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EXPRESSING $f \in \mathscr{D}$ AS A DIFFERENCE OF TWO POSITIVE FUNCTIONS $f_1, f_2 \in \mathscr{D}$

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The following unsolved problem was published in American Mathematical Monthly (7, 1975)*). Is it possible to express every function

$$f \in \mathscr{D} = \{f \in C^\infty(\mathrm{E}_1);\ \mathrm{f}(x) = 0 \text{ for each } x \in (-\infty, 0\rangle \cup \langle 1, \infty)\}$$

as a difference of two positive functions $f_1, f_2 \in \mathscr{D}$?

We shall prove here that the answer is affirmative. Let E_1 denote the space of real numbers, $\mathrm{C}^\infty(\mathrm{K}) = \{f : \mathrm{K} \to \mathrm{E}_1,\ f$ have continuous derivatives of all orders$\}$, where $\mathrm{K} = \mathrm{E}_1$ or $\mathrm{K} = \langle 0, 1\rangle$.

Let us denote by h an arbitrary function satisfying the following conditions:

1. $h \in \mathscr{D}$,
2. $\mathrm{h}(x) = \mathrm{h}(1 - x)$ for each $x \in \langle 0, 1\rangle$,
3. $\mathrm{h}(x) > 0$ for each $x \in (0, 1)$,
4. h is increasing on $\langle 0, \frac{1}{2}\rangle$.

(For example, take the function $\mathrm{h}(x) = e^{-1/x} \cdot e^{1/(x-1)}$ for $x \in (0, 1)$, $\mathrm{h}(x) = 0$ for $x \in \mathrm{E}_1 - (0, 1)$.)

By $h_{\varepsilon_1, a, \varepsilon_2}$ we shall denote an arbitrary function which has the following properties:

1. $\mathrm{h}_{\varepsilon_1, a, \varepsilon_2} \in C^\infty(\mathrm{E}_1)$,
2. $\mathrm{h}_{\varepsilon_1, a, \varepsilon_2}(x) = 1$ for each $x \in (-\infty, \varepsilon_1\rangle \cup \langle \varepsilon_2, \infty)$,
3. $\mathrm{h}^{(i)}_{\varepsilon_1, a, \varepsilon_2}(a) = 0$ for each $i \in N$,
4. $\mathrm{h}_{\varepsilon_1, a, \varepsilon_2}$ is decreasing (increasing) on $\langle \varepsilon_1, a\rangle$ (on $\langle a, \varepsilon_2\rangle$).

*) In the meantime a different solution of this problem was published in American Mathematical Monthly (3, 1977).

STUDENTS' RESEARCH ACTIVITY AT THE FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY. Awarded the 4th prize in the National Students' Research Work Competition, section Mathematical Analysis, in the year 1977. Scientific adviser: Professor V. SOUČEK.

Lemma 1. (On joining of functions.) *Let* $a \in (0, 1)$, $g_1 \in C^\infty\langle 0, a\rangle$, $g_1 \geqq 0$, $g_2 \in C^\infty\langle a, 1\rangle$, $g_2 \geqq 0$ *and* $0 \leqq \delta_1 < a < \delta_2 \leqq 1$. *Then there exists a function* $g \in C^\infty\langle 0, 1\rangle$ *such that*:

(1) $g \geqq g_1$ *on* $\langle 0, a\rangle$, $g \geqq g_2$ *on* $\langle a, 1\rangle$,

(2) $g = g_1$ *on* $\langle 0, \delta_1\rangle$, $g = g_2$ *on* $\langle \delta_2, 1\rangle$.

Proof. Choose $\varepsilon_1, \varepsilon_2$ such that

$$0 \leqq \delta_1 < \varepsilon_1 < a < \varepsilon_2 < \delta_2 \leqq 1 \quad \text{holds}.$$

Define

$$\begin{aligned} \tilde{g}(x) &= g_1(x) \quad \text{for each} \quad x \in \langle 0, a), \\ &= g_2(x) \quad \text{for each} \quad x \in \langle a, 1\rangle \end{aligned}$$

and put $\mathrm{d}(x) = \tilde{g}(x)\, \mathrm{h}_{\varepsilon_1, a, \varepsilon_2}(x)$.

