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GENERALIZATION OF ONE BAER'S THEOREM FOR NETS

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As is well-known, R. BAER has proved in [1] that a projective plane is (P, l)-desarguesian for a point P and a line l if and only if it is (P, l)-transitive ([1], Theorem 6.2). In the present Note I shall generalize this Baer's theorem for nets of degree $\geqq 4$ provided P is a singular point and l the line of singular points.

After finishing the first version of this Note I got acquainted with the book [2] where an analogous problem for finite nets (of degree $\geqq 3$) is considered in Chap. 4. Whereas I restricted myself to the configurative approach, [2] uses above all the algebraic (coordinatizing) methods and the case of degree 3 is not excluded. Our results show that the excluding of 3-nets (where the situation is known: cf. [3], p. 51) leads to a certain simplification, namely that only the Desargues condition is essential while the Reidemeister condition is superfluous and that the hypothesis of semi-regularity of automorphisms (either no point is fixed or all points are fixed) can be omitted.

Finally I wish to remark that I investigated also the influence of various specializations of the minor Desargues condition with respect to a net of degree $\geqq 4$ onto coordinatizing algebras in the paper [4] stimulated by former results of V. D. BELOUSOV.

A *non-trivial net* (briefly: *net*) is defined as a triplet $(\mathscr{P}, \mathscr{L}, (V_\iota)_{\iota \in I})$ where $\mathscr{P}$ is a non-void set, $\mathscr{L}$ a set of some at least two-element subsets of $\mathscr{P}$, I is an index set with $\# I \geqq 3$ and $\iota \mapsto V_\iota$ an injective mapping of I into $\mathscr{P}$ such that the following conditions are satisfied:

(i) $\{V_\iota \mid \iota \in I\} \in \mathscr{L}$,

(ii) $\forall P \in \mathscr{P} \setminus \{V_\iota \mid \iota \in I\} \quad \forall \iota \in I \quad \exists!\, l \in \mathscr{L} \quad P, V_\iota \in l$,

(iii) $\forall a, b \in \mathscr{L};\ a \neq b \quad \#(a \cap b) = 1$,

(iv) $\#(\mathscr{P} \setminus \{V_\iota \mid \iota \in I\}) \geqq 2$ [1]).

Elements of $\mathscr{P}$ are called *points*, elements of $\mathscr{L}$ *lines*, points V_ι, $\iota \in I$, are termed *singular* (but here it will be more convenient to term them *improper*); also the line

[1]) If (iv) is changed to $\#(\{\mathscr{P} \setminus \{V_\iota \mid \iota \in I\}) \leqq 1$ then a trivial net arises.

$\{V_\iota \mid \iota \in I\}$ will be termed *improper* whereas the remaining points and lines will be denoted as *proper*. The cardinality of I is said to be *the degree* of the net.

By $\overline{A_1, \ldots, A_n}$ we write the fact that points $A_1, \ldots, A_n$ lie on the same line. If A, B are distinct points then $\#\{l \in \mathcal{L} \mid A, B \in l\} = 0$ or $=1$; in the latter case the only line through A, B will be designated by AB. If a, b, are distinct lines, then $\#(a \cap b) = 1$; the only common point of a, b will be designated by $a \sqcap b$.

A quadruplet (P, Q, R, S) is called *a parallelogram* if P, Q, R, S are proper points such that $\overline{P, Q, V}$, $\overline{R, S, V}$, $\overline{Q, R, W}$, $\overline{P, S, W}$ hold for suitable improper points $V \neq W$. A triplet (A, B, C) is called *a triangle* if A, B, C are proper points such that either $\overline{A, B, C}$ does not hold or $\overline{A, B, C}$ holds but A, B, C are not mutually distinct.

Now let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota \in I})$ be a net and let α, β, γ be mutually distinct indices. Then *the Reidemeister aondition* of type (α, β, γ) in $\mathcal{N}$ is defined as the following implication: If (P, Q, R, S), (P, Q, Q', P'), (Q, Q', R', R), (P, P', S', S) are parallelograms in $\mathcal{N}$ such that $\overline{P, S, V_\alpha}$, $\overline{P, Q, V_\beta}$, $\overline{P, P', V_\gamma}$ [2]) then also (P', Q', R', S') is a parallelogram.

Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota \in I})$ be a net of degree $\geqq 4$ and δ an index. Then *the Desargues condition* of type (δ) in $\mathcal{N}$ is defined as the following implication: If (A, B, C), (A', B', C) are triangles in $\mathcal{N}$, if (A, B, B', A'), (A, C, C', A') are parallelograms and if $\overline{A, A', V_\delta}$, $\overline{B, C}$ is true[3]) then (B, C, C', B') is a parallelogram, too, or $\overline{B, C, V_\delta}$.

