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INTERIOR REGULARITY OF SOLUTIONS TO SYSTEMS OF VARIATIONAL INEQUALITIES

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Let Ω be a domain in an N-dimensional Euclidean space R^N with a Lipschitzian boundary. We shall denote by $W_k^2(\Omega)$ the well-known Sobolev space with the norm

$$\|u\|_{W_2{}^k(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leqq k} |D^\alpha u|^2 \, dx\right)^{1/2}.$$

Further, let m be a positive integer. Denote by $[W_2^k(\Omega)]^m$ the Cartesian product of $W_2^k(\Omega)$ (m times) with the usual norm, which we shall denote by $\|\cdot\|_{2,k,\Omega}$.

The elements of $[W_2^k(\Omega)]^m$ will be denoted by $u = [u_1, \ldots, u_m]$ ($u_i \in W_2^k(\Omega)$, $i = 1, \ldots, m$).

Let Γ be a given subset of the boundary of Ω. Denote $V = [W_2^1(\Omega)]^m$, $V_\Gamma = \{v \in V;\ v = 0 \text{ on } \Gamma\}$. (We write $v = 0$ if $v_i = 0$ in the sense of traces for $i = 1, \ldots, m$.)

Let $a_t(\xi_1, \ldots, \xi_\varkappa)$ ($t = 1, \ldots, \varkappa$) be real functions of $\varkappa$ variables. Suppose that these functions have measurable bounded derivatives $\partial a_t/\partial \xi_s$ ($t, s = 1, \ldots, \varkappa$). Further, let N_t ($t = 1, \ldots, \varkappa$) be differential operators defined on $[W_2^1(\Omega)]^m$ by the formulas

$$N_t(u) = \sum_{i=1}^m \sum_{j=1}^N c_{i,j}^t \frac{\partial u_i}{\partial \xi_j},$$

where $c_{i,j}^t$ ($i = 1, \ldots, m$, $j = 1, \ldots, N$, $t = 1, \ldots, \varkappa$) are constants. We shall suppose that the following conditions are fulfilled (with $C > 0$):

(1) $$\sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s}(\eta)\, \xi_t \xi_s \geqq C \sum_{t=1}^{\varkappa} \xi_t^2 \quad \text{for each } \xi, \quad \eta \in R^\varkappa;$$

(2) $$\int_\Omega \sum_{t=1}^{\varkappa} (N_t(v))^2 \, dx \geqq C \|v\|_{2,1,\Omega}^2 \quad \text{for each} \quad v \in V_\Gamma.$$

The condition (1) is the usual ellipticity, the condition (2) is an inequality of Korn's type (cf. [2]).

Define an operator $A : V \to V^*$ *) by

(3) $$\langle Au, v\rangle = \sum_{t=1}^{\varkappa} a_t(N_t(u))\, N_t(v)\, \mathrm{d}x\,.$$

Consider given elements $u_0, \psi \in V$, $u_0 \geqq \psi$ on Ω. (We write $u \geqq \psi$ on Ω if $u_i \geqq \psi_i$ almost everywhere on Ω, $i = 1, \ldots, m$.) Denote

$$K = \{v \in V;\ v - u_0 \in V_\Gamma,\ v \geqq \psi \text{ on } \Omega\}\,.$$

For a given element $f = [f_1, \ldots, f_m] \in [L_2(\Omega)]^m$ we shall seek an element u such that

(4) $$u \in K\,,$$

(5) $$\int_\Omega \sum_{t=1}^{\varkappa} a_t(N_t(u))\, N_t(v - u)\, \mathrm{d}x \geqq \int_\Omega \sum_{r=1}^{m} f_r(v_r - u_r)\, \mathrm{d}x \quad \text{for each} \quad v \in K\,.$$

The last condition can be writen as

(6) $$\langle Au, v - u\rangle \geqq \langle f^*, v - u\rangle \quad \text{for all} \quad v \in K\,,$$

where the functional $f^* \in V^*$ is defined by $\langle f^*, v\rangle = \int_\Omega \sum_{r=1}^{m} f_r v_r\, \mathrm{d}x$.

