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EXTENSION OF A HOMEOMORPHISM OF A TOPOLOGICAL CIRCUMFERENCE

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In the present paper we prove that any homeomorphic mapping h of a topological circumference $T \subset \mathbf{S}$ into $\mathbf{S}$ ($\mathbf{S}$ being the closed Gaussian plane) can be extended to a homeomorphic mapping of the whole $\mathbf{S}$ onto $\mathbf{S}$. The only more advanced results used from the topology of the plane are the Jordan theorem and the theorem on the θ-curves.

The result just formulated is of crucial importance for the topology of the plane as well as for its applications e.g. in the theory of functions of a complex variable. It implies immediately a.o. the theorem on accessibility of all points of the boundary $\partial\Omega$ of a Jordan region Ω from the region as well as other analogous theorems (concerning the outer topological properties of topological circumferences) useful in the theory of conformal mappings and other fields.

However, the application of the above theorem in the mentioned direction in elementary courses of the theory of functions is essentially hindered by the fact that the usual proofs given in the literature are based a.o. on the theorem on accessibility of boundary points of a Jordan region Ω from Ω (cf. e.g. [1], pp. 374–381). The present paper offers an almost elementary proof of the theorem, showing how the proof from [1] can be modified not only to remove the above mentioned drawback but also to avoid the use of other theorems, not widely known and difficult to prove.

Let us first introduce the necessary notation, definitions and theorems. The *closed Gaussian plane* is denoted by $\mathbf{S}$, the *open Gaussian plane* by $\mathbf{E}$. The *boundary* of a set $M \subset \mathbf{S}$ will be denoted by ∂M, the *closure* by $\overline{M}$. If $\emptyset \neq M \subset \mathbf{E}$, then

$$\operatorname{diam} M = \sup_{z', z'' \in M} |z' - z''| ;$$

besides, we put $\operatorname{diam} \emptyset = 0$. If $\operatorname{diam} M < \infty$, then M is said to be a *bounded* set. If $\emptyset \neq M_i \subset \mathbf{E}$ for $i = 1, 2$, then

$$\operatorname{dist}(M_1, M_2) = \inf_{z' \in M_1, z'' \in M_2} |z' - z''| .$$

If for instance $M_1 = \{a\}$ is a one-point set, then we write dist (a, M_2) instead of dist $(\{a\}, M_2)$.

If $M \subset \mathbf{S}$ is either a closed or an open set and if $p, q \in \mathbf{S} - M$ are two points, we say *M separates the points p, q in $\mathbf{S}$*, if p, q belong to different components of the set $\mathbf{S} - M$. (Cf. [2], p. 108.)

Let us recall the Janiszewski theorem (see e.g. [2], p. 172): *If the sets M_1, M_2 ($\subset \mathbf{S}$) are either both closed or both open, if p, q are two points from $\mathbf{S} - (M_1 \cup M)$, if neither M_1 nor M_2 separates the points p, q in $\mathbf{S}$ and the intersection $M_1 \cap M_2$ is connected, then the set $M_1 \cup M_2$ does not separate the points p, q in $\mathbf{S}$, either.*

For any $\varepsilon \in (0, \infty)$ and $z \in \mathbf{E}$ the set

$$U(z, \varepsilon) = \{z' \in \mathbf{E};\ |z' - z| < \varepsilon\}$$

is called an *ε-neighbourhood* (briefly: a *neighbourhood*) of the point z; for each $M \subset \mathbf{E}$ we put

$$U(M, \varepsilon) = \bigcup_{z \in M} U(z, \varepsilon)\,.$$

A *segment* with *end-points* $a \neq b$ from $\mathbf{E}$ is the set

$$u(a, b) = \{z;\ z = a + t(b - a),\ t \in \langle 0, 1 \rangle\}\ ;$$

the corresponding *open segment* is defined to be

$$o(a, b) = u(a, b) - \{a, b\}\ ^{1})\,.$$

The points $z \in o(a, b)$ are called the *interior points* of the segment $u(a, b)$.

An *arc* is a homeomorphic image of a segment. The images of the end-points of the segment in the corresponding homeomorphic mapping are called the *end-points* of the arc. If L is an arc with end-points a, b, then

$$\tilde{L} = L - \{a, b\}$$

defines the corresponding *open arc*. The term *topological circumference* stands for a homeomorphic image of a circumference. Under a *polygonal line* we shall mean here an arc or a topological circumference which is the union of a finite number of segments.

A region (i.e., a connected open set) whose boundary is a topological circumference is called a *Jordan region*. We shall use the term *polygon* for a bounded Jordan region whose boundary is a polygonal line, or for the closure of such a region; the actual meaning will be always clear from the context. Any maximal segment contained in $\partial\Omega$ will be called a *side* of the polygon Ω while each end-point of any one of its sides will be called its *vertex*.

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[1]) $\{a, b\}$ is the two-point set consisting of the points a, b.

The *Jordan theorem* (see [2], p. 108) asserts that for every topological circumference $T \subset \mathbf{S}$, the set $\mathbf{S} - T$ is the union of two disjoint Jordan regions whose common boundary is T. If $T \subset \mathbf{E}$ then one of these regions contains the point ∞ (we shall denote it by Ext T) while the other one is bounded (and will be denoted by Int T).

The theorem on θ-curves will be applied in the following form (see [2], p. 184): *If Ω is a Jordan region, L an arc with end-points a, b in $\partial\Omega$ and satisfying $\tilde{L} \subset \Omega$, if M_1, M_2 are arcs with the same end-points a, b for which $M_1 \cup M_2 = \partial\Omega$, then $\Omega - L$ is the union of two disjoint Jordan regions Ω_1, Ω_2 which fulfil $\partial\Omega_i = M_i \cup L$ for $i = 1, 2$.*

The concept of a net (cf. [1], p. 374) will be of importance for us. Let Ω be a bounded Jordan region, let T_k $(2 \leqq k \leqq n,\ n \geqq 1)$ be arcs with end-points a_k, b_k. The sequence

$$(1) \qquad T_1 = \partial\Omega\,,\ T_2, \ldots, T_n$$

is called a (*n-term*) *net* in $\bar{\Omega}$ if

$$(2) \qquad \tilde{T}_k \subset \Omega$$

and

$$(3) \qquad T_k \cap (T_1 \cup \ldots \cup T_{k-1}) = \{a_k, b_k\}$$

for each $k = 2, \ldots, n$.

The following assertion is easily proved (by induction with respect to n) in virtue of the theorem on θ-curves (cf. [1], p. 374):

Lemma 1. *Let* (1) *be a net in $\bar{\Omega}$. Then the set $\Omega - \bigcup_{k=2}^{n} T_k$ has exactly n components, say $\Omega_1, \ldots, \Omega_n$, and*

$$(4) \qquad \bigcup_{k=1}^{n} \partial\Omega_k = \bigcup_{k=1}^{n} T_k\,.$$

The boundary of the only unbounded component of the set $\mathbf{S} - \bigcup_{k=2}^{n} T_k$ is $\partial\Omega$. Each point $z \in \partial\Omega - \bigcup_{k=2}^{n} T_k$ belongs to the closure of exactly one component of the set $\Omega - \bigcup_{k=2}^{n} T_k$.

Another concept needed in the sequel is the *linear accessibility*: Let Ω be a region, $a \in \partial\Omega$. We say that the point a is linearly accessible from Ω, if there exists a segment $u(a, b)$ such that $u(a, b) - \{a\} \subset \Omega$.

The following lemma is easily verified (cf. [2], p. 527):

Lemma 2. *If Ω is a bounded region then the set Z of all points from $\partial\Omega$ which are linearly accessible from Ω is dense in $\partial\Omega$.*

Let us recall another fact which we believe to be well-known:

Lemma 3. *Each point of the boundary of a polygon is linearly accessible from it.*

The following result will be of importance as well:

Lemma 4. *Let* (1) *be a net in* $\bar{\Omega}$ *with* $T_2, \ldots, T_n$ *polygonal lines. Then every point* $z \in \bigcup_{k=2}^{n} T_k \cap \Omega$ *is linearly accessible from each component of the set* $\Omega - \bigcup_{k=2}^{n} T_k$ *to whose boundary it belongs.*

Proof. If $z \in \bigcup_{k=2}^{n} T_k \cap \Omega$, then for every sufficiently small neighbourhood $U(z, \varepsilon)$ of the point z the set $U(z, \varepsilon) - \bigcup_{k=2}^{n} T_k$ is the union of a finite number of open circular sectors and each component of the set $\Omega - \bigcup_{k=2}^{n} T_k$ whose boundary contains the point z includes also one of these sectors. Obviously, the point z is linearly accessible from any one of these sectors.