Lemma 1. *Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota \in I})$ be a net of degree $\geqq 4$ and δ an index. If $\mathcal{N}$ satisfies the Desargues condition of type (δ) then $\mathcal{N}$ satisfies also the Reidemeister condition of type (δ, ξ, η) for all ξ, η such that δ, ξ, η are mutually distinct.*

Proof. Let the points $P, Q, R, S, P', Q', R', S'$ satisfy the assumptions of the Reidemeister condition of type (δ, ξ, η) in $\mathcal{N}$ for arbitrarily chosen ξ, η. Choose another index $\zeta \neq \delta, \xi, \eta$ which is possible since $\mathcal{N}$ has degree at least 4. Then the points $P, P'V_\xi \sqcap PV_\zeta, P', S, (P'V_\xi \sqcap PV_\zeta) V_\delta \sqcap SV_\zeta, S'$ satisfy the assumptions of the Desargues condition of type (δ) in $\mathcal{N}$ so that $\overline{(P'V_\xi \sqcap PV_\zeta) V_\delta \sqcap SV_\zeta, S', V_\xi}$. Further, consider the points $P, Q, PV_\zeta \sqcap QV_\eta, S, R, SV_\zeta \sqcap (PV_\zeta \sqcap QV_\eta) V_\delta$. These points satisfy the assumptions of the Desargues condition of type (δ) in $\mathcal{N}$, too, so that $\overline{R, SV_\zeta \sqcap (PV_\zeta \sqcap QV_\eta) V_\delta, V_\eta}$. Consequently $\overline{R', SV_\zeta \sqcap (PV_\zeta \sqcap QV_\eta) V_\delta, V_\eta}$. Finally, also the points $PV_\zeta \sqcap QV_\eta$, $P'V_\xi \sqcap PV_\zeta$, Q', $SV_\zeta \sqcap (PV_\zeta \sqcap QV_\eta) V_\delta$, $(P'V_\xi \sqcap \sqcap PV_\zeta) V_\delta \sqcap SV_\zeta$, R satisfy the assumptions of the Desargues condition of type (δ) in $\mathcal{N}$ so that $\overline{(P'V_\xi \sqcap PV_\zeta) V_\delta \sqcap SV_\zeta, R', V_\xi}$. The conclusions of the first and last application of the Desargues condition of type (δ) in $\mathcal{N}$ imply $\overline{S', R', V_\xi}$. ■

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2) We shall also say more briefly that points $P, Q, R, S, P', Q', R', S'$ (in this arrangement) satisfy the assumptions of the Reidemeister condition of type (α, β, γ) in $\mathcal{N}$.

3) We shall say more briefly that points A, B, C, A', B', C' (in this arrangement) satisfy the assumptions of the Desargues condition of type (δ) in $\mathcal{N}$.

Lemma 2. *Let* $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota\in I})$ *be a net of degree* ≥ 4 *satisfying the Desargues condition of type* (δ) *for some* δ. *If* $(1, 2, 2', 1')$, $(1, 3, 3', 1')$, $(2, 4, 4', 2')$ *are parallelograms in* $\mathcal{N}$ *with* $\overline{1, 1', V_\delta}$, $\overline{3, 4}$ *and with* $\overline{3, 4, V_\delta} \Rightarrow 3 = 4$ *then* $(3, 4, 4', 3')$ *is a parallelogram.*

Proof. Let the points $1, 2, 3, 4, 1', 2', 3', 4'$ satisfy the assumptions of Lemma 2. If $\overline{1, 2, 3, 4}$ then $(3, 4, 4', 3')$ is trivially a parallelogram. So let $\overline{1, 2, 3, 4}$ be not true. Further let $(1, 2, 4, 3)$ be a parallelogram. Consider the points $1, 2, 4, 3, 1', 2', 4', 3'$. These points satisfy the assumptions of the Reidemeister condition of type (δ, ξ, η) for suitable ξ, η. By Lemma 1 this Reidemeister condition is valid in $\mathcal{N}$ so that $(3, 4, 4', 3')$ is a parallelogram as required. Now let $\overline{1, 2, 3, 4}$ be not true and let $(1, 2, 4, 3)$ be not a parallelogram. Then for at least one of the pairs $(1, 2)$, $(3, 4)$; $(1, 3)$, $(2, 4)$ there is a proper point 5 such that α) $\overline{1, 2, 5}$, $\overline{3, 4, 5}$ or β) $\overline{1, 3, 5}$, $\overline{2, 4, 5}$, respectively. Let us consider the case α): If a is the line through $1, 2, 5$ and b the line through $3, 4, 5$ then $a \neq b$. Now $(1, 3, 5)$ and $(2, 4, 5)$ are necessarily triangles. Let $5'$ be such that $(1, 5, 5', 1')$ is a parallelogram. Moreover, the points $1, 3, 5, 1', 3', 5'$ as well as $2, 4, 5, 2', 4', 5'$ satisfy the assumptions of the Desargues condition of type (δ) in $\mathcal{N}$ so that $3', 4', 5'$ lie on the line which possesses the same improper point as b. But then $(3, 4, 4', 3')$ is a parallelogram. The case β) can be dealt with similarly. ∎