It is easy to show that the set K is convex and closed in V and that the operator A is bounded, continuous, strictly monotone on K (i.e. $\langle Au - Av, u - v\rangle > 0$ for $u, v \in K$, $u \neq v$) and coercive on K (i.e. $\lim_{\substack{\|u\| \to \infty \\ u \in K}} (\langle Au, u - v_0\rangle / \|u\|_{2,1,\Omega}) = +\infty$ for a certain $v_0 \in K$). This follows from the assumptions (1), (2).) Hence, the existence and unicity of the solution of our problem follows from the general theory of variational inequalities which is developed for example in the book [3]. Here we shall deal with the interior regularity of the solution. Namely, we shall prove the following result:

Theorem. *Suppose* $\psi \in [W_2^3(\Omega)]^m$. *Let* u *be a solution of the problem* (4), (5), *let* Ω' *be a subdomain of* Ω *such that* $\bar{\Omega}' \subset \Omega$. *Then* $u \in [W_2^2(\Omega')]^m$.

This result was proved by J. Frehse in [1] for a special class of operators N_t and for $u_0 = 0$. We shall present here another proof, which is based on penalty method and applies to the general case.

Let us consider a continuous, bounded and monotone operator $\beta : V \to V^*$ such that $\beta(v) = 0$ if and only if $v \in K$, i.e. the so called penalty operator correspond-

*) We denote by V^* the dual space to V; the duality between V and V^* is denoted by $\langle .,. \rangle$.

ing to the set K. Then for each positive ε and $f \in V^*$ there exists a unique solution $u^\varepsilon \in V$ of the equation*)

$$(7) \qquad Au^\varepsilon + \frac{1}{\varepsilon} \beta u^\varepsilon = f$$

and, moreover, $u^\varepsilon \rightharpoonup u$ (if $\varepsilon \to 0+$), where u is a solution of the problem (4), (6) ($\rightharpoonup$ denotes the weak convergence). Especially, u^ε are bounded in the norm of V and $(1/\varepsilon)\, \beta u^\varepsilon$ are bounded in the norm of V^*. This holds for a general Banach space V, a convex closed set $K \subset V$ and a bounded, continuous, strictly monotone and coercive (on K) operator $A : V \to V^*$ (see [3]).

In our special case, it is not convenient to introduce the penalty operator with respect to K directly in the space V. But if we set $K_0 = \{v \in V_\Gamma;\ v + u_0 \in K\}$ and write $w = u - u_0$, then the conditions (4), (6) are equivalent to

$$(8) \qquad w \in K_0\,,$$

$$(9) \qquad \langle A(u_0 + w), v - w \rangle \geqq \langle f, v - w \rangle \quad \text{for each} \quad v \in K_0\,.$$

Define an operator $\beta : V_\Gamma \to V_\Gamma^*$ by

$$\langle \beta(w), v \rangle = - \int_\Omega \sum_{r=1}^{m} (u_{0,r} + w_r - \psi_r)^- \, v_r \, dx \quad \text{for} \quad u, v \in V_\Gamma\,.$$

It is easy to verify that β has all the properties declared above (for V_Γ instead of V and K_0 instead of K).

We can write an operator $A_{u_0} : V_\Gamma \to V_\Gamma^*$ (defined by $A_{u_0}(v) = A(u_0 + v)$) instead of A in (9). Hence we obtain from the above that for each $\varepsilon > 0$ there exists $w^\varepsilon \in V_\Gamma$ such that

$$A(u_0 + w^\varepsilon) + \frac{1}{\varepsilon} \beta(w^\varepsilon) = f\,.$$

This means (by setting $u^\varepsilon = u_0 + w^\varepsilon$) that there exists $u^\varepsilon \in V$ such that

$$(10) \qquad A(u^\varepsilon) + \frac{1}{\varepsilon} \beta(u^\varepsilon - u_0) = f$$

in the space V_Γ^*, i.e.

$$(11) \qquad \int_\Omega \sum_{t=1}^{\varkappa} a_t(N_t(u^\varepsilon))\, N_t(v)\, dx - \frac{1}{\varepsilon} \int_\Omega \sum_{r=1}^{m} (u_r^\varepsilon - \psi_r)^- \, v_r \, dx = \int_\Omega \sum_{r=1}^{m} f_r v_r \, dx$$

for each $v \in V_\Gamma$. Moreover, u^ε are bounded in V, $(1/\varepsilon)\, \beta(u^\varepsilon - u_0)$ are bounded in V_Γ^* (but need not be bounded in V^*!).

*) The so called equation with the penalty corresponding to the problem (4), (6).

