Jaroslav Barták
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# THE LYAPUNOV STABILITY OF THE TIMOSHENKO TYPE EQUATION

JAROSLAV BARTÁK, Praha

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The purpose of this paper is the investigation of the global exponential stability, respectively the stability of the zero solution of the equation

$$u''''(t) + a\, u'''(t) + (b_1 A^{1/2} + b_2 I)\, u''(t) + (c_1 A^{1/2} + c_2 I)\, u'(t) + + (d_1 A + d_2 A^{1/2} + d_3 I)\, u(t) = 0 \tag{1}$$

where $A$ is a selfadjoint, strictly positive linear operator in a Hilbert space $H$; $I$ is the identity operator in $H$; $a, b_1, b_2, c_1, c_2, d_1, d_2, d_3$ are real constants.

Under the solution of (1) we understand a function $u$ from the space $\mathscr{U} = \{u : \langle 0, \infty) \to H \mid u^{(j)} \in C(\mathscr{D}(u), \mathscr{D}(A^{(4-j)/4})),\ j = 0, 1, 2, 3\}$, fulfilling the equation (1) on $\langle 0, \infty)$.

Let us define the norm $\|\cdot\|_{\mathscr{D}(A) \times \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})}$ by the relation

$$\|u(t)\|_{\mathscr{D}(A) \times \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})} =$$
$$= \|(u(t), u'(t), u''(t), u'''(t))\|_{\mathscr{D}(A) \times \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})} =$$
$$= [\|A\, u(t)\|^2 + \|A^{3/4}\, u'(t)\|^2 + \|A^{1/2}\, u''(t)\|^2 + \|A^{1/4}\, u'''(t)\|^2]^{1/2}$$

for $u \in \mathscr{U}$ and $t \in \langle 0, \infty)$, ($\|\cdot\|$ is the norm in the space $H$).

**Definition 1.** We say that the solution $v(t)$ of the equation (1) is *stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A) \times \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})}$ if to arbitrarily chosen $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ so that the following implication holds:

$$\|u(0) - v(0)\|_{\mathscr{D}(A) \times \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})} \leqq \delta(\varepsilon) \Rightarrow$$
$$\Rightarrow \|u(t) - v(t)\|_{\mathscr{D}(A) \times \mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/2}) \times \mathscr{D}(A^{1/4})} \leqq \varepsilon$$

for $t \geqq 0$ and for every solution $u(t)$ of the equation (1).

**Definition 2.** We say that the solution $v(t)$ of the equation (1) is *exponentially stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$ if there exist positive numbers $\delta, K, \alpha$ so that the following implication holds:

$$\|u(0) - v(0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \leqq \delta \Rightarrow$$

$$\Rightarrow \|u(t) - v(t)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})} \leqq Ke^{-\alpha t} \cdot$$

$$\cdot \|u(0) - v(0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$$

for $t \geqq 0$ and for every solution $u(t)$ of the equation (1).

If $\delta = +\infty$ in addition, we speak about the *global exponential stability*.

Let $u(t)$ be a solution of (1) and let the following initial conditions be fulfilled:

$$u(0) = \varphi_0\,, \quad u'(0) = \varphi_1\,, \quad u''(0) = \varphi_2\,, \quad u'''(0) = \varphi_3\,, \tag{2}$$

where $\varphi_i \in \mathscr{D}(A^{1-i/4})$, $i = 0, \ldots, 3$.

Let us assume that

(3) the solution of (1) fulfilling (2) is unique.

The problem of the uniqueness is studied in [1], [2].

Let us denote $E(s)$ a spectral resolution of the identity corresponding to the operator $A$, $\delta = \inf \sigma(A)$. By the assumptions on the operator $A$, we have

$$\delta > 0\,. \tag{4}$$

Let us write the solution of (1) fulfilling (2) in the form (we shall show that this is possible)

$$u(t) = \sum_{i=0}^{3} \int_{\delta}^{\infty} m_i(t, s)\,\mathrm{d}E(s)\,\varphi_i\,, \tag{5}$$

where $m_i(t, s)$, $(i = 0, \ldots, 3)$ are solutions of

$$m''''(t, s) + a\,m'''(t, s) + (b_1 s^{1/2} + b_2)\,m''(t, s) + (c_1 s^{1/2} + c_2)\,m'(t, s) + \\ + (d_1 s + d_2 s^{1/2} + d_3)\,m(t, s) = 0 \tag{6}$$

fulfilling the initial conditions

$$m_i^{(k)}(0, s) = \delta_i^k\,, \quad i, k = 0, \ldots, 3\,, \quad s \geqq \delta\,. \tag{7}$$

The symbol of derivative means the derivative with respect to the variable $t$; $s \geqq \delta$ is a parameter.