By an *automorphism* of a net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota\in I})$ we mean a permutation π of $\mathcal{P}$ such that every singular point is fixed under π and $\{X^\pi \mid X \in l\}$ is contained in a line of $\mathcal{N}$ for every $l \in \mathcal{L}$. For such a π it follows $\{X^\pi \mid X \in l\}, \{X^{\pi^{-1}} \mid X \in l\} \in \mathcal{L}$ for all $l \in \mathcal{L}$. Thus π induces a permutation $\hat{\pi}$ of $\mathcal{L}$ with $l^{\hat{\pi}} := \{X^\pi \mid X \in l\}$ for all $l \in \mathcal{L}$. If $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota\in I})$ is a net and α an index then an *α-automorphism* of $\mathcal{N}$ is an automorphism π of $\mathcal{N}$ such that $l^{\hat{\pi}} = l$ for every $l \in \mathcal{L}$ through V_α. If moreover for any two proper points A, A' with $\overline{A, A', V_\alpha}$ there exists an α-automorphism with $A \mapsto A'$ then $\mathcal{N}$ is said to be *α-transitive*. It can be shown that $\mathcal{N}$ is α-transitive if there is a proper line l_0 through V_α such that for any two proper points A, A' on l_0 there exists an α-automorphism with $A \mapsto A'$.

Theorem. *Let* $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota\in I})$ *be a net of degree* ≥ 4 *and* δ *an index. Then* $\mathcal{N}$ *satisfies the Desargues condition of type* (δ) *if and only if it is* δ*-transitive.*

Proof. a) Let $\mathcal{N}$ be δ-transitive and let the points A, B, C, A', B', C' satisfy the assumptions of the Desargues condition of type (δ) in $\mathcal{N}$. If A, B, C are not mutually different then (B, C, C', B') is trivially a parallelogram. If A, B, C are mutually distinct then use a δ-automorphism π with $A^\pi = A'$. Then $(AB)^{\hat{\pi}} = A'B'$, $(AC)^{\hat{\pi}} = A'C'$, $(BV_\delta)^{\hat{\pi}} = BV_\delta$, $(CV_\delta)^{\hat{\pi}} = CV_\delta$, so that $C^\pi = (AC \sqcap CV_\delta)^\pi = A'C' \sqcap C'V_\delta = C'$, $B^\pi = (AB \sqcap BV_\delta)^\pi = (AB)^{\hat{\pi}} \sqcap (BV_\delta)^{\hat{\pi}} = A'B' \sqcap BV_\delta = B'$. Therefore $(BC)^{\hat{\pi}} = B'C'$ and since π is a net automorphism, BC and $B'C'$ must have the same improper point. Consequently (B, C, C', B') is a parallelogram as claimed.

b) Let $\mathcal{N}$ satisfy the Desargues condition of type (δ). Start with an arbitrary couple (A_0, A_0') of proper points such that $\overline{A_0, A_0', V_\delta}$ and define a mapping $\pi_{A_0,A_0'}$: $: \mathcal{P} \to \mathcal{P}$ as follows: 1) Every improper point will be fixed under $\pi_{A_0,A_0'}$. 2) If X is a proper point, then let X' be a point for which an intermediating couple (X_0, X_0^*) exists so that (A_0, X_0, X_0^*, A_0'), (X_0, X, X', X_0^*) are parallelograms. We shall show that X' is thereby determined in a unique way independently of (X_0, X_0^*): Indeed, at least one intermediating couple (X_0, X_0^*) exists because we can take arbitrary indices α, β such that α, β, δ are mutually distinct and put $X_0 := A_0V_\alpha \sqcap XV_\beta$, $X_0^* := A_0'V_\alpha \sqcap X_0V_\delta$ (consequently, $X' := X_0^*V_\beta \sqcap XV_\delta$). Further, the independence of X' of the choice of (X_0, X_0^*) is guaranteed immediately by Lemma 2. So we can declare X' to be the image of X under $\pi_{A_0,A_0'}$.

Now it is clear that $\pi_{A_0,A_0'}$ must be bijective (and thus a permutation of $\mathcal{P}$) as well as that $\{X^{\pi_{A_0,A_0'}} \mid X \in l\} = l$ for every line through V_δ. So it remains to show that also $\{X^{\pi_{A_0,A_0'}} \mid X \in l\} \in \mathcal{L}$ for every $l \in \mathcal{L}$ not through V_δ: Let l be a line not through V_δ (and therefore going through some V_α, $\alpha \neq \delta$). Choose an index $\beta \neq \alpha, \delta$ and put $X_0 := A_0V_\alpha \sqcap l$, $X_0' := A_0V_\alpha \sqcap X_0V_\delta$. If X is an arbitrary proper point of l then construct $X^{\pi_{A_0,A_0'}}$ by means of the intermediating couple (X_0, X_0'). We see that if X runs over l then $X^{\pi_{A_0,A_0'}}$ runs over X_0V_α i.e. $\{X^{\pi_{A_0,A_0'}} \mid X \in l\} \in \mathcal{L}$ as required. ∎

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