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*Časopis pro pěstování matematiky*, Vol. 103 (1978), No. 4, 363--364

Persistent URL: `http://dml.cz/dmlcz/117994`

Časopis pro pěstování matematiky, roč. 103 (1978), Praha

# A PROPERTY OF ENTIRE TRANSCENDENTAL FUNCTIONS

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(Received November 17, 1976)

Let $\sum_{n=0}^{\infty} a_n z^n$ be an entire transcendental function and $g$ and $h$ two distinct complex numbers. In this paper it is shown that the set of all complex numbers for which a truncated part of $\sum_{n=0}^{\infty} a_n z^n$ takes on the value $g$ or $h$ has infinitely many accumulation points.

First we prove:

**Lemma.** *Let $v \neq 0$ be a zero of the entire transcendental function*

$$f(z) = -g + \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

*where $g$ is a complex number. Then in every neighborhood of $v$ there exists a zero $w$ of the truncated polynomial*

$$p_k(z) = -g + \sum_{n=0}^{k} a_n z^n \quad \textit{for some} \quad k < \infty \tag{2}$$

*such that $w \neq v$.*

Proof. Since $v$ is a zero of the entire transcendental function $f(z)$, we see that there exists a circumference $C$ of positive radius with center at $v$ such that $f(z)$ has no zeros on $C$. But then since $|f(z)|$ is a continuous function on $C$, it has a positive minimum $r$. Thus,

$$|f(z)| \geqq r > 0 \quad \text{for} \quad z \in C . \tag{3}$$

Clearly, $-g + \sum_{n=0}^{\infty} a_n z^n$ has uniform convergence on $C$ and therefore, for some $m < \infty$, in view of (1), (2), (3), we have:

$$|p_m(z)| + |f(z) - p_m(z)| \geqq r \quad \text{with} \quad |f(z) - p_m(z)| < \tfrac{1}{2}r \quad \text{for} \quad z \in C .$$

Consequently, $|p_m(z)| > |f(z) - p_m(z)|$ on $C$. But then since $f(z)$ has a zero in the disk $D$ whose boundary is $C$, by Rouché's theorem [1, p. 157], it follows that $p_m(z)$ must also have at least one zero $u$ in $D$. If $u \neq v$ then we take $k = m$ and $w = u$. If $u = v$ then let $k$ be the smallest natural number larger than $m$ such that $a_k \neq 0$. But then since $v \neq 0$, from the above it follows that $p_k(z)$ has a zero $w$ in $D$ such that $w \neq v$.

Next we prove:

**Theorem.** *Let* $\sum_{n=0}^{\infty} a_n z^n$ *be an entire transcendental function and* $g$ *and* $h$ *two distinct complex numbers. Let*

$$G = \{z \mid g = \sum_{n=0}^{k} a_n z^n \quad \textit{for some} \quad k < \infty\}$$

*and*

$$H = \{z \mid h = \sum_{n=0}^{k} a_n z^n \quad \textit{for some} \quad k < \infty\}$$

*Then the set* $G \cup H$ *has infinitely many accumulation points.*

Proof. Consider the entire transcendental function $f(z)$ given by (1). Since $h \neq g$, by Picard's big theorem [1, p. 341], at least one of the entire transcendental functions $f(z)$ or $-h + g + f(z)$ must have infinitely many distinct zeros. Without loss of generality, let $f(z)$ have infinitely many distinct zeros. But then, by the above Lemma, each such zero is an accumulation point of the set $G$ mentioned in the Theorem.

Thus, the Theorem is proved.

The author thanks Prof. Stuart A. Nelson for helpful discussions.

*Reference*

1] *Saks, S.* and *Zygmund, A.*, Analytic Functions, Warsaw, 1952.

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