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ON THE LATTICE GROUP VALUED MEASURES

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In the paper we study some properties of non-negative lattice group valued measures on topological spaces. Naturally enough, this group is assumed to satisfy a certain regularity condition. Therefore, the first part is devoted to this condition, a generalization of the Alexandroff theorem being proved here. The second part is concerned with the product of measures and the third one with the Kolmogoroff consistency theorem.

1

Let G be an Abelian lattice ordered group, i.e. and Abelian group which is a lattice and which satisfies the implication: $x < y \Rightarrow x + z < y + z$. A group valued submeasure μ is a mapping $\mu : \mathscr{R} \to G$, where $\mathscr{R}$ is a ring of subsets of a space X, non-decreasing, subadditive, $\mu(\emptyset) = 0$ and upper semicontinuous in $\emptyset$ (i.e. $A_n \searrow \emptyset \Rightarrow \mu(A_n) \searrow 0$). An additive submeasure is called measure. (Of course, every measure is σ-additive.)

Definition 1. An Abelian lattice ordered group G is *weakly regular**) if it satisfies the following condition: Let $a \in G$, $a > O$ and let $a_n^i \searrow O$ $(i \to \infty)$, then there are such $i_1, i_2, \ldots$ that

$$a \leqq \sum_{j=1}^{n} a_j^{i_j}$$

for no n.

As an example of a weakly regular group let us take the additive group R of all real numbers. In this case it suffices to choose i_k such that

$$a_k^{i_k} < \frac{a}{2^k}.$$

*) We say "weakly" since there is a stronger notion of regularity used in [5].

Then

$$a \leqq \sum_{j=1}^{n} a_j^{i_j}$$

for some n implies

$$a \leqq \sum_{j=1}^{n} \frac{a}{2^j} < a ,$$

which is impossible.

Now let us present two less trivial examples.

Example 1. Every linearly ordered group is weakly regular. First we construct the sequence $\{i_j\}_{j=1}^{\infty}$. Since $a_1^i \searrow O$ $(i \to \infty)$, it is also $2a_1^i = a_1^i + a_1^i \searrow O$, hence there is i_1 such that $2a_1^{i_1} < a$. Similarly there is i_2 such that $4a_2^{i_2} < a$ and generally there is i_k such that $2^k a_k^{i_k} < a$. If $a \leqq \sum_{j=1}^{n} a_j^{i_j}$ then

$$\begin{aligned} 2^n a &\leqq 2^n a_1^{i_1} + 2^n a_2^{i_2} + \ldots + 2^n a_n^{i_n} = \\ &= 2^{n-1}\, 2a_1^{i_1} + 2^{n-2}\, 2^2 a_2^{i_2} + \ldots + 1 \,.\, 2^n a_n^{i_n} < \\ &< 2^{n-1} a + 2^{n-2} a + \ldots + 1 \,.\, a = (2^n - 1)\, a , \end{aligned}$$

which is impossible.

Example 2. Every regular K-space is a weakly regular group. A regular K-space (see [6] Th. VI.5.2) is a linear semiordered space (= Riesz space = K-lineal) which is relatively complete and such that every sequence of convergent sequences has a common regulator of convergence. If $b_n \searrow O$, then $u > O$ is a regulator of convergence of $\{b_n\}_{n=1}^{\infty}$ iff to any number $\varepsilon > 0$ there is n_0 such that $b_n < \varepsilon u$ for every $n \geqq n_0$. Hence $b_n \searrow O$ is false iff to any $u > O$ there is $\varepsilon > 0$ such that for any n_0 there is $n \geqq n_0$ such that $b_n < \varepsilon u$ is false. Now let $a_n^i \searrow O$ $(i \to \infty,\ n = 1, 2, \ldots)$ and let u be the common regulator of convergence of all $\{a_n^i\}_{i=1}^{\infty}$, $n = 1, 2, \ldots$. Given $\varepsilon > 0$ there is i_n such that

$$a_n^{i_n} < \frac{\varepsilon}{2^n} u .$$

If

$$a = \sum_{j=1}^{n_0} a_j^{i_j} < \sum_{j=1}^{n_0} \frac{\varepsilon}{2^j} u < \varepsilon u$$

then $a < \varepsilon u$ for every $\varepsilon < 0$ which is a contradiction since $a > O$.

