Časopis pro pěstování matematiky

Milan Kučera

A new method for obtaining eigenvalues of variational inequalities. Branches of eigenvalues of the equation with the penalty in a special case

Časopis pro pěstování matematiky, Vol. 104 (1979), No. 3, 295--310

Persistent URL: http://dml.cz/dmlcz/118024

Časopis pro pěstování matematiky, roč. 104 (1979), Praha

A NEW METHOD FOR OBTAINING EIGENVALUES OF VARIATIONAL INEQUALITIES. BRANCHES OF EIGENVALUES OF THE EQUATION WITH THE PENALTY IN A SPECIAL CASE

MILAN KUČERA, Praha

(Received April 8, 1977)

INTRODUCTION

We shall consider a real Hilbert space H with an inner product $(\cdot, \cdot)$ and with the corresponding norm $\|\cdot\|$. Let K be a closed convex cone in H with its vertex at the origine and let $A : H \to H$ be a linear completely continuous symmetric operator in H. It will be supposed that A has only simple eigenvalues. We shall seek eigenvalues and eigenvectors of the variational inequality

$$\text{(I)} \qquad u \in K\,,$$

$$\text{(II)} \qquad (\lambda u - Au, v - u) \geqq 0 \quad \text{for all} \quad v \in K$$

(i.e., the couples λ, u satisfying (I), (II), λ real, $\|u\| \neq 0$). Analogously as in the papers [4], [5], we shall study the existence of branches λ_ε, u_ε of solutions of the penalty equation[1])

$$\lambda_\varepsilon u_\varepsilon - Au_\varepsilon + \varepsilon\beta u_\varepsilon = 0$$

defined on $\langle 0, +\infty)$ starting at a given eigenvalue λ_0 and a given eigenvector u_0 of the operator A and converging to an eigenvalue λ_∞ and to a new eigenvector u_∞ of (I), (II). While general global results of the bifurcation theory were used in [4], [5], the aim of this paper is to show that in some special case the existence of the above mentioned branches can be proved "elementarily" on the basic of the abstract implicit function theorem. In this case, a better information about the branches $\lambda_\varepsilon, u_\varepsilon$ is obtained. A result of this type was announced in [3]. Unfortunately, only very special examples satisfying the assumptions of the present theory are known to the

[1]) For the properties of the penalty operator β see Section 1.

author (see Examples 5.1, 5.2). Hence the present paper seems to be only a theoretical supplement to [4], where the assumptions are not so restrictive and the nontrivial examples are given (see also [5], where the case of branches starting from multiple eigenvalues λ_0 is studied).

1. TERMINOLOGY, NOTATION

We shall denote by ∂K and K^0 the boundary and the interior of K, respectively. The strong convergence and the weak convergence in H will be denoted by $\to$ and $\rightharpoonup$, respectively.

Definition 1.1. A real λ is said to be an *eigenvalue of the variational inequality* (I), (II) if there exists a nontrivial $u \in H$ satisfying (I), (II). The point u is called an *eigenvector* of (I), (II) corresponding to λ. We shall denote by Λ_V and E_V the set of all eigenvalues and eigenvectors of (I), (II), respectively.

The set of all eigenvalues and the set of all eigenvectors of the operator A will be denoted by Λ_A and E_A, respectively.

Definition 1.2. We shall write $\lambda \in \Lambda_{V,b}$ [2]) if $\lambda \in \Lambda_V$ and there exists an eigenvector $u \in E_V \cap \partial K$ corresponding to λ. We shall write $\lambda \in \Lambda_b$ and $\lambda \in \Lambda_i$ if $\lambda \in \Lambda_A$ and there exists an eigenvector corresponding to λ and satisfying $u \in E_A \cap \partial K$ and $u \in E_A \cap K^0$, respectively. We shall write $\lambda \in \Lambda_e$ if $\lambda \in \Lambda_A$ and $u \notin K$, $-u \notin K$ for the corresponding eigenvector $u \in E_A$ [3]).

We shall study the eigenvalues $\lambda \in \Lambda_{V,b}$. The other eigenvalues of (I), (II) are not interesting because they are simultaneously eigenvalues of A. More precisely, we have $\Lambda_V \setminus \Lambda_{V,b} = \Lambda_i$. Some basic properties, relations between the sets Λ_V, Λ_A, $\Lambda_{V,b}$, Λ_i, Λ_b, Λ_e and simple examples illustrating the situation are explained in [4, Section 1] in detail.

Remark 1.1. The general considerations will be illustrated by the following example. Let a, b be functions on $\langle 0, 1 \rangle$. Suppose that a is continuously differentiable, $a \geqq \gamma$ for some $\gamma > 0$, b is continuous, $b > 0$ on $\langle 0, 1 \rangle$. Denote by $H = \mathring{W}_2^1(0, 1)$ the well-known Sobolev space of all absolutely continuous functions on $\langle 0, 1 \rangle$ vanishing at 0 and 1, the derivatives of which are square integrable over $\langle 0, 1 \rangle$. Introduce an inner product on H by

$$(u, v) = \int_0^1 au'v' \, dx \quad \text{for all} \quad u, v \in H$$

---

[2]) A point $\lambda \in \Lambda_{V,b}$ and $\lambda \in \Lambda_b$ is said to be a boundary eigenvalue of (I), (II) and a boundary eigenvalue of A, respectively. A point $\lambda \in \Lambda_i$ and $\lambda \in \Lambda_e$ is said to be an internal and external eigenvalue of A, respectively.

[3]) As we have said in Introduction, throughout the whole paper we assume that A has only simple eigenvalues.

(instead of the usual equivalent inner product $(u, v) = \int_0^1 (u'v' + uv)\,dx$). Set $K = \{u \in H;\ u(x_i) \geqq 0,\ i = 1, \ldots, n\}$, where $x_i \in (0, 1)$, $i = 1, \ldots, n$ are given numbers (n is positive integer). Let us define an operator A by

$$(Au, v) = \int_0^1 buv\,dx \quad \text{for all} \quad u, v \in H\,.$$

The problem (I), (II) is equivalent to a boundary value problem for the second order ordinary differential equation with some additional "transmission conditions" at the points x_i; (see [4, Section 1]). If a, b are constants, then both the eigenvalues and eigenvectors of (I), (II) can be calculated elementarily in this special example (see [6, Section 1]).

2. PENALTY OPERATORS

In the sequel, we shall denote by β a penalty operator corresponding to K, i.e. $\beta : H \to H$ will be a nonlinear continuous operator such that

(P) $\beta u = 0$ if and only if $u \in K$.

The following assumptions about the operator β will be considered.

(CC) β is completely continuous;

(D) β is differentiable on $H \setminus K$ in the sense of Fréchet;

(M) β is monotone, i.e. $(\beta u - \beta v,\ u - v) \geqq 0$ for all $u, v \in H$;

(β, K^0) if $u \in K^0$, $v \notin K$, then $(\beta v, u) \neq 0$;

(BC) if $\varepsilon_n > 0$, $u_n \in H$, $\{\varepsilon_n \beta u_n\}$ is bounded, then there exists a strongly convergent subsequence of $\{\varepsilon_n \beta u_n\}$;

(SE) for each fixed $u \in H \setminus K$, $\varepsilon \geqq 0$, a linear operator $\beta'(u)$ is symmetric and all positive eigenvalues of $A - \varepsilon\,\beta'(u)$ are simple.[4]

Moreover, we shall consider the following assumption about the connection between the solution of the nonlinear equation with the penalty and the corresponding linearized equation ($R > 0$, $\Lambda_2 > \Lambda_1$ are given):

(NL) If $\lambda \in \langle \Lambda_1, \Lambda_2 \rangle$, $\varepsilon \in \langle 0, R \rangle$, $u \in H \setminus K$, $v \in H$, $\|u\| = \|v\| = 1$,

(i) $\lambda u - Au + \varepsilon \beta u = 0$,

(ii) $\lambda v - Av + \varepsilon\,\beta'(u)(v) = \mu u$ for some real μ, then $(u, v) \neq 0$.