The following assertion plays an important role in the sequel:

Lemma 5. *Let* $\varepsilon > 0$. *Let us denote by* $\mathcal{S}$ *the family of all squares*

$$C_{m,n} = \{z; (m-1)\,\varepsilon \leqq \operatorname{Re} z \leqq m\varepsilon,\ (n-1)\,\varepsilon \leqq \operatorname{Im} z \leqq n\varepsilon\}\ ^{2)}$$

where m, n *are integers. Let* Ω *be a bounded Jordan region,* M *a straigt line intersecting* Ω. *Then there exists a finite family* $\mathcal{L}$ *of segments which satisfies the following four conditions*:

(4_1) $L \in \mathcal{L} \Rightarrow L \subset M$;

(4_2) $L \in \mathcal{L},\ L = u(a, b) \Rightarrow o(a, b) \subset \Omega,\ a, b \in \partial\Omega$;

(4_3) $L_1, L_2 \in \mathcal{L},\ L_1 \neq L_2 \Rightarrow L_1 \cap \tilde{L}_2 = \emptyset$;

(4_4) *none of the components of the set* $\Omega - \bigcup_{L \in \mathcal{L}} L$ *intersects simultaneously two squares* $C', C'' \in \mathcal{S}$ *which lie in different components of the set* $\mathbf{E} - M$.

Proof. Let $\mathcal{L}^*$ be the family of all segments which are closures of the components of the set $M \cap \Omega$. The family $\mathcal{L}^*$ is either finite or denumerable and satisfies the implications (4_1)–(4_3) with $\mathcal{L}$ replaced by $\mathcal{L}^*$.

If the family $\mathcal{L}^*$ is finite put $\mathcal{L} = \mathcal{L}^*$. Then the implication ($4_4$) is obvious as well: every connected set included in Ω and intersecting both components of the set $\mathbf{E} - M$ intersects also the set $M \cap \Omega = \bigcup_{L \in \mathcal{L}} \tilde{L}$.

It remains to prove the assertion under the assumption that the family $\mathcal{L}^*$ is infinite. In this case let us arrange the segments of the family $\mathcal{L}^*$ in a sequence $L_1, \ldots, L_n, \ldots$ with mutually different terms. Let the endpoints of the segment L_n

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[2]) Re z, Im z stand for the real and imaginary parts of a number $z \in \mathbf{E}$, respectively.

be a_n, b_n. With regard to the fact that the open segments $\tilde{L}_n$ are disjoint and contained in the bounded part $\Omega \cap M$ of the straight line M we have

$$(5) \qquad |a_n - b_n| \to 0 .$$

Let us choose an arc X of the topological circumference $\partial\Omega$ which does not intersect M, and let Y be the arc of the topological circumference $\partial\Omega$ with the same endpoints as X, which satisfies $X \cup Y = \partial\Omega$. Let φ be a homeomorphic mapping of the interval $\langle 0, 1 \rangle$ onto Y. Denoting

$$(6) \qquad \alpha_n = \varphi_{-1}(a_n), \quad \beta_n = \varphi_{-1}(b_n)\ ^{3})$$

we can assume that $\alpha_n < \beta_n$ for all n since the denotation of the endpoints of the segments L_n is immaterial in the sequel. We have $\beta_n - \alpha_n \to 0$ by (5) and in virtue of the uniform continuity of the function φ_{-1}. Hence, with regard to the uniform continuity of φ,

$$(7) \qquad \operatorname{diam} \varphi(\langle \alpha_n, \beta_n \rangle) \to 0 .$$

The theorem on θ-curves implies (for every n)

$$(8) \qquad \Omega - L_n = \Omega_{n1} \cup \Omega_{n2}$$

where Ω_{n1}, Ω_{n2} are disjoint Jordan regions with

$$(9) \qquad \partial\Omega_{n1} = L_n \cup \varphi(\langle \alpha_n, \beta_n \rangle) .$$

Since

$$\operatorname{diam} \bar{\Omega}_{n1} = \operatorname{diam} \partial\Omega_{n1} \leqq \operatorname{diam} L_n + \operatorname{diam} \varphi(\langle \alpha_n, \beta_n \rangle) ,$$

we conclude by (5) and (7) that

$$(10) \qquad \operatorname{diam} \bar{\Omega}_{n1} \to 0 .$$

Let Z_1, Z_2 be open half-planes determined by the straight line M. Let us denote by W_i ($i = 1, 2$) the union of all squares $C_{m,n} \in \mathcal{S}$ which are contained in Z_i and intersect Ω. The sets W_i (being finite unions of compact sets $C_{m,n}$) are compact. If one of them is empty then our assertion is trivial as the empty family may be taken for L. Therefore, let $W_1 \neq \emptyset \neq W_2$; then

$$(11) \qquad \operatorname{dist}(\Omega \cap M, W_i) > 0 \quad \text{for} \quad i = 1, 2 .$$

Since $\bar{\Omega}_{n1} \cap (\Omega \cap M) \neq \emptyset$ for every n, by (10) and (11) there exists a positive integer p such that

$$(12) \qquad \bar{\Omega}_{n1} \cap (W_1 \cup W_2) = \emptyset \quad \text{for all} \quad n > p .$$

[3]) The symbol φ_{-1} denotes, of course, the inverse mapping to φ.

Put $\mathscr{L} = \{L_1, \ldots, L_p\}$ and assume that there exists a component K of the set $\Omega - \bigcup_{L \in \mathscr{L}} L = \Omega - \bigcup_{n=1}^{p} L_n$ which intersects both W_1 and W_2. Let us choose points $z_i \in K \cap W_i$ $(i = 1, 2)$. As K is a region, there exists an arc N contained in K, with the end-points z_1, z_2. Further, there exists an $\eta > 0$ such that $\overline{U(N, \eta)} \subset K$. With regard to (10) there exists a $q > p$ such that diam $\bar{\Omega}_{n1} < \eta$ for all $n > q$. This implies

$$(13) \qquad \bar{\Omega}_{n1} \cap N = \emptyset \quad \text{for all} \quad n > q\,.$$

It is easy to see that the set

$$(\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n = (\mathbf{S} - \Omega) \cup \overline{\bigcup_{n>q} L_n}$$

is a continuum[4]. This continuum does not separate the points z_1, z_2 in $\mathbf{S}$, since its complement

$$\mathbf{S} - ((\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n) = \Omega - \bigcup_{n>q} L_n \supset \Omega - \bigcup_{n>q} \bar{\Omega}_{n1}$$

contains the connected set N. If the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n$ did not separate the points z_1, z_2 in $\mathbf{S}$, then the same would hold according to the Janiszewski theorem also for the set

$$(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{\infty} L_n = ((\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n) \cup ((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n)\,,$$

since the intersection

$$((\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n) \cap ((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n) = \mathbf{S} - \Omega = \overline{\operatorname{Ext} \partial\Omega}$$

is connected. However, this is not the case, since $M \subset (\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{\infty} L_n$ and M separates the points z_1, z_2 in $\mathbf{S}$[5]. Consequently,

(14) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n$ separates the points z_1, z_2 in $\mathbf{S}$.

On the other hand, we know that

(15) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{p} L_n$ does not separate the points z_1, z_2 in $\mathbf{S}$ (since the connected set $K \subset \Omega - \bigcup_{n=1}^{p} L_n = \mathbf{S} - ((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{p} L_n)$ contains these points).

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[4]) That is, a compact connected set.

[5]) Let us note that (12) implies that $z_i \in \Omega - \bigcup_{n>p} \bar{\Omega}_{n1}$, hence also $z_i \in \Omega - \bigcup_{n=1}^{\infty} L_n$ for $i = 1, 2$.

Let r be the least positive integer such that

(16′) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r} L_n$ separates the points z_1, z_2 in $\mathbf{S}$.

Then $p < r \leqq q$ and

(16″) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r-1} L_n$ does not separate the points z_1, z_2 in $\mathbf{S}$.

The set $\mathbf{S} - \bar{\Omega}_{r1} = \operatorname{Ext} \partial\Omega_{r1}$ is connected and according to (12) contains both z_1 and z_2. Consequently, $\bar{\Omega}_{r1}$ does not separate these points in $\mathbf{S}$. Further, we have

$$((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r-1} L_n) \cap \bar{\Omega}_{r1} = ((\mathbf{S} - \Omega) \cap \bar{\Omega}_{r1}) \cup (\bigcup_{n=1}^{r-1} L_n \cap \bar{\Omega}_{r1}) = \varphi(\langle\alpha_r, \beta_r\rangle) \cup \Lambda\,,$$

where Λ is the union of all segments L_n $(n = 1, \ldots, r - 1)$ satisfying $\tilde{L}_n \subset \Omega_{r1}$. The end-points of such segments belong to $(\mathbf{S} - \Omega) \cap \bar{\Omega}_{r1} = \varphi(\langle\alpha_r, \beta_r\rangle)$ and therefore the union $\varphi(\langle\alpha_r, \beta_r\rangle) \cup \Lambda$ is connected. The Janiszewski theorem implies that the set

$$(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r-1} L_n \cup \bar{\Omega}_{r1}$$

does not separate the points z_1, z_2 in $\mathbf{S}$, either. The less does so the smaller set $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r} L_n$; however, this contradicts (16′).

This completes the proof of Lemma 5.