In the paper we shall consider only regular measures.

Definition 2. Let $\mathscr{C}$ be a family of subsets of a set X. We say that $\mathscr{C}$ is *a compact family* if $\mathscr{C}$ is closed under finite intersections and every decreasing sequence of non-empty sets of $\mathscr{C}$ has a non-empty intersection.

Definition 3. Let $\mathscr{R}$ be a ring of subsets of a set X, $\mathscr{C} \subset \mathscr{R}$, $\mathscr{C}$ a compact family. Let $\mu : \mathscr{R} \to G$ be a lattice group valued submeasure. We say that μ is *inner regular* if to any $E \in \mathscr{R}$ there are such sets $C_n \in \mathscr{C}$ $(n = 1, 2, \ldots)$ that $C_n \subset C_{n+1} \subset E$ $(n = 1, 2, \ldots)$ and

$$\mu(E - C_n) \searrow 0 .$$

The following theorem is a generalization of the Alexandroff theorem. Various other generalizations in the real-valued case are found in [4].

Theorem 1. *Let G be weakly regular, σ-complete,*) Abelian lattice-ordered group. Let X be a topological space, $\mathscr{R}$ a ring of subsets of X. Let $\mu : \mathscr{R} \to G$, $\mu(\emptyset) = O$ be monotone, subadditive and inner regular. Then μ is upper semicontinuous in $\emptyset$.*

Proof. Let $A_n \searrow \emptyset$ (i.e. $A_n \supset A_{n+1}$ $(n = 1, 2, \ldots)$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$). We want to prove that $\mu(A_n) \searrow O$. Let us prove it indirectly. Since G is σ-complete, there is $a > O$ such that $\mu(A_n) \geqq a$. Since μ is inner regular, to any n there are $C_n^i \in \mathscr{C}$ $(i = 1, 2, \ldots)$ such that

$$C_n^i \subset C_n^{i+1} \subset A_n \quad (i = 1, 2, \ldots)$$

and

$$\mu(A_n - C_n^i) \searrow O \quad (i \to \infty) .$$

Put

$$a_n = \mu(A_n) , \quad a_n^i = \mu(A_n - C_n^i)$$

and choose $i_1, i_2, \ldots$ by Definition 1. Now put

$$D_1 = C_1^{i_1},\ D_2 = C_2^{i_2} \cap C_1^{i_1}, \ldots, D_n = C_n^{i_n} \cap C_{n-1}^{i_{n-1}} \cap \ldots \cap C_1^{i_1}, \ldots .$$

Then $D_n \in \mathscr{C}$, $D_n \supset D_{n+1}$ $(n = 1, 2, \ldots)$. We prove that $D_n \neq \emptyset$ $(n = 1, 2, \ldots)$:

If $D_n = \emptyset$, then

$$a \leqq \mu(A_n) \leqq \mu((\bigcup_{j=1}^{n} (A_j - C_j^{i_j})) \cup (\bigcap_{j=1}^{n} C_j^{i_j})) \leqq$$

$$\leqq \sum_{j=1}^{n} \mu(A_j - C_j^{i_j}) + \mu(D_n) = \sum_{j=1}^{n} \mu(A_j - C_j^{i_j}) = \sum_{j=1}^{n} a_j^{i_j}$$

which is impossible.

Since $D_n \supset D_{n+1}$, $D_n \in \mathscr{C}$, $D_n \neq \emptyset$ $(n = 1. 2, \ldots)$, we have $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$. But $D_n \subset A_n$ $(n = 1, 2, \ldots)$, hence also $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$, which is a contradiction.

*) I.e. every bounded countable set has the supremum.