In the sequel we shall use the following notation: Let e be a vector in the direction of the i-th coordinate axis in R^N, $\|e\|_{R^N} = 1$; if v is a real or vector function, $h \neq 0$ a real number, then v_h denotes the function defined by $v_h(x) = v(x - he)$. Moreover, we set

$$\Delta_h(v) = \frac{v_h - v}{h}\,.$$

Proof of Theorem. Let u^ε be a solution of the equation (10). Consider an arbitrary element $v \in V$ such that $\operatorname{supp} v \subset \Omega'$ (i.e. v lies in the closure of $[\mathscr{D}(\Omega')]^m$ in V). Then we have $v \in V_\Gamma$ and for h sufficiently small also $v_{-h} \in V_\Gamma$. Hence (11) holds for v as well as for v_{-h} instead of v. Thus

$$(12)\quad \int_\Omega \sum_{t=1}^{\varkappa} a_t(N_\tau(u^\varepsilon))\, N_t(v_{-h} - v)\, \mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega \sum_{r=1}^{m} (u_r^\varepsilon - \psi_r)^- \,(v_{r,-h} - v_r)\, \mathrm{d}x =$$
$$= \int_\Omega \sum_{r=1}^{m} f_r(v_{r,-h} - v_r)\, \mathrm{d}x\,.$$

The same equality holds for $v - v_h$ (instead of $v_{-h} - v$) and by a translation of h

$$(13)\quad \int_\Omega \sum_{t=1}^{\varkappa} a_t(N_\tau(u^\varepsilon_{-h}))\, N_t(v_{-h} - v)\, \mathrm{d}x -$$
$$- \frac{1}{\varepsilon}\int_\Omega \sum_{r=1}^{m} (u^\varepsilon_{r,-h} - \psi_{r,-h})^- \,(v_{r,-h} - v_r)\, \mathrm{d}x =$$
$$= \int_\Omega \sum_{r=1}^{m} f_{r,-h}(v_{r,-h} - v_r)\, \mathrm{d}x\,.$$

By adding the two equalities we obtain

$$(14)\quad \int_\Omega \sum_{t=1}^{\varkappa} [a_t(N_\tau(u^\varepsilon_{-h})) - a_t(N_\tau(u^\varepsilon))]\,.\,N_t(v_{-h} - v)\, \mathrm{d}x -$$
$$- \frac{1}{\varepsilon}\int_\Omega \sum_{r=1}^{m} (u^\varepsilon_{r,-h} - \psi_{r,-h})^- - (u^\varepsilon_r - \psi_r)^- \,(v_{r,-h} - v_r)\, \mathrm{d}x =$$
$$= \int_\Omega \sum_{r=1}^{m} (f_{r,-h} - f_r)\,(v_{r,-h} - v_r)\, \mathrm{d}x\,.$$

Further, we shall consider a domain Ω^* such that $\bar{\Omega}' \subset \Omega^*$, $\bar{\Omega}^* \subset \Omega$. There exists a real function $\Phi \in \mathscr{D}(\Omega^*)$ such that $\Phi \equiv 1$ on Ω'. We shall set $v = \Phi^2\,.\,u^\varepsilon$ in (14). Now, we obtain from (14)

$$(15)\quad \langle A(u^\varepsilon_{-h}) - A(u^\varepsilon),\ (\Phi^2 u^\varepsilon)_{-h} - \Phi^2 u^\varepsilon\rangle -$$

$$-\frac{1}{\varepsilon}\int_\Omega \sum_{r=1}^m [(u^\varepsilon_{r,-h}-\psi_{r,-h})^- - (u^\varepsilon_r-\psi_r)^-] \cdot ((\Phi^2 u^\varepsilon_r)_{-h} - \Phi^2 u^\varepsilon_r)\,\mathrm{d}x =$$
$$= \int_\Omega \sum_{r=1}^m (f_{r,-h}-f_r)((\Phi^2 u^\varepsilon_r)_{-h} - \Phi^2 u^\varepsilon_r)\,\mathrm{d}x\,.$$

We shall show that there exist constants C_1, C_2, C_3 such that $C_1 > 0$ and

$$C_1 \|\Delta_{-h}(u^\varepsilon)\,\Phi\|^2_{2,1,\Omega} \leqq C_2 \|\Delta_{-h}(u^\varepsilon)\,\Phi\|_{2,1,\Omega} + C_3\,. \tag{16}$$