Lemma 6. *Let Ω be a bounded Jordan region and let $\varepsilon > 0$. Then:*

1. *There exist segments $L_1, \ldots, L_q$ (with $q \geqq 0$) such that*

(17) $$\partial\Omega,\ L_1, \ldots, L_q \quad \text{is a net in} \quad \bar{\Omega}\,;$$

(18) *every component of the set $\Omega - \bigcup_{n=1}^{q} L_n$ has a diameter less than ε.*

2. *If Ω is a polygon and the set Z is dense in $\partial\Omega$, then it is possible to choose the segments $L_1, \ldots, L_q$ so that, in adition to* (17) *and* (18), *the following two conditions hold:*

(19) *every point $z \in \partial\Omega$ belong to at most one set L_n $(1 \leqq n \leqq q)$;*

(20) $$L_n \cap \partial\Omega \subset Z \quad \text{for all} \quad n = 1, \ldots, q\,.$$

Proof. 1. Let Ω be a bounded Jordan region, $\varepsilon > 0$. Let us denote by $\mathscr{S}$ the family of all squares

$$C_{m,n} = \{z;\ \tfrac{1}{3}(m - 1)\,\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}m\varepsilon,\ \tfrac{1}{3}(n - 1)\,\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}n\varepsilon\}$$

where m, n are integers.

Let $\mathscr{M}_1$ be the family of all straight lines

(21) $$\{z;\ \operatorname{Im} z = \tfrac{1}{6}(2k - 1)\,\varepsilon\}$$

where k is an integer satisfying

(22) $$\{z \in \Omega;\ \operatorname{Im} z \geqq \tfrac{1}{3}k\varepsilon\} \neq \emptyset \neq \{z \in \Omega;\ \operatorname{Im} z \leqq \tfrac{1}{3}(k-1)\,\varepsilon\},$$

and let $\mathcal{M}_2$ be the family of all straight lines

(23) $$\{z;\ \operatorname{Re} z = \tfrac{1}{6}(2j-1)\,\varepsilon\}$$

where j is an integer satisfying

(24) $$\{z \in \Omega;\ \operatorname{Re} z \geqq \tfrac{1}{3}j\varepsilon\} \neq \emptyset \neq \{z \in \Omega;\ \operatorname{Re} z \leqq \tfrac{1}{3}(j-1)\,\varepsilon\}.$$

Put $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. It follows from the boundedness of the set Ω that the family $\mathcal{M}$ is finite.

According to Lemma 5 we have to every straight line $M \in \mathcal{M}$ a finite system $\mathcal{L}(M)$ of segments satisfying

(25_1) $L \in \mathcal{L}(M) \Rightarrow L \subset M$;

(25_2) $L \in \mathcal{L}(M),\ L = u(a, b) \Rightarrow o(a, b) \subset \Omega,\ a, b \in \partial\Omega$;

(25_3) $L', L'' \in \mathcal{L}(M),\ L' \neq L'' \Rightarrow L' \cap \bar{L}'' = \emptyset$;

(25_4) none of the components of the set $\Omega - \bigcup_{L \in \mathcal{L}(M)} L$ intersects simultaneously two squares $C', C'' \in \mathcal{S}$ which belong to different components of the set $\mathbf{E} - M$.

Let K be one of the components of the set

(26) $$\Omega - \bigcup_{M \in \mathcal{M}} \bigcup_{L \in \mathcal{L}(M)} L.$$

Then there exists a maximal integer k such that $\{z \in K;\ \operatorname{Im} z \leqq \tfrac{1}{3}(k-1)\,\varepsilon\} = \emptyset$. It follows from its definition that $K \cap C_{m_1,k} \neq \emptyset$ for a certain integer m_1. If it were $\{z \in K;\ \operatorname{Im} z \geqq \tfrac{1}{3}(k+1)\,\varepsilon\} \neq \emptyset$, then $K \cap C_{m_2,n} \neq \emptyset$ for some two numbers m_2 and $n \geqq k+2$. The straight line $M = \{z; \operatorname{Im} z = \tfrac{1}{6}(2k+1)\,\varepsilon\}$ would belong to $\mathcal{M}_1$. However, this is not possible, since K is part of a component K' of the set $\Omega - \bigcup_{L \in \mathcal{L}(M)} L$ and according to (25_4) K' does not intersect both squares $C_{m_1,k}$, $C_{m_2,n}$, as they belong (in virtue of the inequality $n \geqq k+2$) to different components of the set $\mathbf{E} - M$. Thus we have proved that

(27) $$K \subset \{z;\ \tfrac{1}{3}(k-1)\,\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}(k+1)\,\varepsilon\}$$

for k suitably chosen; we can prove similarly that

(27′) $$K \subset \{z;\ \tfrac{1}{3}(j-1)\,\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}(j+1)\,\varepsilon\}$$

for j suitably chosen. The last two inclusions imply immediately

$$\operatorname{diam} K \leqq \operatorname{diam} \{z; \tfrac{1}{3}(j-1)\,\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}(j+1)\,\varepsilon\,,$$
$$\tfrac{1}{3}(k-1)\,\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}(k+1)\,\varepsilon\} = \varepsilon\sqrt{\tfrac{8}{9}} < \varepsilon\,.$$

Consequently,

(28) if K is a component of the set (26), then $\operatorname{diam} K < \varepsilon$.

Let us now arrange all segments of the family $\bigcup_{M\in\mathscr{M}_1} \mathscr{L}(M)$ in a sequence $L_1, \ldots, L_p$ with mutually different terms (so that $p \geqq 0$). With regard to $(25_1)-(25_3)$ and to the fact that the straight lines from $\mathscr{M}_1$ are disjoint we obtain that

(29) the sequence $\partial\Omega, L_1, \ldots, L_p$ is a net in $\bar{\Omega}$.

On the basis of the families $\mathscr{L}(M)$, where $M \in \mathscr{M}_2$, we form new systems $\mathscr{L}^*(M)$ in the following way: For every $L \in \bigcup_{M\in\mathscr{M}_2} \mathscr{L}(M)$, let

$$L \cap \left(\partial\Omega \cup \bigcup_{n=1}^{p} L_n\right)^{6)} = \{a_0^L, \ldots, a_{r(L)}^L\} \tag{30}$$

where

$$\operatorname{Im} a_0^L < \operatorname{Im} a_1^L < \ldots < \operatorname{Im} a_{r(L)}^L\,. \tag{31}$$

Let the family $\mathscr{L}^*(M)$ contain exactly all the segments

$$u(a_0^L, a_1^L),\ u(a_1^L, a_2^L),\ \ldots,\ u(a_{r(L)-1}^L, a_{r(L)}^L)$$

where $L \in \mathscr{L}(M)$.

If all segments of the family $\bigcup_{M\in\mathscr{M}_2} \mathscr{L}^*(M)$ are arranged in a sequence $L_{p+1}, \ldots, L_q$ with mutually different terms, then $q \geqq p$ and it is evident that (17) holds. Further, obviously

$$\bigcup_{n=p+1}^{q} L_n = \bigcup_{M\in\mathscr{M}_2} \bigcup_{L\in\mathscr{L}(M)} L\,,$$

hence also

$$\bigcup_{n=1}^{q} L_n = \bigcup_{M\in\mathscr{M}} \bigcup_{L\in\mathscr{L}(M)} L\,. \tag{32}$$

In virtue of (28), (18) is proved.

2. Now let Ω be a polygon and Z a set dense in $\partial\Omega$. Without any loss of generality we may assume that

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6) This set is finite, since $L \cap \partial\Omega$ is a two-point set according to (25_2) and L is ortogonal to each segment $L_1, \ldots, L_p$.

(33) any vertex of the polygon Ω belongs neither to a line $\{z; \operatorname{Re} z = \frac{1}{6}l\varepsilon\}$ nor to a line $\{z; \operatorname{Im} z = \frac{1}{6}l\varepsilon\}$ with l odd,

(34) no point of the form $\frac{1}{6}(l_1 + il_2)\varepsilon$ with l_1, l_2 odd belongs to $\partial\Omega$,

since, if necessary, Ω may be suitably translated without affecting essentially our argument.[7])

Let the symbols $C_{m,n}$, $\mathcal{S}$, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}$ have the same meaning as in Part 1 of the proof. According to (33) the set $M \cap \partial\Omega$ is finite for every $M \in \mathcal{M}$; let

$$(35) \qquad M \cap \partial\Omega = \{a_1^M, a_2^M, \ldots, a_{r(M)}^M\}$$

where

$$(36_1) \qquad \operatorname{Re} a_1^M < \operatorname{Re} a_2^M < \ldots < \operatorname{Re} a_{r(M)}^M \quad \text{if} \quad M \in \mathcal{M}_1\,,$$

$$(36_2) \qquad \operatorname{Im} a_1^M < \operatorname{Im} a_2^M < \ldots < \operatorname{Im} a_{r(M)}^M \quad \text{if} \quad M \in \mathcal{M}_2\,.$$

Further, let us denote by X_k^M the side of the polygon Ω which contains the point a_k^M; since the point a_k^M is not a vertex of the polygon Ω there is exactly one such side, it is not parallel to M, and the point a_k^M is its interior point. Let $P^+(M)$, $P^-(M)$ be the components of the set $\mathbf{E} - M$. Let $W^+(M)$ and $W^-(M)$ be the unions of all squares $C_{m,n} \in \mathcal{S}$ which intersect Ω and satisfy $C_{m,n} \subset P^+(M)$ and $C_{m,n} \subset P^-(M)$, respectively.