Now we want to prove a theorem on the product of two measures. Usually the product of two measures μ, ν is defined as such a measure λ in the cartesian product that

$$\lambda(E \times F) = \mu(E)\,\nu(F)$$

for all E, F from the corresponding domains. However, in our general group G we need not have any product. Hence we shall assume that there are given three groups G_1, G_2, G and a mapping

$$\pi : G_1 \times G_2 \to G$$

satisfying some conditions. We shall need the following three simple conditions:

1. $\pi(a + b, c) = \pi(a, c) + \pi(b, c)$, $\pi(a', b' + c') = \pi(a', b') + \pi(a', c')$ for all $a, b, a' \in G_1$, $c, b', c' \in G_2$.

2. If $a \geqq O$, $b \geqq O$, $a \in G_1$, $b \in G_2$, then $\pi(a, b) \geqq O$.

3. If $a_n \searrow O$, $b_n \searrow O$, $a_n \in G_1$, $b_n \in G_2$ $(n = 1, 2, \ldots)$ then $\pi(a_n, b_n) \searrow O$.

Theorem 2. *Let $\mathscr{R}_1$ or $\mathscr{R}_2$ be rings of subsets of X_1 or X_2 respectively. Let $\mu : \mathscr{R}_1 \to G_1$, $\nu : \mathscr{R}_2 \to G_2$ be inner regular measures. Let G be weakly regular, σ-complete, Abelian, lattice-ordered group. Then there is exactly one G-valued measure λ defined on the ring $\mathscr{R}$ generated by the family $\mathscr{D} = \{E \times F;\ E \in \mathscr{R}_1, F \in \mathscr{R}_2\}$ and such that*

$$\lambda(E \times F) = \pi(\mu(E), \nu(F))$$

for all $E \in \mathscr{R}_1$, $F \in \mathscr{R}_2$.

Proof. Define first $\lambda_0 : \mathscr{D} \to G$ by the formula $\lambda_0(E \times F) = \pi(\mu(E), \nu(F))$. Evidently λ_0 is additive, monotone, $\lambda_0(\emptyset) = O$. Hence we can extend λ_0 to a function $\lambda : \mathscr{R} \to G$ by the formula

$$\lambda\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \lambda_0(A_i)$$

where A_i are disjoint sets from $\mathscr{D}$ $(i = 1, \ldots, n)$. The function λ is also additive, non-negative (and therefore monotone and subadditive). It suffices to prove that λ is upper semicontinuous in $\emptyset$.

Let $\mathscr{C}_1, \mathscr{C}_2$ be compact families of subsets of X_1 or X_2 respectively. Let $\mathscr{C}$ consist of all finite unions of sets of the form $C \times D$ where $C \in \mathscr{C}_1$, $D \in \mathscr{C}_2$. Then $\mathscr{C}$ is a compact family.

Now let $A \in \mathscr{R}$. Then $A = \bigcup_{i=1}^{m} A_i = \bigcup_{i=1}^{m} (E_i \times F_i)$, where A_i are pairwise disjoint.

Since $E_i \in \mathscr{R}_1$ and μ is inner regular, there are $C_i^n \in C_1$ $(n = 1, 2, \ldots)$ such that

$$C_i^n \subset C_i^{n+1} \subset E_i \quad (n = 1, 2, \ldots)$$

and

$$\mu(E_i - C_i^n) \searrow O \quad (n \to \infty).$$

Similarly there are $D_i^n \in \mathscr{C}_2$ $(n = 1, 2, \ldots)$ such that

$$D_i^n \subset D_i^{n+1} \subset F_i \quad (n = 1, 2, \ldots)$$

and

$$\nu(F_i - D_i^n) \searrow O \quad (n \to \infty).$$

By the third property of π we have

$$\lambda(A_i - C_i^n \times D_i^n) = \lambda((E_i \times F_i) - (C_i^n \times D_i^n)) \leqq$$
$$\leqq \pi(\mu(E_i - C_i^n), \nu(F_i)) + \pi(\mu(E_i), \nu(F_i - D_i^n)) \searrow O \quad (n \to \infty).$$