We show by mathematical induction that $d \in C^\infty\langle 0, 1\rangle$. The "bad" point is a. The first step is easy. Now suppose that $d \in C^{n-1}\langle 0, 1\rangle$. We have

$$\lim_{x\to a} \mathrm{d}^{(n)}(x) = \lim_{x\to a} \sum_{i=1}^{n} \binom{n}{i} \tilde{g}^{(i)}(x)\, \mathrm{h}^{(n-i)}_{\varepsilon_1, a, \varepsilon_2}(x) = 0\,.$$

According to the well known theorem $d \in C^n\langle 0, 1\rangle$. Let us denote

$$\tilde{\mathrm{h}}(x) = \mathrm{h}\left(\frac{x - \delta_1}{\delta_2 - \delta_1}\right), \quad n = \min_{x \in \langle \varepsilon_1, \varepsilon_2\rangle} \tilde{\mathrm{h}}(x)$$

$m = \max_{x \in \langle \varepsilon_1, \varepsilon_2\rangle} \tilde{g}(x)$ and finally, put $g(x) = (d(x) + 1)\, ((m/n)\, \tilde{\mathrm{h}}(x) + 1) - 1$. Clearly $g \in C^\infty\langle 0, 1\rangle$ and $g(x) = d(x) = \tilde{g}(x)$ for $x \in \langle 0, 1\rangle - \langle \delta_1, \delta_2\rangle$. If $x \in \langle \delta_1, \delta_2\rangle - \langle \varepsilon_1, \varepsilon_2\rangle$ then $((m/n)\, \mathrm{h}(x) + 1) \geqq 1$ and $g \geqq d = \tilde{g}$. If $x \in \langle \varepsilon_1, \varepsilon_2\rangle$ then $((m/n) \,.\, \mathrm{h}(x) + 1) \geqq m + 1$ and $g \geqq m \geqq \tilde{g}$, hence the proof is complete.

Lemma 2. *Let* $\mathrm{f}(x) \in \mathscr{D}$, *then* $\mathrm{f}(x)/x^m \in \mathscr{D}$ *for every* $m \in N$.

Proof. Using the well known formula we have

$$\left(\frac{\mathrm{f}(x)}{x^m}\right)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \mathrm{f}^{(n-i)}(x) \left(\frac{1}{x^m}\right)^{(i)} =$$

$$= \sum_{i=0}^{n} \binom{n}{i} (-1)^i\, \mathrm{f}^{(n-i)}(x)\, \frac{m(m+1)\ldots(m+i-1)}{x^{m+i}}\,.$$

We see that to prove $\lim_{x\to 0} (\mathrm{f}(x)/x^m)^{(n)} = 0$ it is sufficient to show that $\lim_{x\to 0} \mathrm{f}(x)/x^k = 0$ for all $f \in \mathscr{D}$, $k \in N$.

However,

$$\lim_{x\to 0^+}\left|\frac{f(x)}{x^k}\right| = \lim_{x\to 0^+}\left|\frac{f(x)-f(0)}{x^k}\right| = \lim_{x\to 0^+}\left|\frac{x\,f'(\xi_x^1)}{x^k}\right| \quad (\text{where } 0 < \xi_x^1 < x) =$$

$$= \lim_{x\to 0^+}\left|\frac{x(f'(\xi_x^1)-f'(0))}{x^k}\right| = \lim_{x\to 0^+}\left|\frac{x\xi_x^1\,f''(\xi_x^2)}{x^k}\right| \quad (\text{where } 0 < \xi_x^2 < \xi_x^1 < x) =$$

$$= \ldots = \lim_{x\to 0^+}\left|\frac{x\xi_x^1 \ldots \xi_x^{k-1}\,f^{(k)}(\xi_x^k)}{x^k}\right| \quad (\text{where } 0 < \xi_x^k < \ldots < \xi_x^1 < x) \leqq$$

$$\leqq \lim_{x\to 0^+}|f^{(k)}(\xi_x^k)| = 0\,.$$

Lemma 3. *There exists a family of segments* $\{U_n\}_{n\in N}$ *satisfying the following conditions*:

1. $\bigcup_{n\in N} U_n \subset (0, 1)$.
2. *Denote*

$$U_n = \langle a_n, b_n\rangle\,,\quad \varepsilon_n = b_n - a_n\,,$$
$$U'_n = \langle a'_n, b'_n\rangle = \langle a_n + \varepsilon_n/3,\; b_n - \varepsilon_n/3\rangle\,.$$

