Charles S. Kahane

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ON THE DENSITY OF THE DILATIONS AND TRANSLATES OF FUNCTION IN L_1

CHARLES KAHANE, Nashville

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A well known theorem of WIENER [2] asserts that linear combinations of the translates $k(\xi + t)$ of a fixed function $k(\xi)$ in L_1 are dense in L_1, provided that the Fourier transform of k never vanishes on the real axis. Suppose that in addition to the translates of k we also allow dilations:

$$k(\delta^{-1}[\xi + t] \qquad (\delta > 0) \tag{1}$$

and then ask, for which k's is the span of this set of functions dense in L_1? Clearly, the span will be dense if k is the characteristic function of an interval – this just amounts to the fact that the class of step functions vanishing outside of a finite interval is dense in L_1. More generally we have the following result.

Theorem. *The necessary and sufficient condition for the set of functions* (1) *to have a dense span in* L_1 *is that* $\int k(\xi)\,\mathrm{d}\xi \neq 0$.

Proof. The necessity follows from the observation that if $\int k(\xi)\,\mathrm{d}\xi = 0$, then the integral of any linear combination of the functions (1) will also be zero:

$$\int \sum_{j=1}^{n} a_j\, k(\delta_j^{-1}[\xi + t_j])\,\mathrm{d}x = \left(\sum_{j=1}^{n} a_j \delta_j\right) \int k(\xi)\,\mathrm{d}\xi = 0\,.$$

Consequently, it would be impossible to approximate any function f in L_1 with a non-vanishing integral by such combinations of the functions (1).

To prove the sufficiency we employ a well-known criterion, based on the Hahn-Banach theorem, for denseness of the span of a set of elements $\mathscr{S}$ in a normed linear space X (cf. [1], p. 65); namely that the only bounded linear functional on X vanishing for each of the elements of $\mathscr{S}$ be the identically zero functional. By the Riesz representation theorem, all linear functionals $I(f)$ on L_1 are known to have the form $I(f) = \int f(\xi)\, g(\xi)\,\mathrm{d}\xi$ where g is a bounded measurable function. It will, therefore, be sufficient to show that if the relations

$$\int k\left(\frac{x-\xi}{\delta}\right) g(\xi)\,\mathrm{d}\xi = 0$$

hold for all x and all $\delta > 0$, then $g(\xi)$ must be zero almost everywhere. In turn, this will be an immediate consequence of the following result regarding approximations of the identity which is of some interest in itself.

Lemma. *Let $k \in L_1$ and $g \in L_\infty$, then*

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int k\left(\frac{x-\xi}{\delta}\right) g(\xi)\,\mathrm{d}\xi = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int k\left(\frac{\xi}{\delta}\right) g(x-\xi)\,\mathrm{d}\xi = g(x) \int k(\xi)\,\mathrm{d}\xi \tag{2}$$

holds on the Lebesgue set of g.

Proof. We recall the definition of the Lebesgue set of g as the set of all points x at which the relation

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{|\xi| < \delta} |g(x-\xi) - g(x)|\,\mathrm{d}\xi = 0$$

holds.

We will first prove (2) under the additional assumption that k is a bounded measurable function vanishing outside of a finite interval, say $k(\xi) = 0$ for $|\xi| \geqq a$. Letting $\|k\|_\infty$ denote the essential supremum of k, a straightforward computation then yields

$$\left|\frac{1}{\delta} \int k(\xi/\delta)\, g(x-\xi)\,\mathrm{d}\xi - g(x) \int k(\xi)\,\mathrm{d}\xi\right| = \frac{1}{\delta} \left|\int_{|\xi| \leqq a\delta} k(\xi/\delta)\,[g(x-\xi) - g(x)]\,\mathrm{d}\xi\right| \leqq$$

$$\leqq (a\|k\|_\infty)\left(\frac{1}{a\delta} \int_{|\xi| \leqq a\delta} |g(x-\xi) - g(x)|\,\mathrm{d}\xi\right);$$

from which (2) follows immediately in this case.

Suppose now that $k \in L_1$. To reduce this situation to the one just considered, we construct a sequence of bounded measurable functions $k^{(n)}$, $n = 1, 2, \ldots$, vanishing outside of $[-n, +n]$, and which converge to k in the L_1 norm:

$$\|k^{(n)} - k\|_1 \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3}$$

Setting

$$\gamma_n = \int k^{(n)}(\xi)\,\mathrm{d}\xi \quad \text{and} \quad \gamma = \int k(\xi)\,\mathrm{d}\xi\,,$$

this implies that

$$\gamma_n \to \gamma \quad \text{as} \quad n \to \infty\,. \tag{4}$$

If we now introduce the abbreviations

$$k_\delta^{(n)}(x) = \frac{1}{\delta}\, k^{(n)}\left(\frac{x}{\delta}\right) \quad \text{and} \quad k_\delta(x) = \frac{1}{\delta}\, k\left(\frac{x}{\delta}\right),$$

and use the notation $(u * v)(x)$ to denote $\int u(x - \xi)\, v(\xi)\, d\xi = \int u(\xi)\, v(x - \xi)\, d\xi$, the convolution of u and v evaluated at x, we may write

$$\left|(k_\delta * g)(x) - \gamma\, g(x)\right| \leqq \left|(k_\delta^{(n)} * g)(x) - \gamma_n\, g(x)\right| + \\ + \left|([k_\delta - k_\delta^{(n)}] * g)(x)\right| + \left|(\gamma_n - \gamma)\, g(x)\right| .$$

For the middle term we have the estimate

$$\left|([k_\delta - k_\delta^{(n)}] * g)(x)\right| \leqq \left\|k_\delta - k_\delta^{(n)}\right\|_1 \left\|g\right\|_\infty = \left\|k - k^{(n)}\right\|_1 \left\|g\right\|_\infty ;$$

hence

$$\left|(k_\delta * g)(x) - \gamma\, g(x)\right| \leqq \\ \leqq \left|(k_\delta^{(n)} * g)(x) - \gamma_n\, g(x)\right| + \left[\left\|k - k^{(n)}\right\|_1 + \left|\gamma_n - \gamma\right|\right] \left\|g\right\|_\infty .$$

Since each of the $k^{(n)}$'s is a bounded measurable function vanishing outside of a finite interval, by what has already been proven, the first term on the right of the preceding estimate tends to zero as $\delta \downarrow 0$, for x in the Lebesgue set of g. Thus, on the Lebesgue set of g

$$\varlimsup_{\delta \downarrow 0} \left|(k_\delta * g)(x) - \gamma\, g(x)\right| \leqq \left[\left\|k - k^{(n)}\right\|_1 + \left|\gamma_n - \gamma\right|\right] \left\|g\right\|_\infty ;$$

so that in view of (3) and (4), the desired result (2) then follows by letting $n \to \infty$.

References

[1] *N. Dunford* and *J. T. Schwartz,* Linear operators, Part I, Interscience, New York, 1958.

[2] *N. Wiener,* Tauberian theorems, Annals of Math. *33* (1932), 1—100.

Author's address: Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37235, U.S.A.