Remark 2.1. Let us consider the example from Remark 1.1. Introduce penalty operators β_α ($\alpha \in \langle 0, 1 \rangle$ is a parameter) defined by the formula

$$(2.1) \qquad (\beta_\alpha u, v) = -\sum_{i=1}^{n} |u(x_i)|^\alpha\, u^-(x_i)\, v(x_i) \quad \text{for all} \quad u, v \in H\,,$$

[4]) We denote by β' the Fréchet derivative of β.

where u^- denotes the negative part of u. It is easy to see that β_α fulfils the assumptions (P), (CC), (M), (β, K^0), (BC) for each $\alpha \in \langle 0, 1\rangle$ but in the case $n > 1$ thc condition (D) is satisfied only for $\alpha > 0$. Notice that for $\alpha = 0$ the operator β_0 is positive homogeneous (i.e. $\beta_0(tu) = t\,\beta_0(u)$ for $t > 0$, $u \in H$) which is a very useful property for the further considerations. Unfortunately, we can use the operator β_0 only in the case $n = 1$ when (D) is satisfied also for $\alpha = 0$. In the case $n > 1$, we shall approximate β_0 by β_α with small α (see Sections 4, 5).

Remark 2.2. Let us show that the assumption (SE) is fulfilled in the example from Remark 1.1 with the operator β_α defined by (2.1) (for an arbitrary $\alpha \in (0, 1\rangle$ fixed and in the case $n = 1$ also for $\alpha = 0$). First, we shall consider the case $\alpha > 0$. We have

$$(\beta'_\alpha(u)(v), w) = (1 + \alpha) \sum_{i=1}^{n} u^-(x_i)^\alpha\, v(x_i)\, w(x_i)$$

for all $u, v, w \in H$. The first part of the condition (SE) is obvious. Now, let λ be a positive eigenvalue and v the corresponding eigenvector of the operator $A - \varepsilon\,\beta'(u)$ (with fixed $u \in H$ and $\varepsilon > 0$). Then

$$\int_0^1 (\lambda a v' w' - b v w)\,\mathrm{d}x + \sum_{i=1}^{n} c_i\, v(x_i)\, w(x_i) = 0 \quad \text{for all} \quad w \in H \tag{2.2}$$

where $c_i = \varepsilon(1 + \alpha)\, u^-(x_i)^\alpha$. Denoting by χ_{x_i} the characteristic function of the interval $\langle 0, x_i\rangle$, the integration by parts yields

$$\int_0^1 \left[\lambda a v' + \int_0^x b v\,\mathrm{d}t - \sum_{i=1}^{n} c_i\, v(x_i)\, \chi_{x_i}\right] w'\,\mathrm{d}x = 0 \quad \text{for all} \quad w \in H$$

and this implies

$$\lambda a v' + \int_0^x b v\,\mathrm{d}t - \sum_{i=1}^{n} c_i\, v(x_i)\, \chi_{x_i} = C$$

with some constant C. It is clear from here that v has a continuous second derivative on (x_i, x_{i+1}), $i = 0, \ldots, n$, and

$$v'(x_i-) - v'(x_i+) = \frac{c_i\, v(x_i)}{\lambda\, a(x_i)}\,, \tag{2.3}$$

where we denote $v'(x_i\pm) = \lim_{x \to x_i\pm} v'(x)$. If we have two eigenvectors v_1, v_2, then we can choose a nontrivial combination $v = t_1 v_1 + t_2 v_2$ such that $v'(0) = 0$. It follows from (2.2) that v is a solution of the equation $\lambda(a v')' + b v = 0$ on the intervals (x_i, x_{i+1}). Hence in virtue of the initial conditions it is $v \equiv 0$ on $(0, x_1)$. This together with the continuity of v gives $v(x_1) = 0$, (2.3) implies the continuity of v' at x_1 and $v'(x_1) = 0$. An analogous argument can be used for (x_1, x_2) etc. After a finite number of steps we obtain $v \equiv 0$ on $\langle 0, 1\rangle$, i.e. v_1, v_2 are linearly dependent and λ is a simple eigenvalue. Analogously for $\alpha = 0$, $n = 1$ but for $u(x_1) \neq 0$ only.

Remark 2.3. It follows from (CC) that $\beta'(u)$ is a completely continuous linear mapping for each $u \in H \setminus K$.

Remark 2.4. If β is positive homogeneous and satisfies (CC), (SE), then (NL) is fulfilled with $R = +\infty$, $\Lambda_1 = 0$, $\Lambda_2 = +\infty$. Indeed, we have $\beta'(u)(u) = \beta(u)$, i.e. $v = u$ is a solution of (ii) with $\mu = 0$ and $\mu \neq 0$ in (ii) is impossible as is seen from the Fredholm alternative and (SE).

3. BRANCHES OF EIGENVALUES AND EIGENVECTORS

Lemma 3.1. *Let us suppose that the assumptions* (CC), (D), (SE) *and* (NL) *are fulfilled with some* $R > 0$, $\Lambda_2 > \Lambda_1 > 0$. *Consider real numbers* $\varepsilon_0 \in \langle 0, R \rangle$, $\lambda_{\varepsilon_0} \in \langle \Lambda_1, \Lambda_2 \rangle$ *such that there exists* $u_{\varepsilon_0} \in H$ *satisfying the conditions* $\|u_{\varepsilon_0}\| = 1$,

$$\lambda_{\varepsilon_0} u_{\varepsilon_0} - A u_{\varepsilon_0} + \varepsilon_0 \beta u_{\varepsilon_0} = 0 \,. \tag{3.1}$$

Then there exists $\delta > 0$ *and a neighbourhood* U *of the point* $[\lambda_{\varepsilon_0}, u_{\varepsilon_0}]$ *in the space* $\mathbb{R} \times H$ *such that for each* $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$ *there is a unique couple* $[\lambda_\varepsilon, u_\varepsilon] \in U$ *satisfying the conditions*

$$\|u_\varepsilon\| = 1 \,, \tag{a}$$

$$\lambda_\varepsilon u_\varepsilon - A u_\varepsilon + \varepsilon \beta u_\varepsilon = 0 \,. \tag{b}$$

Moreover, the functions λ_ε, u_ε *defined in this way are differentiable on* $(\varepsilon_0 - \delta, \varepsilon_0 + \delta)$.

Proof. Let us define a mapping $F : \mathbb{R} \times H \times \mathbb{R} \to H \times \mathbb{R}$ by

$$F(\lambda, u, \varepsilon) = [\lambda u - A u + \varepsilon \beta u, \; \|u\|^2 - 1] \,.$$

This mapping is Fréchet differentiable and for its partial derivative with respect to the variables $[\lambda, u]$ we have

$$F'_{[\lambda,u]}(\lambda, u, \varepsilon)(\mu, v) = [\mu u + \lambda v - A v + \varepsilon \, \beta'(u)(v), \, 2(u, v)] \,.$$

Let us show that $F'_{[\lambda,u]}(\lambda_{\varepsilon_0}, u_{\varepsilon_0}, \varepsilon_0)$ as a linear mapping of $\mathbb{R} \times H$ into $H \times \mathbb{R}$ is an isomorphism. Let $[w, \xi] \in H \times \mathbb{R}$ be an arbitrary point. We shall show that there exist $\mu_1, \mu_0 \in \mathbb{R}$, $v_1, v_0 \in H$, $v_0 \neq 0$ such that

$$\mu_1 u_{\varepsilon_0} + \lambda_{\varepsilon_0} v_1 - A v_1 + \varepsilon_0 \, \beta'(u_{\varepsilon_0})(v_1) = w \,,$$

$$\mu_0 u_{\varepsilon_0} + \lambda_{\varepsilon_0} v_0 - A v_0 + \varepsilon_0 \, \beta'(u_{\varepsilon_0})(v_0) = 0 \,.$$

If λ_{ε_0} is an eigenvalue of the linear completely continuous operator $A - \varepsilon_0 \, \beta'(u_{\varepsilon_0})$ (see Remark 2.3) corresponding to an eigenvector $\bar{v}$, then we set $v_0 = \bar{v}$ and $\mu_0 = 0$. The existence of μ_1, v_1 is a consequence of the fact that the operator $\lambda_{\varepsilon_0} I - A +$