It will be clear from here that the norms $\|\Delta_{-h}(u^\varepsilon)\,.\,\Phi\|_{2,1,\Omega}$ are bounded, especially, the norms $\|\Delta_{-h}(u^\varepsilon)\|_{2,1,\Omega'}$ are bounded (independently of ε, h). We have $\Delta_{-h}(u^\varepsilon) \underset{\varepsilon\to 0+}{\rightharpoonup} \Delta_{-h}(u)$ in V for each fixed positive h and therefore the norms $\|\Delta_{-h}(u)\|_{2,1,\Omega'}$ $(h>0)$ are bounded, too. That means that there exists a weakly convergent sequence $\Delta_{-h_n}(u)$ $(h_n \to 0)$. Simultaneously, $\Delta_{-h}(u) \rightharpoonup \partial u/\partial x_i$ (if $h\to 0$) in $[L_2(\Omega)]^m$, because $u \in [W^1_2(\Omega)]^m$. This implies $\Delta_{-h_n}(u) \rightharpoonup \partial u/\partial x_i$ in $[W^1_2(\Omega')]^m$. In particular, $\partial u/\partial x_i \in [W^1_2(\Omega')]^m$, i.e. $u \in [W^2_2(\Omega')]^m$ (because the index i was arbitrary). Hence it is sufficient for the proof of Theorem to show that (16) holds.

First, we shall estimate the left hand side in (15). By using the identity $\Delta_{-h}(\Phi^2 u^\varepsilon) = \Phi^2\,\Delta_{-h}(u^\varepsilon) + u^\varepsilon_{-h}\,\Delta_{-h}(\Phi^2)$ we obtain

$$\frac{1}{h^2}\langle A(u^\varepsilon_{-h}) - A(u^\varepsilon),\, (\Phi^2 u^\varepsilon)_{-h} - \Phi^2 u^\varepsilon\rangle = \tag{17}$$
$$= \int_\Omega \sum_{t,s=1}^{\varkappa} \int_0^1 \frac{\partial a_t}{\partial \xi_s}(N_\tau(u^\varepsilon) + \varrho(N_\tau(u^\varepsilon_{-h}) -$$
$$- N_\tau(u^\varepsilon)))\,\mathrm{d}\varrho\, N_s(\Delta_{-h}(u^\varepsilon))\, N_t[\Delta_{-h}(\Phi^2 u^\varepsilon)]\,\mathrm{d}x = I_1 + I_2\,,$$

where

$$I_1 = \int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Delta_{-h}(u^\varepsilon))\,.\,N_t(\Phi^2\,\Delta_{-h}(u^\varepsilon))\,\mathrm{d}\varrho\,\mathrm{d}x\,,$$

$$I_2 = \int_\Omega \int_0^1 \sum_{s,t=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Delta_{-h}(u^\varepsilon))\, N_t(u^\varepsilon_{-h}\,\Delta_{-h}(\Phi^2))\,\mathrm{d}\varrho\,\mathrm{d}x\,.$$

(We do not write the arguments of the functions $\partial a_t/\partial \xi_s$ depending on ϱ; it is the same as in (17).) By using the formulas

$$N_t(\Phi^2\,\Delta_{-h}(u^\varepsilon)) = \Phi\, N_t(\Phi\,\Delta_{-h}(u^\varepsilon)) + \sum_{i=1}^m \sum_{j=1}^N c^t_{i,j} \frac{\partial \Phi}{\partial x_j}\,\Phi\,\Delta_{-h}(u^\varepsilon_i)\,, \tag{18}$$

$$N_s(\Delta_{-h}(u^\varepsilon))\,\Phi = N_s(\Phi\,\Delta_{-h}(u^\varepsilon)) - \sum_{i=1}^m \sum_{j=1}^N c^s_{i,j} \frac{\partial \Phi}{\partial x_j}\,\Delta_{-h}(u^\varepsilon_i) \tag{19}$$

we obtain

$$(20)\qquad I_1 = \int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Phi\, \Delta_{-h}(u^\varepsilon))\, N_t(\Phi\, \Delta_{-h}(u^\varepsilon))\, d\varrho\, dx -$$

$$- \int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_t(\Phi\, \Delta_{-h}(u^\varepsilon)) \left[\sum_{i=1}^{m} \sum_{j=1}^{N} c^s_{i,j}\, \Delta_{-h}(u^\varepsilon_i) \frac{\partial \Phi}{\partial x_j} \right] d\varrho\, dx +$$

$$+ \int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Delta_{-h}(u^\varepsilon))\, \Phi \left[\sum_{i=1}^{m} \sum_{j=1}^{N} c^t_{i,j}\, \Delta_{-h}(u^\varepsilon_i) \frac{\partial \Phi}{\partial x_j} \right] d\varrho\, dx\,.$$