   Then there exists $\delta > 0$ *such that* $(0, \delta) \subset \bigcup_{n\in N} U'_n$.
3. *There exists* $k \in N$ *such that for each* $x \in (0, \delta)$ *it holds* card $\{n, x \in U_n\} \leqq k$
4. *Define* $\Phi(x) = \sup\{y \in (0, 1),\; y \in \bigcup_{n\in N,\, x\in U_n} U_n\}$. *Then* $\lim_{x\to 0}\Phi(x) = 0$.
5. *There exists* $l \in N$ *such that for every* $n \in N$ *it holds*

$$\frac{1}{\varepsilon_n} \leqq \left(\frac{1}{b_n}\right)^l.$$

Proof. Put $U'_n = \langle 1/(n+1), 1/n\rangle$ for $n \in N$, $a'_n = 1/(n+1)$, $b'_n = 1/n$,

$$a_n = \frac{n-1}{n(n+1)}\,,\quad b_n = \frac{n+2}{n(n+1)}\,,\quad \varepsilon_n = \frac{3}{n(n+1)}\,.$$

First we find a suitable number $n_0 \in N$ such that the sequences $\{a_n\}$, $\{b_n\}$ are for $n > n_0$ decreasing. It is sufficient to investigate the functions $(x-1)/x(x+1)$ and $(x+2)/x(x+1)$ for $x \to \infty$. Further, let n_1 denote a positive integer such that for all $n > n_1$ it holds

$$(*)\qquad \frac{1}{\varepsilon_n} = \frac{n(n+1)}{3} \leqq \left(\frac{n(n+1)}{n+2}\right)^3 = \left(\frac{1}{b_n}\right)^3.$$

For the family $\{U_n\}$ we take the set of segments $\langle a_n, b_n\rangle$, where $n \geqq \max(n_0, n_1)$. The conditions 1 and 2 are obviously satisfied. The inequality $(*)$ implies the property 5 with $l = 3$.

For fixed $x \in (0, \delta)$ we shall study the set

$$\mathrm{N}(x) = \{n \in N,\ n \geqq \max(n_0, n_1),\ a_n \leqq x \leqq b_n\}\,.$$

This requires to solve the following inequalities.

$$\frac{n-1}{n(n+1)} \leqq x$$

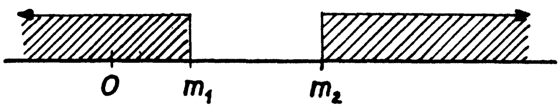
$$m_{1,2} = \frac{(1-x) \pm \sqrt{[(x-1)^2 - 4x]}}{2x}$$

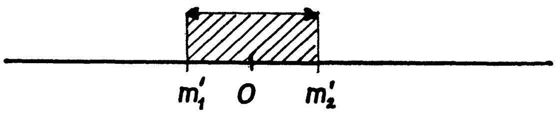
$$x \leqq m_1 \quad \text{or} \quad x \geqq m_2$$

$$x \leqq \frac{n+2}{n(n+1)}$$

$$m'_{1,2} = \frac{(1-x) \pm \sqrt{[(x-1)^2 + 8x]}}{2x}$$

$$m'_1 \leqq x \leqq m'_2$$





We show that there does not exist $n \in \mathrm{N}(x)$ such that $m'_1 \leqq n \leqq m_1$. Assume the contrary. Denote $i \in \mathrm{N}(x)$, $m'_1 \leqq i \leqq m_1$ and choose $j, k \in N$ such that $m_1 < j < m_2$ and $m_2 \leqq k \leqq m'_2$. This is possible since

$$(**) \qquad \xi(x) = m'_2 - m_2 = \frac{\sqrt{[(x-1)^2 + 8x]} - \sqrt{[(x-1)^2 - 4x]}}{2x} =$$

$$= \frac{6}{\sqrt{[(x-1)^2 + 8x]} + \sqrt{[(x-1)^2 - 4x]}} \to 3 \quad (x \to 0)$$

and

$$m_2 - m_1 = \frac{\sqrt{[(x-1)^2 - 4x]}}{x} \to \infty \quad (x \to 0)\,.$$

We obtain $x \in U_i \cap U_k$ & $x \notin U_j$ together with $i < j < k$ and this is contradiction since $\{a_n\}, \{b_n\}$ are decreasing. It follows from $(**)$ that there exists a suitable number $\eta \in (0, 1)$ such that $\xi(x) \leqq 4$ for each $x \in (0, \eta)$ and this proves the property 3.