We shall prove that for every sufficiently small $\delta > 0$ the following five conditions hold:

(37_1) for any $M \in \mathcal{M}$, there is no vertex of the polygon Ω in the strip $\overline{U(M, \delta)}$;

(37_2) $\overline{U(M, \delta)} \cap (W^+(M) \cup W^-(M)) = \emptyset$ for each $M \in \mathcal{M}$;

(37_3) if $M_1 \neq M_2$ and either $M_1, M_2 \in \mathcal{M}_1$ or $M_1, M_2 \in \mathcal{M}_2$, then $U(M_1, \delta) \cap$ $\cap\, U(M_2, \delta) = \emptyset$;

(37_4) if $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$, then $U(M_1, \delta) \cap U(M_2, \delta) \cap \partial\Omega = \emptyset$;

(37_5) given arbitrary points $b_k^M \in X_k^M \cap U(M, \delta)$ (where $M \in \mathcal{M}$, $1 \leqq k \leqq r(M)$), then the angle between the segment $u(b_{k-1}^M, b_k^M)$ (where $1 < k \leqq r(M)$) and the line M is non-negative and less than $\frac{1}{2}\pi$.

With regard to (33) we find that (37_1) holds for all sufficiently small $\delta > 0$. As $W^+(M) \cup W^-(M)$ is a compact set disjoint with M, (37_2) holds for all sufficiently small $\delta > 0$ as well. If $0 < \delta < \frac{1}{6}\varepsilon$, then (37_3) holds, too. Further, (34) implies that

[7]) It is sufficient to use a translation vector v which is parallel neither to the real and imaginary axes nor to any side of the polygon Ω, and whose magnitude is less than (i) the distance of every vertex of the polygon Ω from the union of all straight lines $\{z; \operatorname{Re} z = l\varepsilon/6\}$, $\{z; \operatorname{Im} z = l\varepsilon/6\}$ with l odd, not containing this vertex, (ii) the distance of every point $(l_1 + il_2)\,\varepsilon/6$ with l_1, l_2 odd from any side of the polygon Ω not containing the point in question.

also the condition (37_4) is fulfilled for all $\delta > 0$ sufficiently small. If the points b_k^M are sufficiently close to the points a_k^M, then all conditions concerning angles will be fulfilled as required in (37_5). Taking into account that the segments X_k^M are not parallel to the straight lines M we see that the points $b_k^M \in X_k^M \cap U(M, \delta)$ will be arbitrarily close to the points $a_k^M (\in X_k^M \cap M)$ provided $\delta > 0$ is sufficiently small.

Let us now fix a number $\delta > 0$ so that $(37_1)-(37_5)$ hold. Denote

$$(40_1) \qquad A_0^M = \{z \in M;\ \text{Re}\, z < \text{Re}\, a_1^M\}\,, \quad A_{r(M)}^M = \{z \in M;\ \text{Re}\, z > \text{Re}\, a_{r(M)}^M\}$$

for $M \in \mathcal{M}_1$,

$$(40_2) \qquad A_0^M = \{z \in M;\ \text{Im}\, z < \text{Im}\, a_1^M\}\,, \quad A_{r(M)}^M = \{z \in M;\ \text{Im}\, z > \text{Im}\, a_{r(M)}^M\}$$

for $M \in \mathcal{M}_2$, and let

$$(40_3) \qquad A_k^M = o(a_k^M, a_{k+1}^M)$$

for $M \in \mathcal{M}$ and $1 \leqq k < r(M)$.

Further, let G_k^M (where $M \in \mathcal{M}$, $0 \leqq k \leqq r(M)$) be the component of the set $U(M, \delta) - \bigcup_{k=1}^{r(M)} X_k^M$ containing the set A_k^M. Let us note that each set G_k^M is convex (being the intersection of three or four half-planes).

First, let us show that the number $r(M)$ is even for all $M \in \mathcal{M}$ and

$$(41) \qquad A_0^M \cup A_2^M \cup \ldots \cup A_{r(M)}^M \subset \boldsymbol{S} - \bar{\Omega}\,, \quad A_1^M \cup \ldots \cup A_{r(M)-1}^M \subset \Omega\,.$$

In virtue of its connectedness and of the condition $A_k^M \cap \partial\Omega = \emptyset$ each set A_k^M is contained either in $\boldsymbol{S} - \bar{\Omega}$ or in Ω. Since obviously $A_0^M \cup A_{r(M)}^M \subset \boldsymbol{S} - \bar{\Omega}$, (41) will be proved (together with the evenness of $r(M)$), if we show that

(42) for each $k = 1, \ldots, r(M)$ one of the sets A_{k-1}^M, A_k^M is contained in $\boldsymbol{S} - \bar{\Omega}$, the other one in Ω.

To prove (42), let us choose (with $k = 1, \ldots, r(M)$ fixed) a neighbourhood U of the point a_k^M so small that $U \cap \partial\Omega = U \cap X_k^M$. Then the set $U - \partial\Omega$ is the union of two open semicircles, one of them being contained in $\boldsymbol{S} - \bar{\Omega}$, the other one in Ω according to the Jordan theorem. At the same time it is apparent that one of the sets A_{k-1}^M, A_k^M intersects one of these semicircles while the other set intersects the other one. This implies immediately (42) in virtue of the fact that each set A_j^M is a subset either of $\boldsymbol{S} - \bar{\Omega}$ or of Ω.

If we show that

$$(43) \qquad G_k^M \cap \partial\Omega = \emptyset \quad \text{for every} \quad M \in \mathcal{M} \quad \text{and every} \quad k = 0, \ldots, r(M)\,,$$

then (41) together with the connectedness of the sets G_k^M will imply

$$(44) \qquad G_0^M \cup G_2^M \cup \ldots \cup G_{r(M)}^M \subset \boldsymbol{S} - \bar{\Omega}\,, \quad G_1^M \cup \ldots \cup G_{r(M)-1}^M \subset \Omega\,.$$

Let us suppose that there is a point $z_0 \in G_k^M \cap \partial\Omega$. By (37_1), the point z_0 is an interior point of a certain side X of the polygon Ω, while the end-points of X do not belong to $\overline{G_k^M}$. Hence X intersects ∂G_k^M in two points $z_1 \neq z_2$. Since obviously $X \neq X_j^M$ for $j = 1, \ldots, r(M)$, we have $z_1, z_2 \notin \bigcup_{j=1}^{r(M)} X_j^M$ and, consequently, the points z_1, z_2 belong to the straight lines whose closures form $\partial U(M, \delta)$, and lie in different components of the $\mathbf{E} - M$. This implies that $X \cap M \neq \emptyset$; it is easy to see, with regard to the convexity of the sets G_k^M, that $o(z_1, z_2) \subset G_k^M$ so that the point of intersection of X and M belongs to $A_k^M = G_k^M \cap M$. However, this is a contradiction since $A_k^M \cap \partial\Omega = \emptyset$.

This completes the proof of (43), and thus also of (44).

Since the set Z is dense in $\partial\Omega$, there exist points

(45) $$b_k^M \in \dot{X}_k^M \cap Z \cap U(M, \delta) \left(M \in \mathcal{M},\ 1 \leqq k \leqq r(M)\right).$$

For $M \in \mathcal{M}_1$ let us put

(46_1) $$B_0^M = \{z;\ \operatorname{Re} z < \operatorname{Re} b_1^M,\ \operatorname{Im} z = \operatorname{Im} b_1^M\},$$
$$B_{r(M)}^M = \{z;\ \operatorname{Re} z > \operatorname{Re} b_{r(M)}^M,\ \operatorname{Im} z = \operatorname{Im} b_{r(M)}^M\},$$

for $M \in \mathcal{M}_2$ let

(46_2) $$B_0^M = \{z;\ \operatorname{Re} z = \operatorname{Re} b_1^M,\ \operatorname{Im} z < \operatorname{Im} b_1^M\},$$
$$B_{r(M)}^M = \{z;\ \operatorname{Re} z = \operatorname{Re} b_{r(M)}^M,\ \operatorname{Im} z > \operatorname{Im} b_{r(M)}^M\};$$

for every $M \in \mathcal{M}$ let

(46_3) $$B_k^M = o(b_k^M, b_{k+1}^M) \quad (1 \leqq k < r(M))$$

and

(46_4) $$B^M = \bigcup_{r=0}^{r(M)} \bar{B}_k^M .$$

As evidently $B_k^M \subset G_k^M$, we have, by (44),

(47) $$B_0^M \cup B_2^M \cup \ldots \cup B_{r(M)}^M \subset \mathbf{S} - \bar{\Omega}, \quad B_1^M \cup \ldots \cup B_{r(M)-1}^M \subset \Omega .$$

The set B^M is a topological circumference, hence the set $\mathbf{S} - B^M$ has exactly two components in virtue of the Jordan theorem. Since $B^M \subset \overline{U(M, \delta)}$, one of the components contains the half-plane $P^+(M) - \overline{U(M, \delta)}$, hence (by (37_2)) also the set $W^+(M)$, while the other one contains the half-plane $P^-(M) - \overline{U(M, \delta)}$, and hence also the set $W^-(M)$. This implies that every connected set intersecting both $W^+(M)$ and $W^-(M)$ intersects B^M as well, so that

(48) none of the components of the set $\Omega - B^M$ intersects both $W^+(M)$ and $W^-(M)$.