By the assumptions (1), (2) the first integral is not less than $c_1 \|\Phi\, \Delta_{-h}(u))\|^2_{2,1,\Omega}$. Let us estimate the second integral. The functions $\partial a_t / \partial \xi_s$, $\partial \Phi / \partial x_j$ are bounded. Hence, we obtain by virtue of the Hölder inequality and the inequality $2ab \leqq \delta a^2 + \delta^{-1} b^2$ (holding for arbitrary real a, b and $\delta > 0$) that the second integral is not greater in the absolute value than

$$c_2 \|\Phi\, \Delta_{-h}(u^\varepsilon))\|_{2,1,\Omega}\, \|\Delta_{-h}(u^\varepsilon))\|_{2,\Omega^*} \leqq$$
$$\leqq c_2^2 (\delta \|\Phi\, \Delta_{-h}(u^\varepsilon))\|^2_{2,1,\Omega} + \delta^{-1} \|\Delta_{-h}(u^\varepsilon)\|^2_{2,\Omega^*})\,,$$

where we denote by $\|\cdot\|_{2,\Omega}$ the norm in the space $[L_2(\Omega)]^m$. Let us estimate the last integral in (20). This integral can be rewritten as

$$\int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Phi\, \Delta_{-h}(u^\varepsilon)) \left[\sum_{i=1}^{m} \sum_{j=1}^{N} c^t_{i,j}\, \Delta_{-h}(u^\varepsilon_i) \frac{\partial \Phi}{\partial x_j} \right] d\varrho\, dx -$$

$$- \int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} \left[\sum_{i,j=1}^{N} c^s_{i,j}\, \Delta_{-h}(u^\varepsilon_i) \frac{\partial \Phi}{\partial x_j} \right] \left[\sum_{i=1}^{m} \sum_{j=1}^{N} c^t_{i,j}\, \Delta_{-h}(u^\varepsilon_i) \frac{\partial \Phi}{x_j} \right] d\varrho\, dx\,.$$

The first expression can be estimated in the same way as the second integral in (20), the second is not greater than $c_3 \|\Delta_{-h}(u^\varepsilon)\|^2_{2,\Omega^*}$. Hence we obtain

$$(21)\qquad I_1 \geqq c_1 \|\Phi\, \Delta_{-h}(u^\varepsilon)\|^2_{2,1,\Omega} - c_4\, \delta \|\Phi\, \Delta_{-h}(u^\varepsilon)\|^2_{2,1,\Omega} -$$
$$- c_5 (\delta^{-1} + 1) \|\, \Delta_{-h}(u^\varepsilon)\|^2_{2,\Omega^*}\,.$$

Now we shall estimate the integral I_2. It is easy to see that

$$\Delta_{-h}(\Phi^2) = \Delta_{-h}(\Phi)\, (\Phi + \Phi_{-h})\,,$$
$$N_s[\Delta_{-h}(u^\varepsilon)\, (\Phi + \Phi_{-h})] =$$
$$= N_s[\Delta_{-h}(u^\varepsilon)]\, (\Phi + \Phi_{-h}) + \sum_{i=1}^{m} \sum_{j=1}^{N} c^s_{i,j}\, \Delta_{-h}(u^\varepsilon_i) \frac{\partial}{\partial x_j} (\Phi + \Phi_{-h})\,,$$

$$N_t[u^\varepsilon_{-h}\,\Delta_{-h}(\Phi^2)] = N_t(u^\varepsilon_{-h})\,\Delta_{-h}(\Phi^2) + \sum_{i=1}^{m}\sum_{j=1}^{N} c^t_{i,j} u^\varepsilon_{i,-h} \frac{\partial}{\partial x_j}(\Delta_{-h}\Phi^2)\,.$$

By an easy calculation we obtain from here

$$(22)\qquad I_2 = \int_\Omega\int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s[\Delta_{-h}(u^\varepsilon)(\Phi + \Phi_{-h})]\, N_t[u^\varepsilon_{-h}\,\Delta_{-h}(\Phi)]\,\mathrm{d}\varrho\,\mathrm{d}x -$$

$$- \int_\Omega\int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s[\Delta_{-h}(u^\varepsilon)(\Phi + \Phi_{-h})] \times$$

$$\times \left[\sum_{i=1}^{m}\sum_{j=1}^{N} c^t_{i,j} u^\varepsilon_{i,-h} \frac{\partial}{\partial x_j}(\Delta_{-h}(\Phi))\right] \mathrm{d}\varrho\,\mathrm{d}x -$$

$$- \int_\Omega\int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} \left[\sum_{i=1}^{m}\sum_{j=1}^{N} c^s_{i,j}\,\Delta_{-h}(u^\varepsilon_i) \frac{\partial}{\partial x_j}(\Phi + \Phi_{-h})\right] N_t(u^\varepsilon_{-h}\,\Delta_{-h}(\Phi))\,\mathrm{d}\varrho\,\mathrm{d}x +$$

