then*

(A_1) *the function* F *is infinitely differentiable on* (ω, ∞),

(A_2) $\|(\mathrm{d}^p/\mathrm{d}\lambda^p)\, F(\lambda)\| \leqq Mp!/(\lambda - \omega)^{p+1}$ *for every* $\lambda > \omega$ *and* $p \in \{(0, 1, \ldots\}$,

if and only if there exists a function $f \in (0, \infty) \to E$ *such that*

(B_1) f *is measurable on* $(0, \infty)$,

(B_2) $\|f(t)\| \leqq M\mathrm{e}^{\omega t}$ *for almost every* $t > 0$,

(B_3) $\displaystyle\int_0^\infty \mathrm{e}^{-\lambda\tau} f(\tau)\, \mathrm{d}\tau = F(\lambda)$ *for every* $\lambda > \omega$.

Proof. In view of Theorem 10 it is only to verify that the condition (A_3) of Theorem 10 follows from conditions (A_1) and (A_2).

Indeed we see from (A_1) and (A_2) that

$$\left\|\frac{1}{p!}\left(\frac{p}{t}\right)^{p+1} F^{(p)}\left(\frac{p}{t}\right)\right\| \leqq M \frac{1}{\left(1-\frac{\omega t}{p}\right)^{p+1}} \text{ for every } t>0 \text{ and } p \in\{0,1, \ldots\} \text{ such}$$

that $p > \omega t$.

Since every bounded set in E is relatively weakly compact owing to the reflexivity of E, the above established inequality proves (A_3) of Theorem 10 even for every $t > 0$ because, as well-known, $\frac{1}{\left(1-\frac{\omega t}{p}\right)^{p+1}} \to_{p\to\infty, p>\omega t} e^{\omega t}$ for every $t \in R$.

13. Theorem. *Let $\omega \geqq 0$, $M \geqq 0$, $C \subseteq E$ and $F \in (\omega, \infty) \to E$. If the set C is a convex, symmetric and weakly compact subset of E, then*

(A_1) *the function F is infinitely differentiable on $(0, \infty)$,*

(A_2) $(d^p/d\lambda^p) F(\lambda) \in (Mp!/(\lambda-\omega)^{p+1})\, C$ *for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$,*

if and only if there exists a function $f \in (0, \infty) \to E$ such that

(B_1) *f is measurable on $(0, \infty)$,*

(B_2) *$f(t) \in Me^{\omega t}C$ for almost every $t > 0$,*

(B_3) $\int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau = F(\lambda)$ *for every $\lambda > \omega$.*

Proof. "Only if". Let us denote

(1) $M_0 = \sup_{x\in C} (\|x\|)$.

It follows from (A_2) and (1) that

(2) $\|(d^p/d\lambda^p) F(\lambda)\| \leqq M_0 p!/(\lambda-\omega)^{p+1}$ for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

Further it follows from (A_2) that

(3) $$\frac{1}{p!}\left(\frac{p}{t}\right)^{p+1} F^{(p)}\left(\frac{p}{t}\right) \in \frac{1}{p!}\left(\frac{p}{t}\right)^{p+1} \frac{Mp!}{\left(\frac{p}{t}-\omega\right)^{p+1}}\, C = \frac{M}{\left(1-\frac{\omega t}{p}\right)^{p+1}}\, C$$

for every $t > 0$ and $p \in \{0, 1, \ldots\}$ such that $p > \omega t$.

But since, as well-known,

$$\frac{1}{\left(1-\dfrac{\omega t}{p}\right)^{p+1}} \to_{p\to\infty, p>\omega t} \mathrm{e}^{\omega t}$$

for every $t > 0$, we see from (3) that

(4) the set $\left\{\dfrac{1}{p!}\left(\dfrac{p}{t}\right)^{p+1} F^{(p)}\left(\dfrac{p}{t}\right) : p \in \{0, 1, \ldots\},\ p > \omega t\right\}$ is relatively weakly compact in E.

We see from (A_1), (2) and (4) that the assumptions (A_1), (A_2) and (A_3) of Theorem 10 are fulfilled and consequently there exists a function $f \in (0, \infty) \to E$ such that

(5) f is measurable on $(0, \infty)$,

(6) $\|f(t)\| \leqq M_0 \mathrm{e}^{\omega t}$ for almost every $t > 0$,

(7) $\displaystyle\int_0^\infty \mathrm{e}^{-\lambda\tau} f(\tau)\, \mathrm{d}\tau = F(\lambda)$ for every $\lambda > \omega$.

Using Proposition 4 we obtain from (5) that there is an $S \subseteq (0, \infty)$ such that

(8) the set $(0, \infty) \setminus S$ is measurable of measure zero,

(9) $\displaystyle\frac{1}{h}\int_0^\infty \|f(t + \tau) + f(t - \tau) - 2f(t)\|\, \mathrm{d}\tau \to_{h\to 0_+} 0$ for every $t \in S$.

By Proposition 8 we see from (5), (6), (7) and (9) that

(10) $\displaystyle l(f(t)) = \lim_{p\to\infty, p>\omega t} \frac{(-1)^p}{p!}\left(\frac{p}{t}\right)^{p+1} l\left(F^{(p)}\left(\frac{p}{t}\right)\right)$ for every $t \in S$ and $l \in E^*$.