If for any $M \in \mathscr{M}$, the symbol $\mathscr{L}(M)$ stands for the system of all segments $\bar{B}_1^M$, $\bar{B}_3^M$, $\bar{B}_{r(M)-1}^M$, then the system is obviously disjoint and, by (47),

(49) $$\Omega \cap B^M = \bigcup_{L \in \mathscr{L}(M)} \tilde{L},$$

so that, by (48),

(50) none of the components of the set $\Omega - \bigcup_{L \in \mathscr{L}(M)} L$ intersects (for any $M \in \mathscr{M}$) both sets $W^+(M)$, $W^-(M)$.

Hence it follows similarly as in Part 1 of the proof that

(51) to every component K of the set $\Omega - \bigcup_{M \in \mathscr{M}} \bigcup_{L \in \mathscr{L}(M)} L$ there exist integers j, k such that $K \subset \{z; \frac{1}{3}(j-1)\,\varepsilon \leqq \operatorname{Re} z \leqq \frac{1}{3}(j+1)\,\varepsilon,\ \frac{1}{3}(k-1)\,\varepsilon \leqq \operatorname{Im} z \leqq \frac{1}{3}(k+1)\,\varepsilon\}$,

and consequently

(52) $\operatorname{diam} K < \varepsilon$ for every component K of the set $\Omega - \bigcup_{M \in \mathscr{M}} \bigcup_{L \in \mathscr{L}(M)} L$.

Now let us arrange all segments of the system $\bigcup_{M \in \mathscr{M}} \mathscr{L}(M)$ in a sequence $L_1, \ldots, L_p$ with mutually different terms. Similarly as in Part 1 of the proof

(53) the sequence $\partial\Omega, L_1, \ldots, L_p$ is a net in $\bar{\Omega}$.

If $L \in \mathscr{L}(M)$ with $M \in \mathscr{M}_2$, then the set

(54) $$L \cap (\partial\Omega \cup \bigcup_{n=1}^{p} L_n)$$

is finite since $L \cap \partial\Omega$ is a two-point set and the segments L, L_n $(1 \leqq n \leqq p)$ are not parallel (as, by (37_5), the angle between the segment L and the imaginary axis as well as the angle between any one of the segments L_n and the real axis is less than $\frac{1}{4}\pi$).

Consequently, for every $M \in \mathscr{M}_2$ a system $\mathscr{L}^*(M)$ can be defined analogously as in Part 1 of the proof; we arrange the segments of the system $\bigcup_{M \in \mathscr{M}_2} \mathscr{L}^*(M)$ in a sequence $L_{p+1}, \ldots, L_q$ with mutually different terms as before.

Again, (17) and (18) hold by an analogous argument. Two different segments L', L'' belonging either both to $\bigcup_{M \in \mathscr{M}_1} \mathscr{L}(M)$ or both to $\bigcup_{M \in \mathscr{M}_2} \mathscr{L}(M)$ do not intersect (which is a consequence of the construction of the systems $\mathscr{L}(M)$ – cf. (37_3)). If $L' \in \mathscr{L}(M_1)$, $L'' \in \mathscr{L}(M_2)$ with $M_1 \in \mathscr{M}_1$, $M_2 \in \mathscr{M}_2$, then $L' \cap L''$ is part of $U(M_2, \delta) \cap \cap U(M_2, \delta)$ which is a set disjoint with $\partial\Omega$ according to (37_4). Hence it follows that the condition (19) is satisfied. The validity of the condition (20) follows from the fact that every point from $L_n \cap \partial\Omega$ with $1 \leqq n \leqq q$ is one of the points b_k^M.

This completes the proof of Part 2 of Lemma 6.

Definition. Let Ω and Ω^* be bounded Jordan regions. Suppose that

(55) $$\partial\Omega = T_1, T_2, \dots, T_n \quad \text{is a net in} \quad \bar{\Omega},$$

(55*) $$\partial\Omega^* = T_1^*, T_2^*, \dots, T_n^* \quad \text{is a net in} \quad \bar{\Omega}^*.$$

A homeomorphic mapping h of the set $\bigcup_{n=1}^{q} T_n$ onto the set $\bigcup_{n=1}^{q} T_n^*$ is said to be *regular*[8]) if it is possible to number the components $\Omega_1, \dots, \Omega_q$ of the set $\Omega - \bigcup_{n=1}^{q} T_n$ and the components $\Omega_1^*, \dots, \Omega_q^*$ of the set $\Omega^* - \bigcup_{n=1}^{q} T_n^*$ in such a way that

(56) $$h(\partial\Omega_n) = \partial\Omega_n^* \quad \text{for} \quad n = 1, \dots, q.$$

Lemma 7. *Let Ω be a bounded Jordan region, Ω^* a polygon. Let f be a homeomorphic mapping of $\partial\Omega$ onto $\partial\Omega^*$. Then:*

1. *Under the assumption (55) there exist polygonal lines $T_2^*, \dots, T_q^*$ such that (55*) holds and the mapping f can be extended to a regular homeomorphic mapping F of the set $\bigcup_{n=1}^{q} T_n$ onto the set $\bigcup_{n=1}^{q} T_n^*$.*

2. *Let Z denote the set of all points from $\partial\Omega$ which are linearly accessible from Ω; let $Z^* = f(Z)$, $f^* = f_{-1}$. Let us further assume that (55*) is satisfied,*

(57) $$\bigcup_{n=2}^{q} T_n^* \cap \partial\Omega^* \subset Z^*,$$

and

(58) *there are no two arcs T_m^*, T_n^*, $2 \leqq m < n \leqq q$, with a common point in $\partial\Omega^*$.*

Then there exist polygonal lines $T_2, \dots, T_q$ such that (55) is satisfied and that the mapping f^* can be extended to a regular homeomorphic F^* mapping of the set $\bigcup_{n=1}^{q} T_n^*$ onto the set $\bigcup_{n=1}^{q} T_n$.

Proof. Let the assumptions of Part 1 of Lemma 7 be satisfied; we shall proceed by induction with respect to q.

For $q = 1$ it is sufficient to notice that every homeomorphic mapping of $\partial\Omega$ onto $\partial\Omega^*$ is regular.

Let the assertion analogous to the above one hold for each $(q - 1)$-term net. If (55) is satisfied, then

(59) $$\partial\Omega = T_1, \dots, T_{q-1}$$

[8]) Cf. [1], p. 376.

is a $(q-1)$-term net so that there exist polygonal lines $T_2^*, \ldots, T_{q-1}^*$ such that $\partial\Omega^* = T_1^*, T_2^*, \ldots, T_{q-1}^*$ is a net in $\bar{\Omega}^*$ and the mapping f can be extended to a regular homeomorphic mapping g of the set $\bigcup_{n=1}^{q-1} T_n$ onto $\bigcup_{n=1}^{q-1} T_n^*$. This means that numbering suitably the components $G_1, \ldots, G_{q-1}$ of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$ and the components $G_1^*, \ldots, G_{q-1}^*$ of the set $\Omega^* - \bigcup_{n=1}^{q-1} T_n^*$ we have

$$g(\partial G_n) = \partial G_n^* \quad \text{for} \quad n = 1, \ldots, q-1\,. \tag{60}$$

According to the definition of a net, T_q is an arc contained in $\bar{\Omega}$ with its end-points a, b in $\bigcup_{n=1}^{q-1} T_n$ while $\mathring{T}_q \cap \bigcup_{n=1}^{q-1} T_n = \emptyset$. This implies that $\mathring{T}_q$ is part of a certain component of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$. Without any loss of generality we may assume that this component is G_{q-1}. Then $a, b \in \partial G_{q-1}$. Since G_{q-1}^* is a polygon, the points $g(a), g(b) \in \in \partial G_{q-1}^*$ are linearly accessible from G_{q-1}^* (see Lemma 3); consequently, there exists an arc T_q^* with end-points $g(a), g(b)$ which is a polygonal line and satisfies $\mathring{T}_q^* \subset G_{q-1}^*$. The topological circumference ∂G_{q-1} is the union of two arcs M_1, M_2 with end-points a, b. Denoting $M_i^* = g(M_i)$ for $i = 1, 2$ we conclude that M_i^* are two arcs with end-points $g(a), g(b)$ whose union is ∂G_{q-1}^*. In virtue of the theorem on the θ-curves we have

$$G_{q-1} - T_q = \Omega_{q-1} \cup \Omega_q\,, \quad G_{q-1}^* - T_q^* = \Omega_{q-1}^* \cup \Omega_q^*\,, \tag{61}$$

where Ω_{q-1}, Ω_q as well as $\Omega_{q-1}^*, \Omega_q^*$ are disjoint Jordan regions with boundaries

$$\partial\Omega_{q-1} = T_q \cup M_1\,, \quad \partial\Omega_q = T_q \cup M_2\,, \quad \partial\Omega_{q-1}^* = T_q^* \cup M_1^*\,, \quad \partial\Omega_q^* = T_q^* \cup M_2^*\,, \tag{62}$$

respectively. Let h be a homeomorphic mapping of the arc T_q onto the arc T_q^* satisfying $h(a) = g(a)$, $h(b) = g(b)$. Then we may put

$$F = \begin{cases} g & \text{in} \quad \bigcup_{n=1}^{q-1} T_n\,, \\ h & \text{in} \quad T_q\,, \end{cases} \tag{63}$$

and F is evidently a homeomorphic mapping of $\bigcup_{n=1}^{q} T_n$ onto $\bigcup_{n=1}^{q} T_n^*$ which is an extension of f.