$$+ \int_\Omega\int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} \left[\sum_{i=1}^{m}\sum_{j=1}^{N} c^s_{i,j}\,\Delta_{-h}(u^\varepsilon_i) \frac{\partial}{\partial x_j}(\Phi + \Phi_{-h})\right] \times$$

$$\times \left[\sum_{i=1}^{m}\sum_{j=1}^{N} c^t_{i,j} u^\varepsilon_{i,-h} \frac{\partial}{\partial x_j}(\Delta_{-h}(\Phi))\right] \mathrm{d}\varrho\,\mathrm{d}x +$$

$$+ \int_\Omega\int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Delta_{-h}(u^\varepsilon) \left[\sum_{i=1}^{m}\sum_{j=1}^{N} c^t_{i,j} u^\varepsilon_{i,-h} \frac{\partial}{\partial x_j}(\Delta_{-h}(\Phi^2))\right] \mathrm{d}\varrho\,\mathrm{d}x\,.$$

The functions $\partial a_t/\partial \xi_s$, $\Delta_{-h}(\Phi)$ are bounded. Hence we can use the same argument to estimate the first integral in (22) as in the case of the second integral in (20). Moreover, if we use the identity $\Delta_{-h}(u^\varepsilon)(\Phi + \Phi_{-h}) = 2\Delta_{-h}(u^\varepsilon)\,\Phi + (u^\varepsilon_{-h} - u^\varepsilon)\,\Delta_{-h}(\Phi)$, we obtain that the first integral is not greater (in the absolute value) than

$$c_7\|\Delta_{-h}(u^\varepsilon)(\Phi + \Phi_{-h})\|_{2,1,\Omega^*}\,\|u^\varepsilon_{-h}\|_{2,1,\Omega^*} \leqq$$

$$\leqq c_7^2\delta\|\Delta_{-h}(u^\varepsilon)(\Phi + \Phi_{-h})\|^2_{2,1,\Omega} + c_7^2\delta^{-1}\|u^\varepsilon_{-h}\|^2_{2,1,\Omega^*} \leqq$$

$$c_8\delta\|\Delta_{-h}(u^\varepsilon)\,\Phi\|^2_{2,1,\Omega} + c_8\delta^{-1}\|u^\varepsilon\|^2_{2,1,\Omega}\,.$$

The second integral can be estimated in the same way. The third integral is not greater than

$$c_9\|\Delta_{-h}(u^\varepsilon)\|^2_{2,\Omega^*} + c_9\|u^\varepsilon\|^2_{2,1,\Omega}\,,$$

the fourth integral is not greater than

$$c_{10}\|\Delta_{-h}(u^\varepsilon)\|^2_{2,\Omega^*} + c_{10}\|u^\varepsilon\|^2_{2,\Omega}\,.$$

In the case of the last integral, we come back to an expression without any derivatives of the functions a_t. We have

$$(23)\qquad \int_\Omega \int_0^1 \sum_{t,s=1}^{\varkappa} \frac{\partial a_t}{\partial \xi_s} N_s(\Delta_{-h}(u^\varepsilon)) \left[\sum_{i=1}^m \sum_{j=1}^N c_{i,j}^t u_{i,-h}^\varepsilon \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi^2)\right] \mathrm{d}\varrho \, \mathrm{d}x =$$

$$= \frac{1}{h} \int_\Omega \sum_{t=1}^{\varkappa} [a_t(N_\tau(u_{-h}^\varepsilon)) - a_t(N_\tau(u^\varepsilon))] \left[\sum_{i=1}^m \sum_{j=1}^N c_{i,j}^t u_{i,-h}^\varepsilon \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi^2))\right] \mathrm{d}x =$$

$$= \frac{1}{h} \int_\Omega \sum_{t=1}^{\varkappa} a_t(N_\tau(u^\varepsilon)) \sum_{i=1}^m \sum_{j=1}^N c_{i,j}^t \left[u_i^\varepsilon \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi_h^2)) - u_{i,-h}^\varepsilon \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi^2))\right] \mathrm{d}x .$$

Further,

$$(24)\qquad u_i^\varepsilon \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi_h^2)) - u_{i,-h}^\varepsilon \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi^2)) =$$

$$= (u_i^\varepsilon - u_{i,-h}^\varepsilon) \frac{\partial}{\partial x_j} (\Delta_{-h}(\Phi_h^2) + u_{i,-h}^\varepsilon \frac{\partial}{\partial x_j} [\Delta_{-h}(\Phi_h^2 - \Phi^2)] .$$