Put $K_n = \bigcup_{i=1}^{m} (C_i^n \times D_i^n)$ $(n = 1, 2, \ldots)$. Then $K_n \in \mathscr{C}$, $K_n \subset K_{n+1} \subset A$ $(n = 1, 2, \ldots)$

and

$$\lambda(A - K_n) = \lambda(\bigcup_{i=1}^{m} A_i - \bigcup_{i=1}^{m} (C_i^n \times D_i^n)) =$$

$$= \lambda(\bigcup_{i=1}^{m} (A_i - C_i^n \times D_i^n)) = \sum_{i=1}^{m} \lambda(A_i - C_i^n \times D_i^n) \searrow O$$

if $n \to \infty$. Hence λ is regular and the proof is complete.

Remark. A special case of Theorem 2 is Theorem 2 in [3].

3

Let $\{X_t\}_{t \in T}$ be a family of topological spaces. Denote by Γ the set of all finite subsets of T. For any $\alpha \in \Gamma$ put $X_\alpha = \mathop{\times}_{t \in \alpha} X_t$. If $\alpha, \beta \in \Gamma$, $\alpha \supset \beta$ then $\pi_{\alpha\beta}$ denotes the projection $\pi_{\alpha\beta} : X_\alpha \to X_\beta$. Every X_α is a topological space with the product topology and every $\pi_{\alpha\beta}$ is a continuous mapping. Let G be a weakly regular Abelian l-group.

Now we shall assume that we are given a consistent family of inner regular G-valued measures $\{\mu_\alpha\}_{\alpha \in \Gamma}$. Of course, regularity is taken with respect to the compact family of compact subsets of the corresponding space. Hence for every $\alpha \supset \beta$ and every $E \in \mathscr{R}_\beta$ ($\mathscr{R}_\beta$ is the domain of μ_β) we have

$$\pi_{\alpha\beta}^{-1}(E) \in \mathscr{R}_\alpha, \quad \mu_\alpha(\pi_{\alpha\beta}^{-1}(E)) = \mu_\beta(E).$$

In this case the projective limit of the projective system $(X_\alpha, \mathscr{R}_\alpha, \mu_\alpha, \pi_{\alpha\beta})$ is $(X, \mathscr{R}, \mu, \pi_\alpha)$, where $X = \bigtimes_{t\in T} X_t$, π_α is the projection $\pi_\alpha : X \to X_\alpha$, $\mathscr{R} = \{\pi_\alpha^{-1}(E); E \in \mathscr{R}_\alpha, \alpha \in \Gamma\}$, $\mu(A) = \mu(\pi_\alpha^{-1}(E)) = \mu_\alpha(E)$. It is not difficult to prove that the definition of μ is correct (μ does not depend on the choice of α), $\mathscr{R}$ is a ring and that μ is additive, monotone, $\mu(\emptyset) = 0$. The only problem is whether μ is σ-additive, i.e. whether

$$(X, \mathscr{R}, \mu, \pi_\alpha)$$

is the projective limit of the system in the category of measure spaces (see [1], [2]).

Theorem 3. *Let G be a weakly regular, σ-complete, Abelian l-group. The function μ defined above is a measure and $(X, \mathscr{R}, \mu, \pi_\alpha)$ is the projective limit in the category of measure spaces.*

Proof. To prove that μ is σ-additive it suffices to prove that μ is upper semicontinuous in $\emptyset$. Let $\mathscr{C}_\alpha$ denote the family of all compact sets in X_α. Put

$$\mathscr{C} = \{\pi_\alpha^{-1}(E);\ E \in \mathscr{C}_\alpha,\ \alpha \in \Gamma\}\,.$$

Evidently μ is inner regular with respect to $\mathscr{C}$. We prove that $\mathscr{C}$ is a compact family.