$+ \varepsilon_0 \beta'(u_{\varepsilon 0})$ maps H onto $v_0^{\perp}$ [5]) (which follows from the assumption (SE) and the Fredholm alternative) and that $u_{\varepsilon 0} \notin v_0^{\perp}$ (with respect to the assumption (NL)), i.e. $v_0^{\perp} \oplus u_{\varepsilon_0} = H$. If λ_{ε_0} is not an eigenvalue of $A - \varepsilon_0 \beta'(u_{\varepsilon 0})$, then $\lambda_{\varepsilon_0} I - A + \varepsilon_0 \beta'(u_{\varepsilon_0})$ maps H onto H and the existence of μ_0, μ_1, v_0, v_1 follows immediately. Further, the assumption (NL) implies $(u_{\varepsilon_0}, v_0) \neq 0$ and therefore there exists $\eta \in \mathbb{R}$ such that $2(u_{\varepsilon_0}, v_1 + \eta v_0) = \xi$. If we set $\mu = \mu_1 + \eta\mu_0$, $v = v_1 + \eta v_0$, then

$$F'_{[\lambda,u]}(\lambda_{\varepsilon_0}, u_{\varepsilon_0}, \varepsilon_0)(\mu, v) = [w, \xi] .$$

Thus, the mapping $F'_{[\lambda,u]}(\lambda_{\varepsilon_0}, u_{\varepsilon_0}, \varepsilon_0)$ maps $\mathbb{R} \times H$ onto $H \times \mathbb{R}$. Further, suppose

$$F'_{[\lambda,u]}(\lambda_{\varepsilon_0}, u_{\varepsilon_0}, \varepsilon_0)(\mu, v) = 0 \quad \text{for some} \quad \mu, v .$$

Then

$$\mu u_{\varepsilon_0} + \lambda_{\varepsilon_0} v - Av + \varepsilon_0 \beta'(u_{\varepsilon_0})(v) = 0 , \quad (u_{\varepsilon_0}, v) = 0 .$$

The case $v \neq 0$ is not possible with respect to (NL) and it is clear that the case $v = 0$, $\mu \neq 0$ is impossible, too. Thus, we have $\mu = 0$, $v = 0$. Hence, the mapping $F'_{[\lambda,u]}(\lambda_{\varepsilon_0}, u_{\varepsilon_0}, \varepsilon_0)$ is one-to-one, i.e. it is an isomorphism. Now, we can use the abstract implicit function theorem for F at the point $\lambda_{\varepsilon_0}, u_{\varepsilon_0}, \varepsilon_0$ (see [2]). This implies the assertion of Lemma 3.1 because the equation $F(\lambda_\varepsilon, u_\varepsilon, \varepsilon) = 0$ is equivalent to the conditions (a), (b).

Lemma 3.2. *Let the assumptions* (P), (β, K⁰) *be fulfilled. Suppose* $\lambda^{(1)}, \lambda^{(2)} \in \Lambda_i$, $0 < \lambda^{(1)} < \lambda^{(2)}$, $\Lambda_b \cap \langle \lambda^{(1)}, \lambda^{(2)} \rangle = \emptyset$. *Suppose that* $\lambda_\varepsilon, u_\varepsilon$ *are continuous functions*[6]) *on a certain interval* $\langle 0, \varepsilon_1)$ *satisfying the conditions* (a), (b) *for all* $\varepsilon \in \langle 0, \varepsilon_1)$ *and such that* $\lambda^{(1)} < \lambda_0 < \lambda^{(2)}$, $u_0 \notin K$. *Then the following conditions are valid for all* $\varepsilon \in \langle 0, \varepsilon_1)$:

(c) $$u_\varepsilon \notin K ,$$

(d) $$\lambda^{(1)} < \lambda_\varepsilon < \lambda^{(2)} .$$

Proof:[7]) Denote $\varepsilon_0 = \inf \{\tilde{\varepsilon} \in \langle 0, \varepsilon_1); \lambda_\varepsilon$ satisfies (d) on $(0, \tilde{\varepsilon})\}$. We have $\lambda^{(1)} < \lambda_0 < \lambda^{(2)}$, λ_ε is continuous and therefore $\varepsilon_0 > 0$. The condition (c) for the interval $\langle 0, \varepsilon_0 \rangle$ (or $\langle 0, \varepsilon_0)$ in the case $\varepsilon_0 = \varepsilon_1$) follows from the assumption that $\Lambda_b \cap \langle \lambda^{(1)}; \lambda^{(2)} \rangle = \emptyset$. Indeed, if (c) is not true on $\langle 0, \varepsilon_0 \rangle$, then there exists $\tilde{\varepsilon} \in \langle 0, \varepsilon_0 \rangle$ such that $u_{\tilde{\varepsilon}} \in \partial K$. The conditions (P), (b) imply that $\lambda_{\tilde{\varepsilon}}$ is an eigenvalue of A with an eigenvector

[5]) $v_0^{\perp}$ denotes the orthogonal complement to v_0 in H.

[6]) We shall denote by λ_ε a real function and by u_ε an abstract function of the variable ε with values in H.

[7]) Lemma 3.2 is a special case of Lemma 2.3 from [4]. For the completness, we give here the proof corresponding to our special situation. The main ideas are the same as those in [4], but the procedure is simpler.

$u_{\bar{\varepsilon}}$, which is not possible. Now, let us consider $\varepsilon_0 < \varepsilon_1$. Then we have $\lambda_{\varepsilon_0} = \lambda^{(i)}$ with $i = 1$ or $i = 2$. We obtain

$$\lambda^{(i)}u_{\varepsilon_0} - Au_{\varepsilon_0} + \varepsilon_0\beta u_{\varepsilon_0} = 0\,,$$

$$\lambda^{(i)}u^{(i)} - Au^{(i)} = 0\,,$$

where $u^{(i)} \in K^0$ is an eigenvector corresponding to the interior eigenvalue $\lambda^{(i)}$. This implies

$$\lambda^{(i)}(u_{\varepsilon_0}, u^{(i)}) - (Au_{\varepsilon_0}, u^{(i)}) + \varepsilon(\beta u_{\varepsilon_0}, u^{(i)}) = 0\,,$$

$$\lambda^{(i)}(u^{(i)}, u_{\varepsilon_0}) - (Au^{(i)}, u_{\varepsilon_0}) = 0$$

and therefore (with respect to the symmetry of A) we obtain $(\beta u_{\varepsilon_0}, u^{(i)}) = 0$. However, this contradicts the assumption (β, K^0) because we have $u_{\varepsilon_0} \notin K$ (the condition (c) is proved for $\varepsilon \in \langle 0, \varepsilon_0 \rangle$) and $u^{(i)} \in K^0$.

Theorem 3.1. *Suppose that $\lambda^{(1)}, \lambda^{(2)} \in \Lambda_i$, $\lambda^{(0)} \in \Lambda_A$ is an eigenvalue corresponding to an eigenvector $u^{(0)} \notin K$ of A and $0 < \lambda^{(1)} < \lambda^{(0)} < \lambda^{(2)}$, $\Lambda_b \cap \langle \lambda^{(1)}, \lambda^{(2)} \rangle = \emptyset$. Assume that the assumptions* (P), (CC), (D), (β, K^0), (SE) *and* (NL) *are satisfied with some $R > 0$ and $\Lambda_1 = \lambda^{(1)}$, $\Lambda_2 = \lambda^{(2)}$. Then there exist differentiable functions λ_ε, u_ε on $\langle 0, R)$ such that $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$ and* (a), (b), (c), (d) *hold for all $\varepsilon \in \langle 0, R)$.*

Proof. Let us denote

$$E = \{\bar{\varepsilon} \in (0, R\rangle;\ \text{there exist continuous functions } \lambda_\varepsilon, u_\varepsilon \text{ on } \langle 0, \bar{\varepsilon}\rangle \text{ such that}$$
$$\lambda_0 = \lambda^{(0)},\ u_0 = u^{(0)},\ \text{(a)–(d) hold on } \langle 0, \bar{\varepsilon}\rangle\}\,.$$