It is easy to see that the functions $(\partial/\partial x_j)(\Delta_{-h}(\Phi_h^2))$ and $(1/h)\,\Delta_{-h}(\Phi_h^2 - \Phi^2)$ are bounded. Moreover, we have $|a_t(\xi)| \leqq c\|\xi\|_{R^N}$ (for $\xi \in R^N$, $\|\xi\|_{R^N}$ denotes the usual Euclidean norm). This follows from the assumption that the derivatives $\partial a_t/\partial \xi_s$ are bounded. Hence we obtain from (23) and (24) that the last integral in (22) is not greater in absolute value than

$$c_{11}\|u^\varepsilon\|_{2,1,\Omega} (\|\Delta_{-h}(u^\varepsilon)\|_{2,\Omega^*} + \|u^\varepsilon\|_{2,\Omega}) \leqq$$
$$\leqq c_{12}(\|u^\varepsilon\|_{2,1,\Omega}^2 + \|\Delta_{-h}(u^\varepsilon)\|_{2,\Omega^*}^2 + \|u^\varepsilon\|_{2,\Omega}^2) .$$

This together with the previous yields

$$(25)\qquad |I_2| \leqq c_{13}\,\delta \|\Delta_{-h}(u^\varepsilon)\,\Phi\|_{2,1,\Omega}^2 + c_{14}(\|u^\varepsilon\|_{2,1,\Omega}^2 + \|\Delta_{-h}(u^\varepsilon)\|_{2,\Omega^*}^2) ,$$

where $c_{13} > 0$ and c_{14} depends on the choice of δ.

Let us remind that the norms $\|u^\varepsilon\|_{2,1,\Omega}^2$ are bounded. It follows from here that the norms $\|\Delta_{-h}(u^\varepsilon)\|_{2,\Omega^*}$ are bounded, too. Hence, if the number δ is sufficiently small, then we obtain from (21), (25)

$$(26)\qquad \frac{1}{h^2} \langle A(u_{-h}^\varepsilon) - A(u^\varepsilon), (\Phi^2 u^\varepsilon)_{-h} - \Phi^2 u \rangle = I_1 + I_2 \geqq$$
$$\geqq C_1 \|\Phi\, \Delta_{-h}(u^\varepsilon)\|_{2,1,\Omega}^2 - C_2 ,$$

where $C_1 > 0$. (The constants C_1, C_2 depend on the choice of δ.)

Now we shall estimate the member given by the penalty operator in the equation (15). We have

$$(27)\qquad -\frac{1}{\varepsilon h^2} \int_\Omega \sum_{r=1}^m [(u_{r,-h}^\varepsilon - \psi_{r,-h})^- - (u_\varepsilon^r - \psi_r)^-] \times$$

$$\times\left[(u_r^\varepsilon \Phi^2)_{-h} - u_r^\varepsilon \Phi^2\right] \mathrm{d}x =$$

$$= -\frac{1}{\varepsilon h^2}\int_\Omega \sum_{r=1}^m [(u_{r,-h}^\varepsilon - \psi_{r,-h})^- - (u_r^\varepsilon - \psi_r)^-] \times$$

$$\times (u_{r,-h}^\varepsilon - \psi_{r,-h} - u_r^\varepsilon + \psi_r)\, \Phi^2 \,\mathrm{d}x +$$

$$+ \frac{1}{\varepsilon h^2}\int_\Omega \sum_{r=1}^m [(u_{r,-h}^\varepsilon - \psi_{r,-h})^- - (u_r^\varepsilon - \psi_r)^-] \times$$

$$\times\left[-(u_r^\varepsilon \Phi^2)_{-h} + u_{r,-h}^\varepsilon \Phi^2 - \psi_{r,-h}\Phi^2 + \psi_r \Phi^2\right] \mathrm{d}x\,.$$

It is clear that the first expression on the right hand side is nonnegative. (This follows from the monotonicity of the negative part.) The second expression can be written as

$$\frac{1}{\varepsilon h^2}\int_\Omega \sum_{r=1}^m (u_r^\varepsilon - \psi_r)^- \left[-u_r^\varepsilon \Phi^2 + u_r^\varepsilon \Phi_h^2 - \psi_r \Phi_h^2 + \psi_{r,h}\Phi_h^2 +\right.$$

$$\left. + (u_r^\varepsilon \Phi^2)_{-h} - u_{r,-h}^\varepsilon \Phi^2 + \psi_{r,-h}\Phi^2 - \psi_r \Phi^2\right] \mathrm{d}x =$$

$$= \frac{1}{\varepsilon h^2}\langle \beta(u^\varepsilon - u_0), u_{-h}^\varepsilon(\Phi_{-h}^2 - \Phi^2) + u^\varepsilon(\Phi_h^2 - \Phi^2) +$$