Suppose that $\lambda_i = \lambda_i(s)$, $i = 1, \ldots, 4$ are solutions of

$$(8) \qquad \lambda^4(s) + a\,\lambda^3(s) + (b_1 s^{1/2} + b_2)\,\lambda^2(s) + (c_1 s^{1/2} + c_2)\,\lambda(s) + \\ + d_1 s + d_2 s^{1/2} + d_3 = 0\,.$$

For the sake of simplification we shall further use the following notation

$$(9) \qquad b = b_1 s^{1/2} + b_2\,, \quad c = c_1 s^{1/2} + c_2\,, \quad d = d_1 s + d_2 s^{1/2} + d_3\,.$$

Then

$$(10_0) \qquad m_0(t, s) = \sum_{i=1}^{4} \frac{\lambda_i^3 + a\lambda_i^2 + b\lambda_i + c}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,,$$

$$(10_1) \qquad m_1(t, s) = \sum_{i=1}^{4} \frac{\lambda_i^2 + a\lambda_i + b}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,,$$

$$(10_2) \qquad m_2(t, s) = \sum_{i=1}^{4} \frac{\lambda_i + a}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,,$$

$$(10_3) \qquad m_3(t, s) = \sum_{i=1}^{4} \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{4} (\lambda_i - \lambda_j)}\, e^{\lambda_i t}\,,$$

if $\lambda_i - \lambda_j \neq 0$ for $i \neq j$.

It will be advantageous to express the functions $m_i(t, s)$ in the following form:

$$(11_0) \qquad m_0(t, s) = (\lambda_1^3 + a\lambda_1^2 + b\lambda_1 + c) \int_0^t e^{\lambda_1(t-\tau)} \int_0^\tau e^{\lambda_2(\tau-\sigma)} \,. \\ . \int_0^\sigma e^{\lambda_3(\sigma-\varrho)} e^{\lambda_4 \varrho}\, d\varrho\, d\sigma\, d\tau + [\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 + \\ + a(\lambda_1 + \lambda_2) + b] \int_0^t e^{\lambda_2(t-\sigma)} \int_0^\sigma e^{\lambda_3(\sigma-\varrho)} e^{\lambda_4 \varrho}\, d\varrho\, d\sigma + \\ + (\lambda_1 + \lambda_2 + \lambda_3 + a) \int_0^t e^{\lambda_3(t-\varrho)} e^{\lambda_4 \varrho}\, d\varrho + e^{\lambda_4 t}\,,$$

$$(11_1)\qquad m_1(t, s) = (\lambda_1^2 + a\lambda_1 + b)\int_0^t e^{\lambda_1(t-\tau)}\int_0^\tau e^{\lambda_2(\tau-\sigma)}\cdot$$

$$\cdot\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,d\tau + (\lambda_1 + \lambda_2 + a)\cdot$$

$$\cdot\int_0^t e^{\lambda_2(t-\sigma)}\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma + \int_0^t e^{\lambda_3(t-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,,$$

$$(11_2)\qquad m_2(t, s) = (\lambda_1 + a)\int_0^t e^{\lambda_1(t-\tau)}\int_0^\tau e^{\lambda_2(\tau-\sigma)}\cdot$$

$$\cdot\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,d\tau + \int_0^t e^{\lambda_2(t-\sigma)}\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,,$$

$$(11_3)\qquad m_3(t, s) = \int_0^t e^{\lambda_1(t-\tau)}\int_0^\tau e^{\lambda_2(\tau-\sigma)}\int_0^\sigma e^{\lambda_3(\sigma-\varrho)}e^{\lambda_4\varrho}\,d\varrho\,d\sigma\,d\tau\,.$$