It follows from (A_2) that

(11) $\displaystyle\frac{(-1)^p}{p!}\left(\frac{p}{t}\right)^{p+1} F^{(p)}\left(\frac{p}{t}\right) \in \frac{M}{\left(1-\dfrac{\omega t}{p}\right)^{p+1}} C$ for every $t > 0$ and $p \in \{0, 1, \ldots\}$ such that $p > \omega t$.

Now we get from (10) and (11) that

(12) $|l(f(t))| \leqq M\mathrm{e}^{\omega t} \sup\limits_{x\in C} (|l(x)|)$ for every $t \in S$ and $l \in E^*$.

Now we need to prove that

(13) $f(t) \in Me^{\omega t}C$ for every $t \in S$.

If $M = 0$, then (13) is an obvious consequence of (12). Hence we shall suppose $M \neq 0$ and proceed indirectly. If (13) does not hold, then there exists a $t_0 \in S$ such that $f(t_0) \notin Me^{\omega t_0}C$. According to a consequence of Hahn-Banach theorem, we can find an $l_0 \in E^*$ so that

(14) $l_0(f(t_0)) > 1$,

(15) $|l_0(x)| \leqq 1$ for every $x \in Me^{\omega t_0}C$.

In view of supposed $M \neq 0$, the property (15) can be written as

(16) $|l_0(x)| \leqq M^{-1}e^{-\omega t_0}$ for every $x \in C$.

Now we obtain from (12) and (16) that $|l_0(f(t_0))| \leqq Me^{\omega t_0}\, M^{-1}e^{-\omega t_0} = 1$ which contradicts (14) and thus proves (13).

The "only if" part follows from (5), (7), (8) and (13).

"If." Let us first fix a function $f \in (0, \infty) \to E$ satisfying (B_1), (B_2) and (B_3).

It is easy to see that (A_1) holds.

If $M = 0$, then (A_2) is obvious and thus we shall suppose $M \neq 0$ and proceed indirectly. If (A_2) does not hold, then there exist a $\lambda_0 > \omega$ and a $p_0 \in \{0, 1, \dots\}$, such that $F^{(p_0)}(\lambda_0) \notin (Mp_0!/(\lambda_0 - \omega)^{p_0+1})\, C$. According to a consequence of Hahn-Banach theorem we can find an $l_0 \in E^*$ such that

(1) $l_0(F^{(p_0)}(\lambda_0)) > 1$,

(2) $|l_0(x)| \leqq 1$ for every $x \in (Mp_0!/(\lambda_0 - \omega)^{p_0+1})\, C$.

In view of supposed $M \neq 0$ we can write (2) in the form

(3) $|l_0(x)| \leqq (\lambda_0 - \omega)^{p_0+1}/Mp_0!$ for every $x \in C$.

Now we obtain from (B_1), (B_2), (B_3) and (3) that

$$|l_0(F^{(p_0)}(\lambda_0))| = \left|l_0\left(\int_0^\infty e^{-\lambda_0\tau}(-\tau)^{p_0} f(\tau)\, d\tau\right)\right| \leqq \int_0^\infty e^{-\lambda_0\tau}\tau^{p_0}\, |l_0(f(\tau))|\, d\tau \leqq$$
$$\leqq \int_0^\infty e^{-\lambda_0\tau}\tau^{p_0} Me^{\omega\tau} \frac{(\lambda_0 - \omega)^{p_0+1}}{Mp_0!}\, d\tau = 1$$

which contradicts (1) and proves (A_2).

The proof of "if" part is complete.

14. Some comments. The basic result in the problem of representability for numerical exponentially bounded functions is due to D. V. WIDDER and is quoted above as Proposition 11.

The extension of Widder's result to vector-valued functions in reflexive Banach spaces, our Theorem 12, is due to I. MIYADERA [4].

Moreover, Miyadera presented an example showing that the conditions (A_1), (A_2) of Theorem 12 cannot be sufficient for the validity of the mentioned theorem in nonreflexive Banach spaces.

Consequently, an additional condition to (A_1), (A_2) of Theorem 12 is necessary. Our condition (A_3) in Theorem (10) (or (A_3') in Remark 11) seems the most simple and natural one and solves completely the problem in consideration. Miyadera's theorem is then a simple consequence of Theorem 10.

Another possibility to extend Miyadera's result to nonreflexive spaces is given in Theorem 13 which deals with the representability problem by exponentially weakly compactly bounded functions and is also an easy consequence of Theorem 10.

Recently, the representability problem was attacked by D. LEVIATAN [5] (see Theorem 7 in [5]) from rather different point of view. Leviatan proved, among others, that, under conditions (A_1), (A_2) of Theorem 12, the original function f can be found in dual spaces of appropriate subspaces of E^*.

Finally, let us remark that the proof of Proposition 7, given by means of Weierstrass-Stone theorem, may seem a little unadequate because it is too "analytic" and the problem itself is essentially linear. The result follows also easily from Šmuljan's theorem on weakly convergent subsequences of weakly compact sequences and, moreover, a direct purely "linear" proof can be given.

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