It is easy to see that the sets $\Omega - \bigcup_{n=1}^{q} T_n$ and $\Omega^* - \bigcup_{n=1}^{q} T_n^*$ have the components

$$\Omega_1 = G_1, \ldots, \Omega_{q-2} = G_{q-2}, \Omega_{q-1}, \Omega_q$$

and

$$\Omega_1^* = G_1^*, \ldots, \Omega_{q-2}^* = G_{q-2}^*, \Omega_{q-1}^*, \Omega_q^*\,,$$

respectively. With respect to (60), (62) and (63) we have evidently

$$F(\partial\Omega_n) = \partial\Omega_n^* \quad \text{for} \quad n = 1, \ldots, q$$

so that the mapping F is regular.

2. Now let the assumptions of Part 2 of Lemma 7 be satisfied. Again we proceed by induction with respect to q. The assertion for $q = 1$ is evident; let the assertion analogous to the above one hold for any $(q - 1)$-term net.

The induction hypothesis together with (55*), (57) and (58) implies that there exist polygonal lines $T_2, \ldots, T_{q-1}$ such that $\partial\Omega = T_1, T_2, \ldots, T_{q-1}$ is a net in $\bar{\Omega}$ and that the mapping f^* can be extended to a regular homeomorphic mapping g^* of the set $\bigcup_{n=1}^{q-1} T_n^*$ onto the set $\bigcup_{n=1}^{q-1} T_n$. This means that the components $G_1^*, \ldots, G_{q-1}^*$ of the set $\Omega^* - \bigcup_{n=1}^{q-1} T_n^*$ and the components $G_1, \ldots, G_{q-1}$ of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$ can be numbered in such a way that

$$g^*(\partial G_n^*) = \partial G_n \quad \text{for} \quad n = 1, \ldots, q - 1 . \tag{64}$$

The arc T_q^* let have end-points a^*, b^*. By (55*), $\mathring{T}_q^*$ is contained in a certain component of the set $\Omega^* - \bigcup_{n=1}^{q-1} T_n^*$; we may assume that the numbering is chosen so that $\mathring{T}_q^* \subset G_{q-1}^*$. Then also $a^*, b^* \in \partial G_{q-1}^*$. Let us denote further $a = g^*(a^*)$, $b = g^*(b^*)$; then $a, b \in \partial G_{q-1}$.

If $a^* \in \partial\Omega^*$ then $a^* \in Z^*$ according to (57) so that the point $a = g^*(a^*) = f_{-1}(a^*) \in Z$ is linearly accessible from Ω. According to (58) we have $a^* \notin \bigcup_{n=2}^{q-1} T_n^*$; consequently $a \notin \bigcup_{n=2}^{q-1} T_n$ and there exists a segment $u(a, a_1)$ satisfying $u(a, a_1) - \{a\} \subset \Omega - \bigcup_{n=1}^{q-1} T_n$. The connected set $u(a, a_1) - \{a\}$ is part of a certain component of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$ and the point a belongs to its closure. However, in virtue of Lemma 1 the point $a \in \partial\Omega - \bigcup_{n=2}^{q-1} T_n$ belongs to the closure of only one component of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$. Hence $u(a, a_1) - \{a\} \subset G_{q-1}$. If $a^* \in \bigcup_{n=1}^{q-1} T_n^* \cap \Omega^*$ then $a \in \bigcup_{n=1}^{q-1} T_n \cap \Omega$ and a similar segment $u(a, a_1)$ exists by Lemma 4. The existence of a point b_1 such that $u(b, b_1) - \{b\} \subset G_{q-1}$ is shown similarly.

Now it follows easily that there exists an arc T_q which is a polygonal line with end-points a, b and satisfying $\mathring{T}_q \subset G_{q-1}$.

If h^* is a homeomorphic mapping of the arc T_q^* onto T_q with $h^*(a^*) = a$, $h^*(b^*) = b$, then

$$F^* = \begin{cases} g^* & \text{in} \quad \bigcup_{n=1}^{q-1} T_n^* , \\ h^* & \text{in} \quad T_q^* \end{cases}$$

is the required regular homeomorphic extension of the mapping f^*. (The proof is quite analogous to that of a similar assertion for the mapping F in the proof of Part 1 of the lemma.)

Lemma 8. *Denote*

$$Q = \{z;\ |\mathrm{Re}\, z| < 2,\ |\mathrm{Im}\, z| < 2\},\quad Q_1 = \{z;\ |\mathrm{Re}\, z| < 1,\ |\mathrm{Im}\, z| < 1\}\,. \tag{65}$$

Let $-2 < a < b < 2$, *let* M_1, M_2 *be two arcs with end-points* a, b *and such that* $T = M_1 \cup M_2$ *is a topological circumference contained in* Q. *Further, let* $0 \in \mathrm{Int}\, T$,

$$i \max \{\mathrm{Im}\, z;\ z \in T,\ \mathrm{Re}\, z = 0\} \in M_1\,, \tag{66}$$

and

$$(\tilde{M}_1 \cup \tilde{M}_2) \cap (\langle -2, a\rangle \cup \langle b, 2\rangle) = \emptyset\,. \tag{67}$$

Let f be a homeomorphic mapping of the set T onto ∂Q_1 satisfying

$$f(a) = -1\,,\quad f(b) = 1\,, \tag{68}$$

$$f(M_1) = \{z \in \partial Q_1;\ \mathrm{Im}\, z \geqq 0\}\,,\quad f(M_2) = \{z \in \partial Q_1;\ \mathrm{Im}\, z \leqq 0\}\,. \tag{69}$$

Then there exists a homeomorphic mapping F of the set $\mathbf{S}$ onto itself satisfying

$$F \mid T = f\,,\quad F \mid (\mathbf{S} - Q) = \mathrm{Id}\,^{9)}\,. \tag{70}$$

Proof. Let us denote

$$T_1 = T_1^* = \partial Q\,, \tag{71_1}$$

$$T_2 = \langle -2, a\rangle \cup M_1 \cup \langle b, 2\rangle\,, \tag{71_2}$$

$$T_2^* = \langle -2, -1\rangle \cup \{z \in \partial Q_1;\ \mathrm{Im}\, z \geqq 0\} \cup \langle 1, 2\rangle\,,$$

$$T_3 = M_2\,,\quad T_3^* = \{z \in \partial Q_1;\ \mathrm{Im}\, z \leqq 0\}\,. \tag{71_3}$$

It is easy to verify that the sequences T_1, T_2, T_3 and T_1^*, T_2^*, T_3^* are nets in $\bar{Q}$. Moreover, it is evident that the mapping f_1 defined in $\bigcup_{n=1}^{3} T_n$ by

$$f_1(z) = \begin{cases} f(z) & \text{for } z \in T\,, \\ z & \text{for } z \in \partial Q\,, \\ [z - 2(a+1)]/(a+2) & \text{for } z \in \langle -2, a\rangle\,, \\ [z + 2(1-b)]/(2-b) & \text{for } z \in \langle b, 2\rangle \end{cases} \tag{72}$$

[9]) Identical mapping; the symbol | is used for parcial mapping.

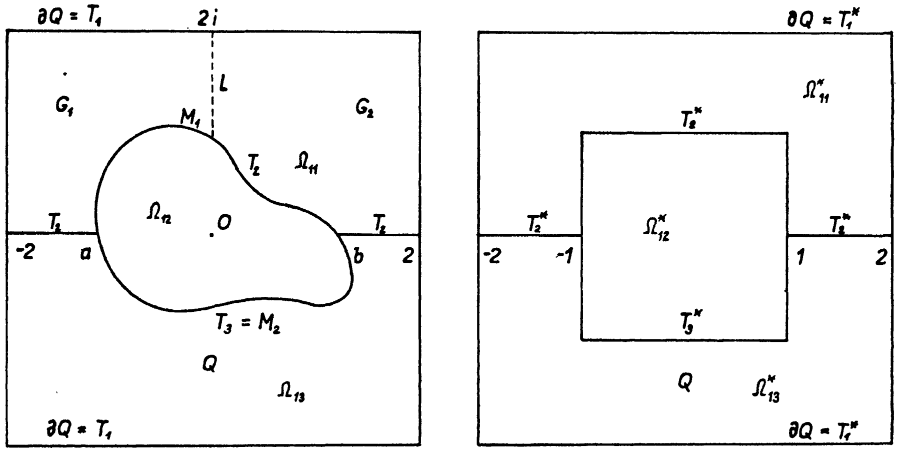
is a homeomorphic mapping of $\bigcup_{n=1}^{3} T_n$ onto $\bigcup_{n=1}^{3} T_n^*$ which is an extension of the mapping f.