**Lemma 1.** *Let the following conditions be fulfilled:*

$$(12)\qquad a > 0\,,$$

$$(13)\qquad c_1s^{1/2} + c_2 > 0 \quad for \quad s \geqq \delta\,,\quad c_1 > 0\,,$$

$$(14)\qquad d_1s + d_2s^{1/2} + d_3 > 0 \quad for \quad s \geqq \delta\,,\quad d_1^2 + d_2^2 > 0\,,$$

$$(15)\qquad a(b_1s^{1/2} + b_2)(c_1s^{1/2} + c_2) - a^2(d_1s + d_2s^{1/2} + d_3) -$$

$$- (c_1s^{1/2} + c_2)^2 > 0 \quad for \quad s \geqq \delta\,,$$

$$(16)\qquad ab_1c_1 - a^2d_1 - c_1^2 > 0\,.$$

*Then there exists a constant $\omega > 0$ such that*

$$(17)\qquad \mathrm{Re}\,\lambda_i(s) \leqq -\omega$$

*for all solutions $\lambda_i(s)$ of the equation (8) and all $s \geqq \delta$.*

Proof. We can easily derive by means of the Hurwitz theorem that the necessary and sufficient conditions that the inequality $\mathrm{Re}\,\lambda_i(s) \leqq -\omega$ (for $s \geqq \delta$) holds are

$$(18_1)\qquad -4\omega + a > 0\,,$$

$$(18_2)\qquad (-4\omega + a)(6\omega^2 - 3a\omega + b) - (-4\omega^3 + 3a\omega^2 - 2b\omega + c) > 0\,,$$

$$(18_3)\qquad (-4\omega + a)(6\omega^2 - 3a\omega + b)(-4\omega^3 + 3a\omega^2 - 2b\omega + c) - $$
$$- (-4\omega + a)^2 (\omega^4 - a\omega^3 + b\omega^2 - c\omega + d) -$$
$$- (-4\omega^3 + 3a\omega^2 - 2b\omega + c)^2 > 0\,,$$

$$(18_4)\qquad \omega^4 - a\omega^3 + b\omega^2 - c\omega + d > 0\,;$$

the inequalities (18) must be fulfilled for all $s \geqq \delta$. It follows from (12) that the condition $(18_1)$ holds for sufficiently small $\omega > 0$. $(18_2)$ follows immediately from (13), (14), $(18_3)$, $(18_4)$. The condition $(18_4)$ is also fulfilled for sufficiently small $\omega > 0$ because of (14). Further it follows from (16) that there exists $S_0 \geqq \delta$ such that $(18_3)$ holds for $s \geqq S_0$. Using (15) we can guarantee also $(18_3)$ on the interval $[\delta, S_0]$, if we consider sufficiently small $\omega > 0$ only.

**Lemma 1A.** *Suppose that it holds* (12), (13), (14), (15). *Then*

$$(19)\qquad \operatorname{Re} \lambda_i(s) \leqq 0$$

*for all solutions* $\lambda_i(s)$ *of the equation* (8) *and all* $s \geqq \delta$.

Proof. It can be proved that to each $S_0 \geqq \delta$ there exists $\omega = \omega(S_0) > 0$ such that (17) holds for all solutions $\lambda_i(s)$ of the equation (8) and all $s \in [\delta, S_0]$ similarly as in the proof of Lemma 1. This proves Lemma 1A.

**Lemma 2.** *There exists a constant* $A_1 > 0$ *such that for each solution* $\lambda_i(s)$ *of the equation* (8) (*which can be written in the form*

$$(20)\qquad \lambda^4(s) + a\,\lambda^3(s) + b\,\lambda^2(s) + c\,\lambda(s) + d = 0$$

*when we use the notation* (9)) *it holds*

$$(21)\qquad |\lambda_i(s)| \leqq A_1 s^{1/4}$$

*for* $s \geqq \delta$.