Clearly $E \neq \emptyset$ because Lemma 3.1 can be used with $\varepsilon_0 = 0$, $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$. It follows from Lemmas 3.1, 3.2 that the set E is open in $\langle 0, R\rangle$. We shall show that E is closed as well. Let $\varepsilon_n \in E$ $(n = 1, 2, \ldots)$, $\varepsilon_n \to \varepsilon_0$. With respect to the conditions (a), (d) holding for each ε_n we can suppose (passing to suitable subsequences if necessary) that $\lambda_{\varepsilon_n} \to \tilde{\lambda}_{\varepsilon_0}$, $u_{\varepsilon_n} \rightharpoonup \tilde{u}_{\varepsilon_0}$ for some real $\tilde{\lambda}_{\varepsilon_0} \in \langle \lambda^{(1)}, \lambda^{(2)} \rangle$ and $\tilde{u}_{\varepsilon_0} \in H$. Moreover, with respect to the complete continuity of A and to the assumption (CC) we can suppose that the sequences $\{Au_{\varepsilon_n}\}$, $\{\beta u_{\varepsilon_n}\}$ are strongly convergent. Now, the equations

$$\lambda_{\varepsilon_n}u_{\varepsilon_n} - Au_{\varepsilon_n} + \varepsilon_n\beta u_{\varepsilon_n} = 0$$

together with (d) imply that $\{u_{\varepsilon_n}\}$ is strongly convergent, i.e. $u_{\varepsilon_n} \to \tilde{u}_{\varepsilon_0}$, and

$$\tilde{\lambda}_{\varepsilon_0}\tilde{u}_{\varepsilon_0} - A\tilde{u}_{\varepsilon_0} + \varepsilon_0\beta\tilde{u}_{\varepsilon_0} = 0\,.$$

Further, Lemma 3.1 can be used for ε_0, $u_{\varepsilon_0} = \tilde{u}_{\varepsilon_0}$, $\lambda_{\varepsilon_0} = \tilde{\lambda}_{\varepsilon_0}$. This Lemma ensures the existence and local unicity of differentiable functions $\tilde{\lambda}_\varepsilon$, $\tilde{u}_\varepsilon$ satisfying (a), (b) on a certain interval $(\varepsilon_0 - \delta, \varepsilon_0 + \delta)$. The local unicity of such functions together with $u_{\varepsilon_n} \to \tilde{u}_{\varepsilon_0}$, $\lambda_{\varepsilon_n} \to \tilde{\lambda}_{\varepsilon_0}$ imply $u_{\varepsilon_n} = \tilde{u}_{\varepsilon_n}$, $\lambda_{\varepsilon_n} = \tilde{\lambda}_{\varepsilon_n}$ for all $n > n_0$, n_0 a suitable positive integer. Further, with respect to the continuity of λ_ε, u_ε on $\langle 0, \varepsilon_0)$, it is necessarily $\lambda_\varepsilon = \tilde{\lambda}_\varepsilon$, $u_\varepsilon = \tilde{u}_\varepsilon$ for all $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0)$ with some $\delta > 0$. Hence, the functions λ_ε, u_ε defined as $\lambda_{\varepsilon_0} = \tilde{\lambda}_{\varepsilon_0}$, $u_{\varepsilon_0} = \tilde{u}_{\varepsilon_0}$ at ε_0 are continuous on $\langle 0, \varepsilon_0 \rangle$. Moreover, the functions λ_ε, u_ε defined as $\lambda_\varepsilon = \tilde{\lambda}_\varepsilon$, $u_\varepsilon = \tilde{u}_\varepsilon$ for $\varepsilon \in \langle \varepsilon_0, \varepsilon_0 + \delta)$ satisfy the assumptions of Lemma 3.2 and we obtain in particular (c), (d) for $\varepsilon = \varepsilon_0$. We have proved that the set E is closed and open in $\langle 0, R \rangle$ and therefore $E = \langle 0, R \rangle$. Moreover, Lemma 3.1 can be used for an arbitrary $\varepsilon \in \langle 0, R \rangle$ which yields the differentiability of the functions λ_ε, u_ε.

Lemma 3.3. *Let $\lambda^{(1)}, \lambda^{(2)} \in \Lambda_i$, $0 < \lambda^{(1)} < \lambda^{(2)}$ and let the operator β satisfy the assumptions* (P), (M), (BC). *Suppose that there exist $\varepsilon_n > 0$, $\lambda_n \in (\lambda^{(1)}, \lambda^{(2)})$ and $u_n \notin K^0$ such that $\|u_n\| = 1$ $(n = 1, 2, \ldots)$, $\varepsilon_n \to +\infty$ and*

$$\lambda_n u_n - A u_n + \varepsilon_n \beta u_n = 0 . \tag{3.2}$$

Then there exists a sequence $\{r_n\}$ of indices such that $r_n \to +\infty$, $\lambda_{r_n} \to \lambda_\infty$, $u_{r_n} \to u_\infty$, where $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(2)})$ and u_∞ is an eigenvector of (I), (II) *corresponding to λ_∞, $u_\infty \in \partial K$. If $\{r_n\}$ is an arbitrary sequence of indices such that $r_n \to +\infty$, $\lambda_{r_n} \to \tilde{\lambda}_\infty$, $u_{r_n} \rightharpoonup \tilde{u}_\infty$ for some $\tilde{\lambda}_\infty$, $\tilde{u}_\infty$, then $\tilde{\lambda}_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(2)})$, $\tilde{u}_\infty$ is an eigenvector of* (I), (II) *corresponding to $\tilde{\lambda}_\infty$ and $u_{r_n} \to \tilde{u}_\infty$, $\tilde{u}_\infty \in \partial K$.*

Proof. See [4], Lemma 2.4.

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be fulfilled with $R = +\infty$ and let the assumptions* (P), (M), (BC) *be satisfied. Then there exist differentiable functions λ_ε, u_ε on $\langle 0, +\infty)$ such that $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$ and* (a)–(d) *hold for each $\varepsilon \in \langle 0, +\infty)$. Moreover, there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to +\infty$, $\lambda_{\varepsilon_n} \to \lambda_\infty^{(0)}$, $u_{\varepsilon_n} \to u_\infty^{(0)}$, where $\lambda_\infty^{(0)} \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(2)})$ and $u_\infty^{(0)} \in \partial K$ is an eigenvector of* (I), (II) *corresponding to $\lambda_\infty^{(0)}$. If $\{\varepsilon_n\}$ is an arbitrary sequence such that $\varepsilon_n > 0$, $\varepsilon_n \to +\infty$, $\lambda_{\varepsilon_n} \to \lambda_\infty$, $u_{\varepsilon_n} \rightharpoonup u_\infty$ for some λ_∞, u_∞, then $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(2)})$, u_∞ is an eigenvector of* (I), (II) *corresponding to λ_∞ and we have $u_{\varepsilon_n} \to u_\infty$, $u_\infty \in \partial K$.*

Proof follows directly from Theorem 3.1 and Lemma 3.3.

Remark 3.1. Theorem 3.2 has no good sense for examples of the types mentioned in Remarks 1.1, 2.1, 2.2, because it is applicable only in the case of a positive homogeneous penalty operator (cf. Remark 2.1, 2.4), and we shall prove a stronger result for this special case (Theorem 3.3). In the cases when a positive homogeneous penalty operator cannot be used directly, we shall approximate it and use Theorem 3.1 with a finite R for its approximations (see Sections 4, 5).