Let us show that f_1 is regular. We have

$$(73) \qquad Q - T_2 = \Omega_1 \cup \Omega_2$$

in virtue of the theorem on the θ-curves, where Ω_1, Ω_2 are disjoint Jordan regions with boundaries

$$(74) \qquad \partial\Omega_1 = T_2 \cup \{z \in \partial Q;\ \operatorname{Im} z \geqq 0\}\,, \quad \partial\Omega_2 = T_2 \cup \{z \in \partial Q;\ \operatorname{Im} z \leqq 0\}\,.$$



Further, we have $\tilde{T}_3 = \tilde{M}_2 \subset \Omega_1 \cup \Omega_2$ so that either $\tilde{M}_2 \subset \Omega_1$ or $\tilde{M}_2 \subset \Omega_2$. Let us denote by L the segment with end-points $2i$, $i \max \{\operatorname{Im} z;\ z \in T,\ \operatorname{Re} z = 0\}$. Then

$$(75) \qquad \Omega_1 - L = G_1 \cup G_2$$

where G_1, G_2 are disjoint Jordan regions satisfying

$$(76) \qquad a \in \bar{G}_1 - \bar{G}_2\,, \quad b \in \bar{G}_2 - \bar{G}_1\,.$$

If it were $\tilde{M}_2 \subset \Omega_1$, then with regard to (75) and (76) necessarily

$$(77) \qquad \tilde{M}_2 \cap G_1 \neq \emptyset \neq \tilde{M}_2 \cap G_2$$

which would imply $\tilde{M}_2 \cap \partial G_1 \neq \emptyset$ as well. However, in virtue of the equality $\partial G_1 \cap \Omega_1 = \tilde{L}$ (which is an easy consequence of the theorem on the θ-curves) and the inclusion $\tilde{M}_2 \subset \Omega_1$, this would yield $\tilde{M}_2 \cap \tilde{L} \neq \emptyset$ which contradicts the assumption (66).

Consequently,

$$(78) \qquad \tilde{T}_3 = \tilde{M}_2 \subset \Omega_2$$

and the theorem on the θ-curves together with (73) yields

$$Q - \bigcup_{n=1}^{3} T_n = \Omega_{11} \cup \Omega_{12} \cup \Omega_{13} \tag{79}$$

where Ω_{1j} ($j = 1, 2, 3$) are the components of the sets on the left-hand side satisfying

$$\partial\Omega_{11} = T_2 \cup \{z \in \partial Q;\ \text{Im}\, z \geqq 0\}, \tag{80_1}$$

$$\partial\Omega_{12} = T, \tag{80_2}$$

$$\partial\Omega_{13} = \langle -2, a\rangle \cup T_3 \cup \langle b, 2\rangle \cup \{z \in \partial Q;\ \text{Im}\, z \leqq 0\}. \tag{80_3}$$

We obtain similarly

$$Q - \bigcup_{n=1}^{3} T_n^* = \Omega_{11}^* \cup \Omega_{12}^* \cup \Omega_{13}^* \tag{81}$$

where the sets on the right-hand side are the components of the set on the left-hand side, and

$$\partial\Omega_{11}^* = T_2^* \cup \{z \in \partial Q;\ \text{Im}\, z \geqq 0\}, \tag{82_1}$$

$$\partial\Omega_{12}^* = \partial Q_1, \tag{82_2}$$

$$\partial\Omega_{13}^* = \langle -2, -1\rangle \cup T_3^* \cup \langle 1, 2\rangle \cup \{z \in \partial Q;\ \text{Im}\, z \leqq 0\}. \tag{82_3}$$

The relations (79), (80_1)–(80_3), (81), (82_1)–(82_3) imply

$$f_1(\partial\Omega_{1j}) = \partial\Omega_{1j}^* \quad \text{for} \quad j = 1, 2, 3, \tag{83}$$

i.e., the regularity of the mapping f_1.

Let us note that diam $Q = \sqrt{32} < 6$ and the more so,

$$\text{diam}\, \Omega_{1j} < 6, \quad \text{diam}\, \Omega_{1j}^* < 6 \quad \text{for} \quad j = 1, 2, 3. \tag{84}$$

Put

$$q(1) = 3. \tag{85}$$

Let us assume that for a positive integer k we have defined nets $T_1, \ldots, T_{q(k)}$ and $T_1^*, \ldots, T_{q(k)}^*$ in $\bar{Q}$, that Ω_{kj} and Ω_{kj}^* with $j = 1, \ldots, q(k)$ are the components of the sets $Q - \bigcup_{n=1}^{q(k)} T_n$ and $Q - \bigcup_{n=1}^{q(k)} T_n^*$, respectively, that f_k is a regular homeomorphic mapping of the set $\bigcup_{n=1}^{q(k)} T_n$ onto the set $\bigcup_{n=1}^{q(k)} T_n^*$ which is an extension of the mapping f_1, and

$$T_1^*, \ldots, T_{q(k)}^* \quad \text{are polygonal lines}, \tag{86}$$

$$\text{diam}\, \Omega_{kj} < 6/k, \quad \text{diam}\, \Omega_{kj}^* < 6/k \quad \text{for} \quad j = 1, \ldots, q(k), \tag{87}$$

$$f_k(\partial\Omega_{kj}) = \partial\Omega_{kj}^* \quad \text{for} \quad j = 1, \ldots, q(k). \tag{88}$$

We see immediately that the above conditions hold for $k = 1$.

In virtue of Lemma 6, Part 1, to every $j = 1, \ldots, q(k)$ there exist segments $L_{j1}, \ldots$ $\ldots, L_{jp(j)}$ such that

$$(89_1) \qquad \partial\Omega_{kj} = L_{j0}, L_{j1}, \ldots, L_{jp(j)} \quad \text{is a net in} \quad \bar{\Omega}_{kj}$$

and

(89_2) every component of the set $\Omega_{kj} - \bigcup_{i=1}^{p(j)} L_{ji}$ has a diameter less than $6/(k + 1)$.

In virtue of Lemma 7, Part 1, there exist polygonal lines $L^*_{j1}, \ldots, L^*_{jp(j)}$ such that

$$(90) \qquad \partial\Omega^*_{kj} = L^*_{j0}, L^*_{j1}, \ldots, L^*_{jp(j)} \quad \text{is a net in} \quad \bar{\Omega}^*_{kj},$$

and a regular homeomorphic mapping Φ_j of the set $\bigcup_{i=0}^{p(j)} L_{ji}$ onto the set $\bigcup_{i=0}^{p(j)} L^*_{ji}$ which is an extension of the mapping $f_k \mid \partial\Omega_{kj}$.

Let us put $r = q(k) + p(1) + \ldots + p(q(k))$, let

$$T_{q(k)+1} = L_{11}, \ldots, T_{q(k)+p(1)} = L_{1p(1)},$$

$$T_{q(k)+p(1)+1} = L_{21}, \ldots, T_{q(k)+p(1)+p(2)} = L_{2p(2)}, \ldots, T_r = L_{q(k)p(q(k))},$$

and similarly

$$T^*_{q(k)+1} = L^*_{11}, \ldots, T^*_r = L^*_{q(k)p(q(k))}.$$

Then it is evident that $T_1, \ldots, T_r$ and $T^*_1, \ldots, T^*_r$ are nets in $\bar{Q}$ and the mapping

$$\Phi = \Phi_j \quad \text{in} \quad \bigcup_{i=0}^{p(j)} L_{ji} \quad (j = 1, \ldots, q(k))$$

is a regular homeomorphic mapping of $\bigcup_{n=1}^{r} T_n$ onto $\bigcup_{n=1}^{r} T^*_n$ which is an extension of the mapping f_k (and thus also of f_1). Numbering suitably the components $X_1, \ldots, X_r$ and $X^*_1, \ldots, X^*_r$ of the sets $Q - \bigcup_{n=1}^{r} T_n$ and $Q - \bigcup_{n=1}^{r} T^*_n$, respectively, we have

$$(91) \qquad \Phi(\partial X_n) = \partial X^*_n \quad \text{for} \quad n = 1, \ldots, r;$$

besides, each of the sets X_n is a component of a certain set $\Omega_{kj} - \bigcup_{i=1}^{p(j)} L_{ji}$ so that

$$(92) \qquad \operatorname{diam} X_n < 6/(k + 1) \quad \text{for} \quad n = 1, \ldots, r.$$

Let us denote by Z_n the set of all points from ∂X_n which are linearly accessible from X_n, let $Z^*_n = \Phi(Z_n)$. Then Z^*_n is dense in ∂X^*_n and by Lemma 6, Part 1 there exist segments Λ^*_{ni}, $1 \leqq i \leqq s(n)$, such that

$$(93_1) \qquad \partial X^*_n = \Lambda^*_{n0}, \Lambda^*_{n1}, \ldots, \Lambda^*_{ns(n)} \quad \text{is a net in} \quad \bar{X}^*_n,$$

(93_2) every component of the set $X^*_n - \bigcup_{i=1}^{s(n)} \Lambda^*_{ni}$ has a diameter less than $6/(k + 1)$,

(93_3) there are no two segments $\Lambda^*_{ni}, \Lambda^*_{nj}$, $1 \leqq i < j \leqq s(n)$, with a common point in ∂X^*_n,

$$(93_4) \qquad \Lambda^*_{ni} \cap \partial X^*_n \subset Z^*_n \quad \text{for} \quad i = 1, \ldots, s(n) .$$

In virtue of Lemma 7, Part 2 there exist polygonal lines Λ_{ni} such that

$$(94) \qquad \partial X_n = \Lambda_{n0}, \Lambda_{n1}, \ldots, \Lambda_{ns(n)} \quad \text{is a net in} \quad \bar{X}_n ,$$

and a regular homeomorphic mapping Ψ_n of the set $\bigcup_{i=0}^{s(n)} \Lambda^*_{ni}$ onto $\bigcup_{i=0}^{s(n)} \Lambda_{ni}$ which is an extension of the mapping $\Phi_{-1} \mid \partial X^*_n$.