Proof. If we put

$$(22)\qquad \lambda = y - \frac{a}{4}$$

we can transform the equation (20) to

$$(23)\qquad y^4 + ey^2 + fy + g = 0$$

where

$$e = b - \frac{3}{8}a^2\,,\quad f = \frac{a^3}{8} - \frac{ab}{2} + c\,,\quad g = -\frac{3}{256}a^4 + \frac{a^2 b}{16} - \frac{ac}{4} + d\,.$$

All solutions of the equation (23) are:

$$(24_1)\qquad y_1 = \tfrac{1}{2}(z_1^{1/2} + z_2^{1/2} + z_3^{1/2}),$$

$$(24_2)\qquad y_2 = \tfrac{1}{2}(z_1^{1/2} - z_2^{1/2} - z_3^{1/2}),$$

$$(24_3)\qquad y_3 = \tfrac{1}{2}(-z_1^{1/2} + z_2^{1/2} - z_3^{1/2}),$$

$$(24_4)\qquad y_4 = \tfrac{1}{2}(-z_1^{1/2} - z_2^{1/2} + z_3^{1/2}),$$

where $z_1$, $z_2$, $z_3$ are solutions of a cubic equation

$$(25)\qquad z^3 + 2ez^2 + (e^2 - 4g)\, z - f^2 = 0\,.$$

We choose values of the square roots such that $z_1^{1/2} \cdot z_2^{1/2} \cdot z_3^{1/2} = -f$. Let us put

$$(26)\qquad z = x - \frac{2}{3}e\,.$$

Then the equation (25) can be transformed to

$$(27)\qquad x^3 + 3px + 2q = 0$$

where

$$p = -\frac{e^2}{9} - \frac{4g}{3}, \quad q = -\frac{e^3}{27} + \frac{4eg}{3} - \frac{f^2}{2}\,.$$

Let us denote

$$(28)\qquad u = \sqrt[3]{(-q + \sqrt{(q^2 + p^3)})}, \quad v = \sqrt[3]{(-q - \sqrt{(q^2 + p^3)})}\,.$$

The square roots are chosen such that $uv = -p$.

Further let us put $\varepsilon = e^{2\pi i/3}$. Then solutions of the equation (27) are

$$(29_1)\qquad x_1 = u + v\,,$$

$$(29_2)\qquad x_2 = \varepsilon u + \varepsilon^2 v\,,$$

$$(29_3)\qquad x_3 = \varepsilon^2 u + \varepsilon v\,.$$

Substituting for $p$, $q$ to (28), we get

$$(30)\qquad u = K_u s^{1/2} + o(s^{1/2})\,, \quad v = K_v s^{1/2} + o(s^{1/2})\,,$$

where $K_u$, $K_v$ are constants and $o(f(s))$ means any expression such that

$$\lim_{s \to +\infty} \frac{o(f(s))}{f(s)} = 0\,.$$

We get from (22), (24), (26), (29), (30)

(31) $$\lambda_i(s) = K_i s^{1/4} + o(s^{1/4}),\; i = 1, \ldots 4,$$

$K_i$ are constants. We can easily find with help of (4) that

(32) to each $S_0 \geqq \delta$ there exists a constant $K(S_0)$ such that $|\lambda_i(s)| \leqq K(S_0)\,\delta^{1/4}$ for $s \in [\delta, S_0]$, $i = 1, \ldots, 4$.

The assertion of the lemma follows immediately from (31), (32).

**Lemma 3.** *Suppose that*

(33) $$d_1 \neq 0,$$

(34) $$b_1^2 - 4d_1 \neq 0.$$

*Then there exist constants $\Lambda_2 > 0$, $S_0 \geqq \delta$ such that*

(35) $$|\lambda_i(s) - \lambda_j(s)| \geqq \Lambda_2 s^{1/4} \quad \text{for} \quad s \geqq S_0,\quad i \neq j,\quad i, j = 1, \ldots, 4.$$

Proof. We use all notations from the proof of Lemma 2. Then

(36) $$\lambda_1 - \lambda_2 = z_2^{1/2} + z_3^{1/2},\quad \lambda_2 - \lambda_3 = z_1^{1/2} - z_2^{1/2},$$
$$\lambda_1 - \lambda_3 = z_1^{1/2} + z_3^{1/2},\quad \lambda_2 - \lambda_4 = z_1^{1/2} - z_3^{1/2},$$
$$\lambda_1 - \lambda_4 = z_1^{1/2} + z_2^{1/2},\quad \lambda_3 - \lambda_4 = z_2^{1/2} - z_3^{1/2}.$$

So if (35) is to be proved it suffices to prove

(37) $$(z_i^{1/2} + z_j^{1/2})\, s^{-1/4} \xrightarrow{(s\to+\infty)} {}^1K_{ij} \neq 0,\quad \text{for}\quad i \neq j,$$
$$(z_i^{1/2} - z_j^{1/2})\, s^{-1/4} \xrightarrow{(s\to+\infty)} {}^2K_{ij} \neq 0,\quad \text{for}\quad i \neq j;$$

the existence of finite limits is clear, cf. (31).