Remark 3.2. Let us consider the situation of Theorem 3.1 or 3.2. It follows from (a), (b) that

$$\lambda_\varepsilon = (A u_\varepsilon, u_\varepsilon) - \varepsilon(\beta u_\varepsilon, u_\varepsilon) .$$

Let us calculate the derivative $\dot{\lambda}_\varepsilon$ of the function λ_ε with respect to ε. Using the symmetry of A, we obtain

$$(3.3) \qquad \dot{\lambda}_\varepsilon = 2(Au_\varepsilon, \dot{u}_\varepsilon) - (\beta u_\varepsilon, u_\varepsilon) - \varepsilon(\beta'(u_\varepsilon)(\dot{u}_\varepsilon), u_\varepsilon) - \varepsilon(\beta u_\varepsilon, \dot{u}_\varepsilon).$$

Let us suppose that the operator β is positive homogeneous. Then we have $\beta'(u)(u) = \beta u$ for each $u \in H$, $u \notin K$. Moreover, let us consider the symmetry condition

$$(\mathrm{S}) \qquad (\beta'(u)(v), w) = (\beta'(u)(w), v) \quad \text{for all} \quad u, v, w \in H\,, \quad u \notin K\,.$$

Then (3.3) together with (b) and (S) gives

$$\dot{\lambda}_\varepsilon = 2\,\lambda_\varepsilon(\dot{u}_\varepsilon, u_\varepsilon) - (\beta u_\varepsilon, u_\varepsilon)\,.$$

But we have

$$2(\dot{u}_\varepsilon, u_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\|u_\varepsilon\|^2) = 0$$

with respect to (a), i.e.

$$\dot{\lambda}_\varepsilon = -\,(\beta u_\varepsilon, u_\varepsilon) \leqq 0\,.$$

Hence, the function λ_ε from Theorem 3.1 or 3.2 is decreasing. Remember that if β is positive homogeneous, then the assumption (NL) is fulfilled with $R = +\infty$, $\Lambda_1 = 0$, $\Lambda_2 = +\infty$ automatically (see Remark 2.4). Moreover, it is easy to see from the proof of Theorems 3.1, 3.2 that with respect to the monotonicity of λ_ε in our special situation, the eigenvalue $\lambda^{(2)}$ is unnecessary in the assumptions, because $\lambda^{(0)}$ itself is an upper bound for λ_ε. More precisely, using this remark, we can prove the following assertion in the same way as Theorem 3.2.

Theorem 3.3. *Let $\lambda^{(0)}$ be an eigenvalue of A corresponding to an eigenvector $u^{(0)} \notin K$, let $\lambda^{(1)} \in \Lambda_i$, $0 < \lambda^{(1)} < \lambda^{(0)}$. Suppose that β is positive homogeneous, the assumptions* (CC), (D), (P), (β, K^0), (SE), (M), (BC), (S) *are fulfilled and $\Lambda_b \cap \langle\lambda^{(1)}, \lambda^{(0)}\rangle = \emptyset$. Then there exist differentiable functions λ_ε, u_ε on $\langle 0, +\infty)$ such that $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$, λ_ε is decreasing and* (a)–(d) *with $\lambda^{(2)} = \lambda^{(0)}$ are valid. We have $\lim_{\varepsilon \to +\infty} \lambda_\varepsilon = \lambda_\infty^{(0)}$, where $\lambda_\infty^{(0)} \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$. There exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to +\infty$, $u_{\varepsilon_n} \to u_\infty^{(0)}$, where $u_\infty^{(0)} \in \partial K$ is an eigenvector corresponding to $\lambda_\infty^{(0)}$. If $\{\varepsilon_n\}$ is an arbitrary sequence, $\varepsilon_n > 0$, $\varepsilon_n \to +\infty$, $u_{\varepsilon_n} \rightharpoonup u_\infty$ for some u_∞, then $u_{\varepsilon_n} \to u_\infty$, $u_\infty \in \partial K$ and u_∞ is again an eigenvector corresponding to $\lambda_\infty^{(0)}$.*

Remark 3.3. Theorems 3.2 and 3.3 give a "new" eigenvector $u_\infty^{(0)}$ of the variational inequality, which is not an eigenvector of A. It follows from the assumption that $\Lambda_b \cap \langle\lambda^{(1)}, \lambda^{(2)}\rangle = \emptyset$ and $\Lambda_b \cap \langle\lambda^{(1)}, \lambda^{(0)}\rangle = \emptyset$, respectively. In general, it does not follow from our consideration that $\lambda_\infty^{(0)}$ is a new eigenvalue: It can happen that $\lambda_\infty^{(0)} \in \Lambda_{V,b} \cap \Lambda_e$, i.e. $\lambda_\infty^{(0)}$ is simultaneously an eigenvalue of A corresponding to the eigenvectors $\tilde{u} \notin K$, $-\tilde{u} \notin K$, $\pm\tilde{u} + u_\infty^{(0)}$, which cannot be eigenvectors of (I), (II) (cf. [4, Remark 1.2, Example 1.1]). Nonetheless, in the examples, Theorem 3.3 is applicable only if K is a halfspace (i.e. $n = 1$ in the examples of the type mentioned in Remark 1.1) and in this case there are only interior eigenvalues of A in $\langle\lambda^{(1)}, \lambda^{(0)}\rangle$ because $\tilde{u} \notin K$, $-\tilde{u} \notin K$ is not possible. Thus, with respect to the simplicity of interior eigenvalues of (I), (II) (see [4, Remark 1.4]), Theorem 3.3 yields also a "new" eigenvalue $\lambda_\infty^{(0)}$ of (I), (II) which is not an eigenvalue of A, i.e. $\lambda_\infty^{(0)} \in \Lambda_{V,b}$, $\lambda_\infty^{(0)} \notin \Lambda_A$.

Remark 3.4. It is clear that if $u \in K^0$, then $-u \notin K$. Hence, we can choose an arbitrary point from Λ_i for $\lambda^{(0)}$ in Theorems 3.2, 3.3. Especially, if the set Λ_i contains infinitely many positive numbers, then the set $\Lambda_{V,b}$ contains infinitely many points converging to zero. Indeed, there are two possibilities: 1) Λ_b contains infinitely many points and these are also in $\Lambda_{V,b}$ (see [4, Remark 1.2]); 2) Λ_b is a finite set and then Theorem 3.2 (or 3.3) can be used for infinitely many triplets $\lambda^{(1)}, \lambda^{(0)}, \lambda^{(2)}$ (or couples $\lambda^{(1)}, \lambda^{(0)}$) of eigenvalues. An arbitrary infinite sequence of eigenvalues of A converges to zero and therefore the resulting sequence of eigenvalues from $\Lambda_{V,b}$ converges to zero as well (because we have either $\lambda_\infty^{(0)} \in (\lambda^{(1)}, \lambda^{(2)})$ or $\lambda_\infty^{(0)} \in (\lambda^{(1)}, \lambda^{(0)})$).

An analogous assertion will be discussed in detail later in a more general case (Theorem 4.3, Remark 4.4).

Remark 3.5. It is easy to see that the assertion of Theorem 3.3 is true if the assumption (SE) is replaced by

(SE′) for each fixed $u \in H \setminus K$, $\varepsilon > 0$, a linear operator $\beta'(u)$ is symmetric and all eigenvalues of $A - \varepsilon\beta'(u)$ lying in $(\lambda^{(1)}, \lambda^{(0)})$ are simple.

Similarly for Theorem 3.2. In this case we can suppose $\lambda^{(1)} < \lambda^{(0)} < 0$ instead of $0 < \lambda^{(1)} < \lambda^{(0)}$ in Theorem 3.3. Similarly for Theorem 3.2 and for all the previous assertions.

4. NON-DIFFERENTIABLE PENALTY OPERATORS

In this section we shall consider a penalty operator β which is not differentiable on $H \setminus K$. We shall suppose that there exists a sequence $\{\beta_n\}$ of differentiable operators in H such that each of them satisfies roughly the assumptions of the previous considerations and such that the following convergence condition is fulfilled for this sequence:

(SCC) if $\{u_n\}$ is bounded, then $\{\beta_n u_n\}$ contains a strongly convergent subsequence; if $u_n \to u$, then $\beta_n u_n \to \beta u$.

Theorem 4.1. *Suppose that $\lambda^{(1)}, \lambda^{(2)} \in \Lambda_i$ and that $\lambda^{(0)} \in \Lambda_A$ is an eigenvalue corresponding to the eigenvector $u^{(0)} \notin K$ of A, $0 < \lambda^{(1)} < \lambda^{(0)} < \lambda^{(2)}$, $\Lambda_b \cap \langle\lambda^{(1)}, \lambda^{(2)}\rangle = \emptyset$. Suppose that β, β_n $(n = 1, 2, \ldots)$ are operators satisfying* (SCC), *β_n for each fixed n fulfils the assumptions* (P), (D), (CC), (β, K⁰), (SE) *and β satisfies the conditions* (P) *and* (β, K⁰). *Suppose that for each $R > 0$ there exists n_0 such that the condition* (NL) *is valid with R, $\Lambda_1 = \lambda^{(1)}$, $\Lambda_2 = \lambda^{(2)}$ and with β_n*

instead of β for each fixed $n \geqq n_0$. Then for each $\varepsilon \geqq 0$ there exists at least one couple λ_ε, u_ε satisfying the conditions (a)–(d).