$$+ (\psi_{-h} - \psi)\, \Phi^2 + (\psi_h - \psi)\, \Phi_h^2\rangle =$$

$$= \frac{1}{\varepsilon h^2}\langle \beta(u^\varepsilon - u_0), u_{-h}^\varepsilon[(\Phi_{-h}^2 - \Phi^2) - (\Phi^2 - \Phi_h^2)] + (u_{-h}^\varepsilon - u^\varepsilon)(\Phi^2 - \Phi_h^2) +$$

$$+ (\psi_h - \psi_r)(\Phi_h^2 - \Phi^2) + (\psi_h - 2\psi + \psi_{-h})\, \Phi^2\rangle =$$

$$= \frac{1}{\varepsilon}\left\langle \beta(u^\varepsilon - u_0), u_{-h}^\varepsilon \frac{\Phi_{-h}^2 - 2\Phi^2 + \Phi_h^2}{h^2} + (u_{-h}^\varepsilon - u^\varepsilon)\frac{(\Phi - \Phi_h)(\Phi_h - \Phi)}{h^2} + \right.$$

$$\left. + \frac{u_{-h}^\varepsilon - u^\varepsilon}{h}\frac{\Phi - \Phi_h}{h} 2\Phi + \frac{(\psi_h - \psi)}{h}\frac{(\Phi_h^2 - \Phi^2)}{h} + \frac{(\psi_h - 2\psi + \psi_{-h})}{h^2}\Phi^2\right\rangle.$$

This expression in the absolute value is not greater than

$$(28) \qquad c_{15}(\|u^\varepsilon\|_{2,1,\Omega} + \|\Delta_{-h}(u^\varepsilon)\,\Phi\|_{2,1,\Omega} + \|\Delta_h(\psi)\|_{2,1,\Omega^*} + \frac{1}{h^2}\|\psi_h - 2\psi + \psi_{-h}\|_{2,1,\Omega^*}\,,$$

because the functionals $(1/\varepsilon)\,\beta(u^\varepsilon - u_0)$ are bounded in the norm of V_Γ^* (independently of ε) and the functions $(1/h^2)(\Phi_{-h}^2 - 2\Phi^2 + \Phi_h^2)$, $(1/h^2)(\Phi - \Phi_h)^2\, (1/h)(\Phi - \Phi_h)$ are bounded. We know that $\|u^\varepsilon\|_{2,1,\Omega}$ are bounded. Moreover, it follows from the assumption $\psi \in [W_2^3(\Omega)]^m$ that the norms $\|\Delta_h\psi\|_{2,1,\Omega^*}$, $(1/h^2)\,\|\psi_h - 2\psi - \psi_{-h}\|_{2,1,\Omega^*}$ are bounded (if $h \to 0$). Hence the expression from (28) is not less than

$$-c_{16}\|\Delta_{-h}(u^\varepsilon)\,\Phi\|_{2,1,\Omega} - c_{17}\,.$$

The right hand side in (15) can be rewritten by means of the identity

$$(29)\qquad (\Phi^2 u^\varepsilon)_{-h} - \Phi^2 u^\varepsilon = \Phi^2_{-h}(u^\varepsilon_{-h} - u^\varepsilon) + (\Phi^2_{-h} - \Phi^2)\, u^\varepsilon =$$
$$= (\Phi^2_{-h} - \Phi^2)\,(u^\varepsilon_{-h} - u^\varepsilon) + \Phi^2(u^\varepsilon_{-h} - u^\varepsilon) + (\Phi^2_{-h} - \Phi^2)\, u^\varepsilon\,.$$

The functions $\Delta_{-h}(\Phi^2)$, Φ, Φ^2 are bounded independently of h. Moreover, it is easy to see that for $f \in [L_2(\Omega)]^m$ we have $\Delta_{-h}(f) \rightharpoonup \partial f/\partial x_i$ in the dual space $([W_2^1(\Omega^*)]^m)^*$. Especially, $\Delta_{-h}(f)$ are bounded in this space.

It follows from here and from (29) that the absolute value of the right hand side in (15) is not greater than

$$c_{17}(\|\Delta_{-h}(u^\varepsilon)\, \Phi\|_{2,1,\Omega} + \|u^\varepsilon\|_{2,1,\Omega})h^2 \leqq c_{18}\|\Delta_{-h}(u^\varepsilon)\, \Phi\|_{2,1,\Omega} h^2\,.$$

Now all the estimates proved yield together the inequality (16). The proof of the theorem is complete.

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