Let us denote $q(k + 1) = r + s(1) + \ldots + s(r)$, let

$$T_{r+1} = \Lambda_{11}, \ldots, T_{r+s(1)} = \Lambda_{1s(1)} ,$$

$$T_{r+s(1)+1} = \Lambda_{21}, \ldots, T_{r+s(1)+s(2)} = \Lambda_{2s(2)}, \ldots, T_{q(k+1)} = \Lambda_{rs(r)} ,$$

and similarly

$$T^*_{r+1} = \Lambda^*_{11}, \ldots, T^*_{q(k+1)} = \Lambda^*_{rs(r)} .$$

The mapping

$$f^*_{k+1} = \Psi_n \quad \text{in} \quad \bigcup_{i=0}^{s(n)} \Lambda^*_{ni} \quad (n = 1, \ldots, r)$$

is then evidently a regular homeomorphic mapping of the set $\bigcup_{n=1}^{q(k+1)} T^*_n$ onto $\bigcup_{n=1}^{q(k+1)} T_n$ which is an extension of the mapping Φ_{-1}, hence also of $(f_k)_{-1}$ and $(f_1)_{-1}$. The mapping $f_{k+1} = (f^*_{k+1})_{-1}$ is a regular homeomorphic mapping of the set $\bigcup_{n=1}^{q(k+1)} T_n$ onto $\bigcup_{n=1}^{q(k+1)} T^*_n$ which is an extension of the mapping Φ, hence also of f_k and f_1. Numbering suitably the components $\Omega_{k+1,j}$ and $\Omega^*_{k+1,j}$ ($j = 1, \ldots, q(k + 1)$) of the sets $Q - \bigcup_{n=1}^{q(k+1)} T_n$ and $Q - \bigcup_{n=1}^{q(k+1)} T^*_n$, respectively, we obtain

$$(95) \qquad f_{k+1}(\partial \Omega_{k+1,j}) = \partial \Omega_{k+1,j} \quad \text{for} \quad j = 1, \ldots, q(k + 1) .$$

It is easily seen from our construction that

$$(96) \qquad T^*_1, \ldots, T^*_{q(k+1)} \quad \text{are polygonal lines} .$$

Since every component $\Omega_{k+1,j}$ of the set $Q - \bigcup_{n=1}^{q(k+1)} T_n$ is part of one of the sets X_n and every component $\Omega^*_{k+1,j}$ of the set $Q - \bigcup_{n=1}^{q(k+1)} T^*_n$ is a component of a certain set $X^*_n - \bigcup_{i=1}^{s(n)} \Lambda^*_{ni}$, we have by (92) and ($93_2$)

$$(97) \qquad \operatorname{diam} \Omega_{k+1,j} < 6/(k + 1) ,$$

$$\operatorname{diam} \Omega^*_{k+1,j} < 6/(k + 1) \quad \text{for} \quad j = 1, \ldots, q(k + 1) .$$

This completes the induction: For every positive integer k we have constructed nets $T_1, \ldots, T_{q(k)}$ and $T_1^*, \ldots, T_{q(k)}^*$ in $\bar{Q}$, components Ω_{kj} and Ω_{kj}^* $(j = 1, \ldots, q(k))$ of the sets $Q - \bigcup_{n=1}^{q(k)} T_n$ and $Q - \bigcup_{n=1}^{q(k)} T_n^*$, respectively, and a regular homeomorphic mapping f_k of the set $\bigcup_{n=1}^{q(k)} T_n$ onto the set $\bigcup_{n=1}^{q(k)} T_n$ which is an extension of the mapping f_{k-1} (where $f_0 = f$) so that (86)–(88) hold.

Now we can show, similarly as in [1], pp. 378–379:

(98_1) the set $\bigcup_{n=1}^{\infty} T_n$, $\bigcup_{n=1}^{\infty} T_n^*$ are dense in $\bar{Q}$,

(98_2) the mapping $f_\omega = f_k$ in $\bigcup_{n=1}^{q(k)} T_n$ $(k = 1, 2, \ldots)$ is uniformly continuous in $\bigcup_{n=1}^{\infty} T_n$ and maps this set onto $\bigcup_{n=1}^{\infty} T_n^*$.

The mapping f_ω can be extended continuously onto the whole $\bar{Q}$ by a well-known theorem[10]). Denoting the resulting mapping by F we can show that F is one-to one[11]), and hence a homeomorphic mapping of $\bar{Q}$ onto itself. The mapping F is an extension of the mapping f_1, hence also of the mapping f, and since $f_1 \mid \partial Q = \mathrm{Id}$, we have $F \mid \partial Q = \mathrm{Id}$ as well. Putting now $F = \mathrm{Id}$ in $\mathbf{S} - Q$ we obtain the required homeomorphic mapping.

Theorem. *Every homeomorphic mapping of a topological circumference into* $\mathbf{S}$ *can be extended to a homeomorphic mapping of* $\mathbf{S}$ *onto itself.*

Proof. Let h be a homeomorphic mapping of a topological circumference $T \subset \mathbf{S}$ into $\mathbf{S}$. Let us choose points $A, B \in \mathbf{E}$ in different components of the set $\mathbf{S} - T$ and let $r \in (0, \infty)$ be so small that $\overline{U(A, r)} \cap T = \emptyset$. Putting

$$\Phi(z) = \frac{r}{z - A} - \frac{r}{B - A} \quad (z \in \mathbf{S}) \tag{99}$$

we have

$$|\Phi(z)| \leqq \left|\frac{r}{z - A}\right| + \left|\frac{r}{B - A}\right| < 2 \quad \text{for} \quad z \in \mathbf{S} - \overline{U(A, r)},$$

so that

$$\Phi(T) \subset Q \ ^{12)}. \tag{100}$$

Since Φ is a homeomorphic mapping $\mathbf{S}$ onto $\mathbf{S}$ and since the points A, B belong to

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10) See e.g. [2], p. 83.

11) The proof is not difficult. We refer the reader to [1], since here the present proof would bring nothing new.

12) Q means the same as in (65); similarly for Q_1.

different components of the set $\mathbf{S} - T$, the points $\infty = \Phi(A)$, $0 = \Phi(B)$ belong to different components of the set $\mathbf{S} - \Phi(T)$. Thus the point 0 belongs to Int $\Phi(T)$.

If we put

$$a = \min \{z \in \Phi(T);\ \mathrm{Im}\, z = 0\}\,, \quad b = \max \{z \in \Phi(T);\ \mathrm{Im}\, z = 0\}\ ^{13})\,,$$

we have $-2 < a < b < 2$. Denoting further by M_1, M_2 the arcs with endpoints a, b which satisfy $M_1 \cup M_2 = \Phi(T)$ we have

$$(\tilde{M}_1 \cup \tilde{M}_2) \cap (\langle -2, a \rangle \cup \langle b, 2 \rangle) = \emptyset\,.$$

Moreover, let us choose the notation so that

$$i \max \{\mathrm{Im}\, z;\ z \in \Phi(T),\ \mathrm{Re}\, z = 0\} \in M_1\,.$$

Certainly there exists a homeomorphic mapping f of the set $\Phi(T)$ onto ∂Q_1 which satisfies (68) and (69). Therefore, by Lemma 8, there exists a homeomorphic mapping F of the set $\mathbf{S}$ onto itself, which is an extension of the mapping f.

The mapping $F \circ \Phi$ is a homeomorphic mapping of $\mathbf{S}$ onto $\mathbf{S}$ which maps the topological circumference T onto ∂Q_1. Similarly, to the topological circumference $h(T)$ there exists a homeomorphic mapping G of the set $\mathbf{S}$ onto itself with $G(h(T)) = \partial Q_1$. The mapping

$$\Psi = G \circ h \circ \Phi_{-1} \circ F_{-1}$$

maps ∂Q_1 homeomorphically onto itself. If it is extended by

$$\Psi(tz) = t\,\Psi(z) \quad \text{for} \quad z \in \partial Q_1\,, \quad t \in \langle 0, \infty)\,, \quad \Psi(\infty) = \infty$$

to the whole $\mathbf{S}$, it is seen immediately that the extended mapping Ψ is a homeomorphic mapping of $\mathbf{S}$ onto $\mathbf{S}$.

Hence it follows that

$$H = G_{-1} \circ \Psi\ ^{14}) \circ F \circ \Phi$$

is a homeomorphic mapping of $\mathbf{S}$ onto itself which is an extension of the mapping h.

References

[1] *Kuratowski K.:* Topologie II, 2nd edition, Warsaw 1952.
[2] *Černý I.:* Elements of analysis in the complex domain, Prague 1967 (in Czech).

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[13]) The right-hand extrema exist since $0 \in \mathrm{Int}\, \Phi(T)$ so that the intersection of the straight line $\{z; \mathrm{Im}\, z = 0\}$ with $\Phi(T)$ is a compact set containing at least one negative and one positive number.

[14]) Ψ stands here, of course, for the extended mapping.