Proof. Let ε be an arbitrary fixed positive number. We can take n_0 such that (NL) holds with $R = \varepsilon + 1$, $\Lambda_1 = \lambda^{(1)}$, $\Lambda_2 = \lambda^{(2)}$ for each β_n, $n \geqq n_0$. Theorem 3.1 implies that there exist $\lambda_{n,\varepsilon}, u_{n,\varepsilon}$ such that $\|u_{n,\varepsilon}\| = 1$, $u_{n,\varepsilon} \notin K$, $\lambda^{(1)} < \lambda_{n,\varepsilon} < \lambda^{(2)}$ and

$$\lambda_{n,\varepsilon} u_{n,\varepsilon} - A u_{n,\varepsilon} + \varepsilon \beta_n u_{n,\varepsilon} = 0 \tag{4.1}$$

($n \geqq n_0$). With respect to the assumption (SCC), there is a subsequence $\{r_n\}$ of indices such that the sequence $\{\beta_{r_n} u_{r_n,\varepsilon}\}$ is strongly convergent. Moreover, we can suppose $\lambda_{r_n,\varepsilon} \to \lambda_\varepsilon$, $u_{r_n,\varepsilon} \rightharpoonup u_\varepsilon$ (as $n \to +\infty$) for some $\lambda_\varepsilon \in \langle \lambda^{(1)}, \lambda^{(2)} \rangle$, $u_\varepsilon \in H$. It follows from here and (4.1) (in virtue of the complete continuity of A) that $u_{r_n,\varepsilon} \to u_\varepsilon$. This implies (a). Passing to limit $n \to +\infty$ in (4.1) and using (SCC) we obtain (b). We have $u_{r_n,\varepsilon} \notin K$, $u_{r_n,\varepsilon} \to u_\varepsilon$ and therefore $u_\varepsilon \notin K^0$. However, the case $u_\varepsilon \in \partial K$ is not possible. Indeed, if $u_\varepsilon \in \partial K$, then we have $\beta u_\varepsilon = 0$ and the equation (b) implies that $\lambda_\varepsilon \in \Lambda_b$, which contradicts the assumption that $\langle \lambda^{(1)}, \lambda^{(2)} \rangle \cap \Lambda_b = \emptyset$. Hence (c) is proved. If (d) is not valid then we have $\lambda_\varepsilon = \lambda^{(i)}$ with $i = 1$ or $i = 2$. Using the same argument as in the proof of Lemma 3.2, we obtain a contradiction, which proves (d).

Theorem 4.2. *Let us consider the situation of Theorem* 4.1. *Moreover, suppose that the operator β satisfies the conditions* (M), (BC). *Then there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$,*

$$\varepsilon_n \to +\infty\,, \quad \lambda_{\varepsilon_n} \to \lambda_\infty^{(0)}\,, \quad u_{\varepsilon_n} \to u_\infty^{(0)}\,, \quad \text{where} \quad \lambda_\infty^{(0)} \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(2)})$$

and $u_\infty^{(0)} \in \partial K$ is the corresponding eigenvector of (I), (II). *If $\{\varepsilon_n\}$ is an arbitrary sequence such that $\varepsilon_n > 0$, $\varepsilon_n \to +\infty$, $\lambda_{\varepsilon_n} \to \lambda_\infty$, $u_{\varepsilon_n} \rightharpoonup u_\infty$, then also $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(2)})$ and u_∞ is the corresponding eigenvector of* (I), (II), *$u_\infty \in \partial K$, $u_{\varepsilon_n} \to u_\infty$.*

Proof follows immediately from Theorem 4.1 and Lemma 3.3.

Remark 4.1. Theorem 4.2 gives a "new" eigenvector $u_\infty^{(0)}$ of the variational inequality, i.e. $u_\infty^{(0)}$ cannot be an eigenvector of the operator A. It is a consequence of the assumption that $\Lambda_b \cap \langle \lambda^{(1)}, \lambda^{(2)} \rangle = \emptyset$. Nonetheles, it does not follow from our considerations that $\lambda_\infty^{(0)}$ is a "new" eigenvalue of (I), (II): it is possible that $\lambda_\infty^{(0)} \in \Lambda_{V,b} \cap \Lambda_e$, i.e. it is possible that there exists an eigenvector $\tilde{u}$ of A corresponding to $\lambda_\infty^{(0)}$, $\tilde{u} \notin K$, $-\tilde{u} \notin K$, $\pm \tilde{u} \neq u_\infty^{(0)}$, which is not simultaneously an eigenvector of (I), (II) (see [4, Remark 1.2, Example 1.1]; cf. Remark 3.3).

Remark 4.2. If we know that $(\lambda^{(1)}, \lambda^{(2)}) \cap \Lambda_e = \emptyset$ (i.e., in particular, $-u^{(0)} \in K^0$), then the situation mentioned in Remark 4.1 is impossible. With respect to the simplicity of interior eigenvalues of the variational inequality (see [4, Remark 1.4]) $\lambda_\infty^{(0)}$ can not be an eigenvalue of A, i.e. Theorem 4.2 gives a "new" eigenvalue and a "new" eigenvector of (I), (II).

Remark 4.3. Let us suppose that the assumptions of Theorem 4.2 are satisfied. Moreover, suppose that $\lambda^{(0)} \in \Lambda_e$ and denote by u the corresponding eigenvector of A. Then we have $u \notin K$, $-u \notin K$. If we set $u^{(0)} = u$ and $u^{(0)} = -u$ in Theorem 4.2, respectively, then we can obtain the same eigenvalue $\lambda_\infty^{(0)}$ and eigenvector $u_\infty^{(0)}$.'

Theorem 4.3. *Let us assume that β, β_n $(n = 1, 2, \ldots)$ are operators satisfying* (SCC), *β_n for each fixed n fulfils the assumptions* (P), (D), (CC), (β, K⁰), (SE) *and β satisfies the conditions* (P), (β, K⁰), (M), (BC). *Assume that for each $R > 0$, $\Lambda_2 > \Lambda_1 > 0$ there exists n_0 such that the condition* (NL) *is valid with R, Λ_1, Λ_2 and with β_n instead of β for each fixed $n \geqq n_0$. Suppose that the set $\Lambda_i \cup \Lambda_b \cap (0, +\infty)$ contains infinitely many points. Then there exists an infinite sequence of boundary eigenvalues of* (I), (II) *converging to zero.*

Proof. If Λ_b contains an infinite sequence, then the assertion is clear. Let us suppose that Λ_b is a finite point set. Under our assumptions, there is an infinite sequence $\{\lambda_n\} \subset \Lambda_i$, $\lambda_n > \lambda_{n+1}$, $n = 1, 2, \ldots$. For each n there exists an eigenvalue $\lambda_n^v \in \Lambda_{V,b} \cap (\lambda_{n+2}, \lambda_n)$, $n = 1, 2, \ldots$. Indeed, if (for a given n) there exists $\tilde{\lambda}_n \in \Lambda_b \cap (\lambda_{n+2}, \lambda_n)$, then we can set $\lambda_n^v = \tilde{\lambda}_n$) in virtue of $\Lambda_b \subset \Lambda_{V,b}$. If (for a given n) $\Lambda_b \cap (\lambda_{n+2}, \lambda_n) = \emptyset$, then also $\Lambda_b \cap \langle \lambda_{n+2}, \lambda_n \rangle = \emptyset$ (because $\lambda_n \in \Lambda_i$, $\Lambda_i \cap \Lambda_b = \emptyset$) and we can use Theorem 4.2 with $\lambda^{(1)} = \lambda_{n+2}$, $\lambda^{(0)} = \lambda_{n+1}$, $\lambda^{(2)} = \lambda_n$, $u^{(0)} = -u_{n+1}$, where $u_{n+1} \in K^0$ is an eigenvector of A corresponding to λ_{n+1}, and set $\lambda_n^v = \lambda_\infty^{(0)}$. With respect to the fact that each infinite sequence of eigenvalues of A converges to zero, the resulting sequence of boundary eigenvalues of (I), (II) converges to zero as well.