Let $C_n \in \mathscr{C}$, $C_n \supset C_{n+1}$, $C_n \neq \emptyset$ $(n = 1, 2, \ldots)$. Then $C_n = \pi_{\alpha_n}^{-1}(D_n)$, $D_n \in \mathscr{C}_{\alpha_n}$ $(n = 1, 2, \ldots)$. The set $\bigcup_{n=1}^{\infty} \alpha_n$ is countable. Put

$$\bigcup_{n=1}^{\infty} \alpha_n = \{t_1, t_2, t_3, \ldots\}\,.$$

Consider the sequence

$$\{\pi_{\{t_1\}}(C_n)\}_{n=1}^{\infty}\,.$$

If $t_1 \notin \alpha_n$, then $\pi_{\{t_1\}}(C_n) = X_{t_1}$. If $t_1 \in \alpha_n$, then $\{t_1\} \subset \alpha_n$, hence

$$\pi_{\{t_1\}}(C_n) = \pi_{\{t_1\}}\pi_{\alpha_n}^{-1}(D_n) = \pi_{\alpha_n\{t_1\}}(D_n)$$

and this is a compact subset of X_{t_1}. Moreover, the sequence $\{\pi_{\{t_1\}}(C_n)\}_{n=1}^{\infty}$ is decreasing, therefore

$$\bigcap_{n=1}^{\infty} \pi_{\{t_1\}}(C_n) \neq \emptyset\,.$$

Denote by $x_{t_1}^0$ an element of $\bigcap_{n=1}^{\infty} \pi_{\{t_1\}}(C_n)$ and repeat the procedure with the second coordinate t_2:

$$E_n = \pi_{\{t_2\}}(C_n \cap \pi_{\{t_1\}}^{-1}(\{x_{t_1}^0\}))\,.$$

Then $E_n \supset E_{n+1}$, E_n is closed $(n = 1, 2, \ldots)$ and E_n is compact if $t_2 \in \alpha_n$. Hence

$$\bigcap_{n=1}^{\infty} E_n \neq \emptyset\,.$$

Denote by $x_{t_2}^0$ an element of $\bigcap_{n=1}^{\infty} E_n$. Repeating this procedure we obtain a sequence

$$x_{t_1}^0, x_{t_2}^0, \ldots, x_{t_k}^0, \ldots \quad .$$

such that

$$x_{t_k}^0 \in \bigcap_{n=1}^{\infty} \pi_{\{t_k\}}(C_n \cap \bigcap_{i=1}^{k-1} \pi_{\{t_i\}}^{-1}(\{x_{t_i}^0\})), \quad k = 1, 2, \ldots,$$

hence to any n there is $x \in C_n$ such that

$$x_{t_1} = x_{t_1}^0, x_{t_2} = x_{t_2}^0, \ldots, x_{t_k} = x_{t_k}^0 .$$

Define x^0 by the following formula:

$$(x^0)_t = x_t^0 , \quad \text{if} \quad t \in \bigcup \alpha_n$$

$$(x^0)_t = \text{an arbitrary element of } X_t, \text{ if } t \notin \bigcup \alpha_n .$$

Now we assert that $x^0 \in \bigcap_{n=1}^{\infty} C_n$.

Take arbitrary n and k such that $\alpha_n \subset \{t_1, \ldots, t_k\}$. We know that there is $x \in C_n$ such that

$$x_{t_1} = x_{t_1}^0, \ x_{t_2} = x_{t_2}^0, \ \ldots, \ x_{t_k} = x_{t_k}^0 .$$

Put $\alpha_n = \{t_{j_1}, \ldots, t_{j_m}\}$. Since $x \in C_n$, $\pi_{\alpha_n}(x) \in D_n$, hence

$$(x_{t_{j_1}}^0, \ldots, x_{t_{j_m}}^0) = (x_{t_{j_1}}, \ldots, x_{t_{j_m}}) \in D_n .$$

But it follows that $\pi_{\alpha_n}(x^0) \in D_n$, i.e. $x^0 \in \pi_{\alpha_n}^{-1}(D_n) = C_n$.

We have proved that $\mathscr{C}$ is a compact system. By Theorem 1 μ is upper continuous in $\emptyset$, i.e. μ is a measure.

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