Suppose now that $x \in (0, \eta)$ is a fixed point and investigate the function

$$m(x) = \min\{n, (n-1)/n(n+1) = a_n \leqq x,\ n \geqq \max(n_0, n_1)\}\,.$$

Since $m_1 < \max(n_0, n_1)$ for each $x \in (0, \eta)$ we obtain $m(x) = [m_2] + 1$.

$$\Phi(x) = \frac{m(x) + 2}{m(x)(m(x) + 1)} = \frac{[m_2] + 3}{([m_2] + 1)([m_2] + 2)} \to 0 \quad (x \to 0)$$

while $[m_2] \to \infty$ $(x \to 0)$ and the proof of the condition 4 is complete.

Theorem. *For every $f \in \mathcal{D}$ there exist functions $f_1, f_2 \in \mathcal{D}$, $f_1 \geqq 0$, $f_2 \geqq 0$ such that $f = f_1 - f_2$. We can also say that $\mathcal{D}$ is generated as a vector space by its positive functions.*

Proof. Let $f \in \mathcal{D}$ be an arbitrary function. If we find $g \in \mathcal{D}$, $g \geqq 0$, $g \geqq f$ then we can write $f = g - (g - f)$, $g = f_1$, $g - f = f_2$. To prove our theorem we show that there exist $\delta > 0$ and $\tilde{g} \in \mathcal{D}$ such that $\tilde{g} \in \mathcal{D}$, $\tilde{g} \geqq 0$, $\tilde{g} \geqq f$ on $(-\infty, \delta)$. Since the space $\mathcal{D}$ has the same behavior at 0 and 1 we complete the proof by means of the joining lemma. Let $\{U_n\}_{n \in N}$ be the family of segments satisfying the conditions of the preceding lemma and h our standard function. For all $n \in N$ we define

$$g_n^{(x)} = h\left(\frac{x - a_n}{\varepsilon_n}\right) \frac{\max_{x \in U_n'} |f(x)|}{h(\frac{1}{3})}\,.$$

Clearly $g_n \in \mathcal{D}$. It will be useful to express

$$g_n^{(i)}(x) = \left(\frac{1}{\varepsilon_n}\right)^i h^{(i)}\left(\frac{x - a_n}{\varepsilon_m}\right) \frac{\max_{x \in U_n'} |f(x)|}{h(\frac{1}{3})}$$

and by Lemma 3 we have

$$|g_n^{(i)}(x)| \leqq \frac{\|h^{(i)}\|}{h(\frac{1}{3})} \frac{|f(x_n)|}{b_n^{i.i}} \leqq \frac{\|h^{(i)}\|}{h(\frac{1}{3})} \frac{|f(x_n)|}{x_n^{i.i}}$$

where $\|h^{(i)}\| = \sup_{x \in \langle 0,1 \rangle} |h^{(i)}(x)|$, $|f(x_n)| = \max_{x \in U_n'} |f(x)|$. Put $\tilde{g} = f + \sum_{n \in N} g_n$. By the condition 3 of Lemma 3 $\tilde{g}(x) < \infty$ for all $x \in E_1$. It is easy to see that $\tilde{g} \geqq 0$ and $\tilde{g} \geqq f$ on $(0, \delta)$ where δ is the same as in Lemma 3. The proof will be complete if we prove that $\tilde{g}_+^{(i)}(0) = 0$ for all $n \in N$. But

$$|\tilde{g}^{(i)}(x)| \leqq |f^{(i)}(x)| + \sum_{\substack{n \in N \\ x \in U_n}} |g_n^{(i)}(x)| \leqq$$

$$\leqq |f^{(i)}(x)| + \frac{\|h^{(i)}\|}{h(\frac{1}{3})} \sum_{\substack{n \in N \\ x \in U_n}} \frac{|f(x_n)|}{x_n^{i.i}} \to 0 \quad (x \to 0)$$

by Lemma 3 (conditions 3 and 4) and Lemma 5.

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