Remark 4.4. Suppose that Λ_i contains infinitely many positive numbers and Λ_b is a finite point set. Then Theorem 4.3 ensures the existence of infinitely many "new" eigenvectors of (I), (II), which are not eigenvectors of A. This follows from the proof of Theorem 4.3 and Remark 4.1. If, moreover, Λ_e is a finite point set, then Theorem 4.3 ensures also the existence of an infinite sequence of "new" boundary eigenvalues of (I), (II), which are not eigenvalues of A. This sequence converges to zero. This follows from the proof of Theorem 4.3 and Remark 4.2. It is easy to see that the assumptions can be modified for the investigation of negative eigenvalues (cf. Remark 3.5).

5. VERIFICATION OF THE CONDITION (NL). EXAMPLES

First, we shall consider the general operators β_n, β from the previous section. Moreover, we shall assume that the following conditions are fulfilled:

(5.1) if $\{u_n\}$ is bounded, then there exists a sequence $\{r_n\}$ of indices such that $r_n \to +\infty$, $\beta'_{r_n}(u_n) \to f$ in the strong operator topology, where $f : H \to H$ is a linear completely continuous symmetric operator such that all positive eigenvalues of $A - \varepsilon f$ are simple for each fixed $\varepsilon > 0$.

(5.2) if $\{u_n\}$, $\{v_n\}$ are bounded, then there exists a strongly convergent subsequence of $\{\beta'_{r_n}(u_n)(v_n)\}$.

Remark 5.1. If we consider the situation from Remark 1.1, then we can choose an arbitrary sequence $\{\alpha_n\}$, $\alpha_n \in (0, 1)$, $\alpha_n \to 0$ and take the operators β_{α_n} from Remark 2.1 as β_n $(n = 1, 2, \ldots)$, $\beta = \beta_0$. It is easy to see that (5.1), (5.2) are fulfilled. (The operator f is of the type $(f(u), v) = \sum_{i=1}^{n} c_i\, u(x_i)\, v(x_i)$ and the proof of the simplicity of eigenvalues of $A - \varepsilon f$ is the same as that in Remark 2.2.)

Remark 5.2. Let β be a positive α-homogeneous $(\alpha > 0)$ operator, i.e. $\beta(tu) = t^\alpha \beta u$ for all $t > 0$, $u \in H$. If β is differentiable in the sense of Fréchet at a point $u \in H$, then $\alpha\beta u = \beta'(u)(u)$. This follows from the definition of the derivative.

Lemma 5.1. *Let the operators* β, β_n $(n = 1, 2, \ldots)$ *satisfy the assumptions* (SCC), (5.1), (5.2). *Suppose that the operator* β_n *is positie* $(1 + \alpha_n)$*-homogeneous with* $\alpha_n > 0$, $(n = 1, 2, \ldots)$, $\alpha_n \to 0$. *Then for each* Λ_1, Λ_2, R, $\Lambda_2 > \Lambda_1 > 0$, $R > 0$ *there exists* n_0 *such that* (NL) *is satisfied with* Λ_1, Λ_2, R *and with* β_n *instead of* β *for an arbitrary fixed* $n \geqq n_0$.

Proof. Let us suppose the contrary. Then there exist $\Lambda_2 > \Lambda_1 > 0$ $R > 0$, $\lambda_n \in \langle \Lambda_1, \Lambda_2 \rangle$, $\varepsilon_n \in \langle 0, R \rangle$, $u_n \in H \setminus K$, $v_n \in H$, $\mu_n \in R$, $r_n > 0$ $(n = 1, 2, \ldots)$ such that $\|u_n\| = \|v_n\| = 1$, $(u_n, v_n) = 0$, $r_n \to +\infty$ and

$$\lambda_n u_n - A u_n + \varepsilon_n \beta_{r_n} u_n = 0\,, \tag{5.3}$$

$$\lambda_n v_n - A v_n + \varepsilon_n \beta'_{r_n}(u_n)(v_n) = \mu_n u_n\,. \tag{5.4}$$

We can suppose $\lambda_n \to \lambda$, $\varepsilon_n \to \varepsilon$. With respect to the assumption (SCC) and the complete continuity of A we can suppose that the sequences $\{\beta_{r_n} u_n\}$, $\{A u_n\}$ are strongly convergent and therefore (5.3) implies $u_n \to u$ for some $u \in H$. Using the assumption (5.2) we can suppose that $\{\beta'_{r_n}(u_n)(v_n)\}$ is strongly convergent. The equation (5.4) implies that μ_n is bounded and that we can suppose $v_n \to v$ for some $v \in H$. The operator β_n is supposed to be positive $(1 + \alpha_n)$-homogeneous and therefore (5.3) can be written as

$$\lambda_n u_n - A u_n + \varepsilon_n (1 + \alpha_{r_n})^{-1}\, \beta'_{r_n}(u_n)(u_n) = 0 \tag{5.3'}$$

(see Remark 5.2). With respekt to (5.1) we can suppose $\beta'_{r_n}(u_n) \to f$ and passing to the limit $n \to +\infty$ in (5.3'), (5.4) we obtain

$$\lambda u - A u + \varepsilon f(u) = 0\,, \tag{5.5}$$

$$\lambda v - A v + \varepsilon f(v) = \mu u\,. \tag{5.6}$$

However, the linear symmetric completely continuous operator $A - \varepsilon f$ has only simple eigenvalues (see (5.1)) and therefore $\lambda I - A + \varepsilon f$ maps H onto $u^{\perp}$ by the Fredholm alternative. Hence (5.6) can be satisfied only with $\mu = 0$, $v = tu$. But we have $(u, v) = 0$, $\|u\| = \|v\| = 1$ and this is a contradiction.

Example 5.1. Let us consider the example described in Remark 1.1. First let us consider the case $n = 1$, i.e., K is a halfspace. The assumptions of Theorem 3.3 are fulfilled for the operator $\beta = \beta_0$ defined in Remark 2.1. The assumption (SE) was discussed in Remark 2.2, (NL) follows from Remark 2.4, the other assumptions are obvious. The following assertion follows from here and from Remark 3.3: If $\lambda^{(1)}, \lambda^{(0)}$ $(0 < \lambda^{(1)} < \lambda^{(0)})$ are eigenvalues of A corresponding to eigenvectors $u^{(1)}, u^{(0)}$, $u^{(1)}(x_1) > 0$, $u^{(0)}(x_1) < 0$ and if $u(x_1) \neq 0$ for all eigenvectors corresponding to the eigenvalues $\lambda \in \langle \lambda^{(1)}, \lambda^{(0)} \rangle$ of A, then we obtain a "new" eigenvalue $\lambda_{\infty}^{(0)} \in (\lambda^{(1)}, \lambda^{(0)})$ and the corresponding "new" eigenvector $u_{\infty}^{(0)}$ of (I), (II) such that $u_{\infty}^{(0)}(x_1) = 0$, which are not an eigenvalue and an eigenvector of A. Further, Remark 3.4 implies that there exists a sequence converging to zero of eigenvalues of (I), (II) such that the corresponding eigenvectors satisfy the condition $u(x_1) = 0$.