The conditions (37) will be fulfilled, if

(38) $$\pm \lim_{s\to+\infty} z_i^{1/2} s^{-1/4} \neq \lim_{s\to+\infty} z_j^{1/2} s^{-1/4},\quad \text{for } i \neq j$$

(the existence of finite limits is clear again).

Using (26) we get the following sufficient condition that (38) is fulfilled

(39) $$\lim_{s\to+\infty} x_i s^{-1/2} \neq \lim_{s\to+\infty} x_j s^{-1/2},\quad \text{for } i \neq j,\ i, j = 1, 2, 3.$$

Let us denote

$$\bar{p} = -\frac{b_1^2}{9} - \frac{4}{3}d_1, \quad \bar{q} = -\frac{b_1^3}{27} + \frac{4}{3}b_1d_1,$$

$$\bar{u} = \sqrt[3]{(-\bar{q} + \sqrt{(\bar{q}^2 + \bar{p}^3)})}, \quad \bar{v} = \sqrt[3]{(-\bar{q} - \sqrt{(\bar{q}^2 + \bar{p}^3)})},$$

then

$$\lim_{s \to +\infty} x_1 s^{-1/2} = \bar{u} + \bar{v}, \tag{40}$$

$$\lim_{s \to +\infty} x_2 s^{-1/2} = \varepsilon\bar{u} + \varepsilon^2\bar{v},$$

$$\lim_{s \to +\infty} x_3 s^{-1/2} = \varepsilon^2\bar{u} + \varepsilon\bar{v}.$$

It follows from (40): the condition (39) is fulfilled if

$$\bar{q}^2 + \bar{p}^3 \neq 0. \tag{41}$$

We can easily find that (41) follows from (33), (34).

This proves the lemma.

**Proposition 1.** *Suppose that* (12)–(16), (33), (34) *hold. Then there exist constants* $L > 0$, $\omega > 0$ *such that*

$$\left|m_i^{(k)}(t, s)\, s^{(i-k)/4}\right| \leqq L e^{-\omega t} \tag{42}$$

*for* $t \geqq 0$, $s \geqq \delta$, $i = 0, \ldots, 3$, $k = 0, \ldots, 4$.

Proof. It follows from (10), (17), (21), (35) that (42) is fulfilled for $s \geqq S_0$. If we take into consideration the boundedness of $\lambda_i(s)$ for $s \in [\delta, S_0]$ and use (11), we easily prove that (42) holds on $[\delta, S_0]$, too.

**Proposition 1A.** *Suppose that* (12)–(15), (33), (34) *hold. Then there exists a constant* $L > 0$ *such that*

$$\left|m_i^{(k)}(t, s)\, s^{(i-k)/4}\right| \leqq L \tag{43}$$

*for* $t \geqq 0$, $s \geqq \delta$, $i = 0, \ldots, 3$, $k = 0, \ldots, 4$.

Proof. It is similar to the proof of Proposition 1.

It follows immediately from Proposition 1A:

**Theorem 1.** *Let* (12)–(15), (33), (34) *be fulfilled. Then the function* $u(t)$, *defined by the relation* (5), *is the solution of the equation* (1) *and fulfils the initial conditions* (2).

**Theorem 2.** *Let* (12)–(16), (33), (34) *be fulfilled. Then the zero solution of the equation* (1) *is globally exponentially stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$.

Proof. Using (42) we get from (5)

$$(44_0)\qquad \|Au(t)\|^2 \leqq 4\left\{\int_\delta^\infty |m_0(t,s)|^2\, s^2\, \mathrm{d}\|E(s)\,\varphi_0\|^2 + \int_\delta^\infty |m_1(t,s)\, s^{1/4}|^2\,.\right.$$

$$.\, s^{3/2}\,\mathrm{d}\|E(s)\,\varphi_1\|^2 + \int_\delta^\infty |m_2(t,s)\, s^{1/2}|^2\, s\, \mathrm{d}\|E(s)\,\varphi_2\|^2 +$$

$$\left. + \int_\delta^\infty |m_3(t,s)\, s^{3/4}|^2\, s^{1/2}\, \mathrm{d}\|E(s)\,\varphi_3\|^2\right\} \leqq 4[Le^{-\omega t}]^2\,.$$

$$.\left(\|A\varphi_0\|^2 + \|A^{3/4}\varphi_1\|^2 + \|A^{1/2}\varphi_2\|^2 + \|A^{1/4}\varphi_3\|^2\right) =$$

$$= 4[Le^{-\omega t}]^2\, \|u(0)\|^2_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}\,.$$

We can prove similarly

$$(44_k)\qquad \|A^{1-k/4}\, u^{(k)}(t)\|^2 \leqq 4\, L[e^{-\omega t}]^2\, \|u(0)\|^2_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}\,,$$

$$k = 1, 2, 3\,.$$

If we add $(44_0)-(44_3)$, we get the global exponential stability of the zero solution.

**Theorem 3.** *Let* (12)–(15), (33), (34) *be fulfilled. Then the zero solution of the equation* (1) *is stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$.

The proof is similar to that of Theorem 2.

Remark 1. *Suppose that $v(t)$ is any solution of the equation* (1). *Then under the assumptions of Theorem* 2, *respectively Theorem* 3, *$v(t)$ is globally exponentially stable, respectively stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$.

Proof. Let $u(t)$ be a solution of (1). Then the function $w(t) = u(t) - v(t)$ satisfies equation (1), too. Now our assertion immediately follows from Theorem 2, respectively Theorem 3.

Example. The following problem is often investigated:

$$(45)\qquad \varepsilon_1\varepsilon_2\, u_{tttt}(t,x) + a\varepsilon_1\varepsilon_2\, u_{ttt}(t,x) - (\varepsilon_1 + \varepsilon_2)\, u_{ttxx}(t,x) +$$

$$+ (1 + c\varepsilon_1\varepsilon_2)\, u_{tt}(t,x) - a\varepsilon_2\, u_{txx}(t,x) + a\, u_t(t,x) + u_{xxxx}(t,x) -$$

$$- c\varepsilon_2\, u_{xx}(t,x) + c\, u(t,x) = 0\,,$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $a > 0$, $c$ are real constants,

$$u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0.$$

Using Theorem 3, we get sufficient conditions for the stability of the zero solution of the problem (45).

Put $H = L_2(0, \pi)$ and define the operator $A$ by the relation

$$(46) \quad A\, v(x) = v_{xxxx}(x)\,, \quad \text{for} \quad v \in \mathscr{D}(A) = \{v \in W_2^4(0, \pi) \mid v(0) = v(\pi) = \\ = v_{xx}(0) = v_{xx}(\pi) = 0\}\,,$$

(in the sense of distributions).

We easily find that the operator $A$ is linear, selfadjoint, strictly positive and $\delta = 1$.

Now, we can rewrite our problem into the form

$$(47) \quad u''''(t) + a\, u'''(t) + \{[(\varepsilon_1 + \varepsilon_2)\, A^{1/2} + 1 + c\varepsilon_1\varepsilon_2]/\varepsilon_1\varepsilon_2\}\, u''(t) + \\ + [(a\varepsilon_2 A^{1/2} + a)/\varepsilon_1\varepsilon_2]\, u'(t) + [(A + c\varepsilon_2 A^{1/2} + c)/\varepsilon_1\varepsilon_2]\, u(t) = 0\,.$$

Simple calculations show that the conditions (12)–(15), (33), (34) are fulfilled, if

$$(48) \quad \varepsilon_1 \neq \varepsilon_2\,, \quad c > -(1 + \varepsilon_2)^{-1}\,.$$

**Theorem 4.** *Let* (48) *be fulfilled. Then the zero solution of the problem* (45) *is stable with respect to the norm* $\|\cdot\|_{\mathscr{D}(A)\times\mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/2})\times\mathscr{D}(A^{1/4})}$, (*the operator* $A$ *is defined by* (46)).

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*Author's address:* 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).