Let us consider the case $n > 1$. Choose $\alpha_n \in (0, 1)$ $(n = 1, 2, \ldots)$ such that $\alpha_n \to 0$. If we take the operators β_{α_n} defined by (2.1) as β_n and $\beta = \beta_0$, then these operators satisfy the assumptions of Theorems 4.2, 4.3. The assumption (SE) was verified in Remark 2.2, the assumption concerning the condition (NL) follows from Lemma 5.1 and the other assumptions are obviously fulfilled. It follows from here and from Remarks 4.1, 4.2 that the following assertion is true:

If $\lambda^{(1)}, \lambda^{(0)}, \lambda^{(2)}$ $(0 < \lambda^{(1)} < \lambda^{(0)} < \lambda^{(2)})$ are eigenvalues of A corresponding to eigenvectors $u^{(1)}, u^{(0)}, u^{(2)}$ such that

$$u^{(1)}(x_i) > 0\,, \quad u^{(2)}(x_i) > 0 \quad \text{for all} \quad i = 1, \ldots, n\,,$$
$$u^{(0)}(x_j) < 0 \quad \text{at least for one } j$$

and if there is no eigenvector u corresponding to the eigenvalue $\lambda \in \langle \lambda^{(1)}, \lambda^{(2)} \rangle$ of A and satisfying the conditions

$$u(x_i) \geqq 0 \quad \text{for all} \quad i = 1, \ldots, n\,, \quad u(x_j) = 0 \quad \text{at least for one j}\,,$$

then there exists an eigenvalue $\lambda_{\infty}^{(0)} \in (\lambda^{(1)}, \lambda^{(2)})$ of (I), (II) and a "new" eigenvector $u_{\infty}^{(0)}$ of (I), (II) corresponding to $\lambda_{\infty}^{(0)}$ such that

$$u_{\infty}^{(0)}(x_j) = 0 \quad \text{at least for one } j\,.$$

This eigenvector is not an eigenvector of A. If, moreover, there is no eigenvector of A corresponding to the eigenvalues $\lambda \in \langle \lambda^{(1)}, \lambda^{(2)} \rangle$ satisfying the condition $u(x_i) < 0$ at least for one i; $u(x_j) > 0$ at least for one j (i.e. there are only interior eigenvalues in $\langle \lambda^{(1)}, \lambda^{(2)} \rangle$), then $\lambda_{\infty}^{(0)}$ is a "new" eigenvalue, i.e. it is not an eigenvalue of A. Further, if there is an infinite sequence of eigenvectors of A satisfying the condition

$$u(x_i) > 0 \quad \text{for all} \quad i = 1, \ldots, n\,,$$

then Theorem 4.3 ensures the existence of a sequence of eigenvalues of (I), (I)I converging to zero, such that the corresponding eigenvectors satisfy the condition

$$u(x_i) = 0 \quad \text{at least for one } i\,.$$

Notice that our theory has only a theoretical significance for the example just discussed. If we can calculate the eigenvalues and eigenvectors of A (for example if a, b are constants) then we can calculate also the eigenvalues and eigenvectors of (I), (II), because they can be obtained from the eigenvalues and eigenvectors of the same boundary value problem on the intervals (x_i, x_{i+1}) (see [6, Section 1]).

Example 5.2. Let a, b be real functions on $\langle 0, 1\rangle$. Suppose that a is two-times continuously differentiable, $a \geqq \gamma > 0$ on $\langle 0, 1\rangle$, b is continuously differentiable, $b \geqq 0$ on $\langle 0, 1\rangle$. Set

$$H = \{W_2^2(0, 1);\; u(0) = u(1) = 0\}$$

and introduce the inner product

$$(u, v) = \int_0^1 au''v''\,\mathrm{d}x \quad \text{for all} \quad u, v \in H\,,$$

which is equivalent with the usual inner product on H. Let us define the operator A by the formula

$$(Au, v) = \int_0^1 bu'v'\,\mathrm{d}x \quad \text{for all} \quad u, v \in H\,.$$

Set

$$K = \{u \in H;\; u(x_0) \geqq 0\}\,,$$

where $x_0 \in (0, 1)$ is a given point. The variational inequality (I), (II) is equivalent with a boundary value problem for an ordinary differential equation of the fourth order with certain "transmission conditions" at x_0. This problem describes a beam which is simply fixed on its ends and compressed by a force proportional to the value $1/\lambda$ (cf. [4, Section 4]). We shall use the operator β_0 defined by (2.1) and Theorem 3.3 together with Remark 3.5. It is easy to see that all the assumptions are fulfilled (analogously as in Example 5.1) with the exception of the assumption (SE). Let us assume that a real λ is an eigenvalue of $A - \varepsilon\,\beta'(u)$ for some $\varepsilon > 0$, $u \in H \setminus K$ and that λ is not simple. Then there exist linearly independent vectors w_1, w_2 such that

$$\lambda \int_0^1 aw_i''v''\,\mathrm{d}x - \int_0^1 bw_i'v'\,\mathrm{d}x + \varepsilon\, u(x_0)\, w_i(x_0)\, v(x_0) = 0 \quad \text{for all} \quad v \in H\,,$$

$i = 1, 2$ (cf. Remark 2.2). We can choose a nontrivial combination $w = c_1w_1 + c_2w_2$ such that $w(x_0) = 0$ and thus

$$\lambda \int_0^1 aw''v''\,\mathrm{d}x - \int_0^1 bw'v'\,\mathrm{d}x = 0 \quad \text{for all} \quad v \in H\,,$$

i.e. $\lambda \in \Lambda_A$. Hence the condition (SE') is fulfilled for an arbitrary couple $\lambda^{(1)} = \lambda_{n+1}$, $\lambda^{(0)} = \lambda_n$, where $\{\lambda_n\}$ $(\lambda_{n+1} < \lambda_n)$ denotes the sequence of all eigenvalues of A. Moreover, it is well-known that A has only simple eigenvalues. If $\lambda_{n+1}, \lambda_n \in \Lambda_i$ for some n (i.e. $u_{n+1}(x_0) \neq 0$, $u_n(x_0) \neq 0$ for the corresponding eigenvectors u_n, u_{n+1}) then we can use Theorem 3.3 and Remark 3.5. We obtain a "new" eigenvalue $\lambda_n^\infty \in (\lambda_{n+1}, \lambda_n) \cap (\Lambda_{V,b} \setminus \Lambda_A)$ and a "new" eigenvector $u_n \in (E_V \setminus E_A) \cap \partial K$ for the couples $\lambda_n, \lambda_{n+1} \in \Lambda_i$. Let us remark that the eigenvalues of (I), (II) in this special case can be calculated on the basis of a method explained in [1]. The existence of an infinite sequence of eigenvalues of (I), (II) in the case of a general halfspace follows from [6, Section 3]. The case of the cone $K = \{u \in H;\ u(x_i) \geqq 0,\ i = 1, \ldots, n\}$ is nontrivial for $n > 1$ for the variational inequality of the fourth order, but the special results discussed in this paper cannot be used for it. It is possible to use more general theorems based on the bifurcation theory (see [4, Section 4]).

Added in proof. It is possible to show that the assumption $\Lambda_b \cap \langle \lambda^{(1)}, \lambda^{(2)} \rangle = \emptyset$ can be omited in all assertions in the case of penalty operators of the type (2.1) because the branch u_ε can not meet points $u \in \delta K \cap E_A$. Particularly, Theorem 4.3 ensures the existence of infinitely many points $u \in E_V \setminus E_A$ if Λ_i contains infinitely many points. Analogously in Examples 5.1, 5.2. It wil be explained in more general situation in [5].

REFERENCES

[1] *S. Fučík, J. Milota:* Linear and nonlinear variational inequalities on halfspaces. Comment. Math. Univ. Carol. *16,* 4 (1975), 663–682.

[2] *T. H. Hildebrandt, L. M. Graves:* Implicit function and their differentials in general analysis. Trans. Amer. Math. Soc. *29* (1927), 127–153.

[3] *M. Kučera:* A new method for obtaining eigenvalues of variational inequalities of the special type. Preliminary communication. Comment. Math. Univ. Carol. *18,* 1 (1977), 205–210.

[4] *M. Kučera:* A new method for obtaining eigenvalues of variational inequalities based on bifurcation theory. Časopis pro pěstování matematiky *104* (1979).

[5] *M. Kučera:* A new method for obtaining eigenvalues of variational inequalities. Operators with multiple eigenvalues. To appear.

[6] *M. Kučera, J. Nečas, J. Souček:* The eigenvalue problem for variational inequalities and a new version of the Ljusternik-Schnirelmann theory. In „Nonlinear Analysis", Academic Press, New York—San Francisco—London 1978.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).