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TOLERANCE RELATION ON LATTICES

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E. C. ZEEMAN [3] has defined the tolerance as a binary relation on a set, which is reflexive and symmetric. M. ARBIB [1, 2] has applied this concept to the theory of automata. In [4] and [5], tolerances compatible with algebraic structures are studied.

Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be an algebraic structure with the element set A and the set of operations $\mathscr{F}$. Let ξ be a tolerance on A. The tolerance ξ is compatible with $\mathbf{A}$, if and only if for any n-ary operation $f \in \mathscr{F}$, where n is a positive integer, and for any $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of A such that $(x_i, y_i) \in \xi$ for $i = 1, \ldots, n$ we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi$.

Here we shall study tolerances which are compatible with lattices. Some simple results in this topic are in [4]. From the definition of a tolerance compatible with an algebraic structure it follows that a tolerance ξ is compatible with a lattice L, if and only if for any four elements x_1, x_2, y_1, y_2 of L such that $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$ we have $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$.

Theorem 1. *Let L be a lattice, let ξ be a tolerance compatible with L. Let $(a, b) \in \xi$ for some $a \in L$, $b \in L$. Then for any x and y from interval $\langle a \wedge b, a \vee b \rangle$ we have $(x, y) \in \xi$.*

Proof. From $(a, b) \in \xi$, $(b, b) \in \xi$ (ξ is reflexive) we obtain $(a \wedge b, b \wedge b) = (a \wedge b, b) \in \xi$, $(a \vee b, b \vee b) = (a \vee b, b) \in \xi$. Analogously we obtain $(a \wedge b, a) \in \xi$, $(a \vee b, a) \in \xi$. Further from $(a \wedge b, a) \in \xi$ and $(a \wedge b, b) \in \xi$ we obtain $((a \wedge b) \vee (a \wedge b), a \vee b) = (a \wedge b, a \vee b) \in \xi$. Now let $x \in \langle a \wedge b, a \vee b \rangle$, $y \in \langle a \wedge b, a \vee b \rangle$. From $(a \wedge b, a \vee b) \in \xi$, $(x, x) \in \xi$ we have $((a \wedge b) \vee x, a \vee b \vee x) = (x, a \vee b) \in \xi$ and analogously $(y, a \vee b) \in \xi$. Taking meets, from $(x, a \vee b) \in \xi$, $(a \vee b, y) \in \xi$ we obtain $(x \wedge (a \vee b), y \wedge (a \vee b)) = (x, y) \in \xi$. As x and y were chosen arbitrarily, this holds for any two elements of the interval $\langle a \wedge b, a \vee b \rangle$.

Corollary. *In a lattice L with O and I, for any tolerance ξ compatible with L the following three assertions are equivalent*:

(i) *For some $a \in L$ there exists a complement a' and $(a, a') \in \xi$.*

(ii) $(O, I) \in \xi$.

(iii) *ξ is the universal relation on L.*

O and I denote respectively the least and the greatest element of the lattice.

Theorem 2. *Let B be a Boolean algebra, let ξ be a tolerance compatible with the operations of join and meet in B. Then ξ is a congruence on B.*

Remark. Here we do not suppose a priori that ξ is compatible with the complementation, but this follows from the assertion.

Proof. Let B_0 be the set of all elements $x \in B$ such that $(x, O) \in \xi$. If $x \in B_0$, $y \in B$, then $x \wedge y \in B_0$, because $(x, O) \in \xi$, $(y, y) \in \xi$ implies $(x \wedge y, O) \in \xi$. Therefore B_0 is an ideal of B. Any ideal of a Boolean algebra determines uniquely a congruence on it. Let $\varkappa$ be the congruence determined on B by B_0. We shall prove $\varkappa \subset \xi$. If $a \in B_0$, $b \in B_0$, then $(a, O) \in \xi$, $(O, b) \in \xi$ and this implies $(a, b) \in \xi$. If c, d are elements of the same congruence class of $\varkappa$, then $c = a \vee z$, $d = b \vee z$, where $a \in B_0$, $b \in B_0$, $z \in B$. From $(a, b) \in \xi$, $(z, z) \in \xi$ we obtain $(a \vee z, b \vee z) = (c, d) \in \xi$. Therefore $\varkappa \subset \xi$. Now let $(u, v) \in \xi$, let $\bar{v}$ be the complement of v. From $(u, v) \in \xi$, $(\bar{v}, \bar{v}) \in \xi$ we obtain $(u \wedge \bar{v}, v \wedge \bar{v}) = (u \wedge \bar{v}, O) \in \xi$ and $u \wedge \bar{v} \in B_0$. This means that the class of $\varkappa$ containing u is the complement of the class of $\varkappa$ containing $\bar{v}$ in the Boolean factor-algebra $B/\varkappa$. But obviously also the class of $\varkappa$ containing v is the complement of the class of $\varkappa$ containing $\bar{v}$. As $B/\varkappa$ is also a Boolean algebra, this complement is unique and u and v belong to the same congruence class of $\varkappa$. We have proved $\xi \subset \varkappa$ and therefore $\xi = \varkappa$.

Theorem 3. *Let C be a chain with at least three elements. Then there exist a tolerance ξ compatible with C which is not a congruence.*

Proof. Choose three elements a, b, c of L so that $a \prec b \prec c$. Now let ξ consist of all pairs (x, y), where either both x and y belong to $\langle a, b\rangle$, or both x and y belong to $\langle b, c\rangle$, or $x = y$. This is evidently a tolerance on C. Now let $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$. If all elements x_1, y_1, x_2, y_2 belong to $\langle a, b\rangle$, then also $x_1 \wedge x_2$, $x_1 \vee x_2$, $y_1 \wedge y_2$, $y_1 \vee y_2$ belong to $\langle a, b\rangle$, because the interval $\langle a, b\rangle$ is a sublattice of C; then $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. We proceed analogously if all elements x_1, y_1, x_2, y_2 belong to $\langle b, c\rangle$. If x_1, y_1 belong to $\langle a, b\rangle$ and x_2, y_2 belong to $\langle b, c\rangle$, then $x_1 \wedge x_2 = x_1$, $x_1 \vee x_2 = x_2$, $y_1 \wedge y_2 = y_1$, $y_1 \vee y_2 = y_2$, therefore $x_1 \wedge x_2$, $y_1 \wedge y_2$ belong to $\langle a, b\rangle$, $x_1 \vee x_2$, $y_1 \vee y_2$ belong to $\langle b, c\rangle$ and again $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. If x_1 belongs neither to $\langle a, b\rangle$, nor to $\langle b, c\rangle$, then necessarily $x_1 = y_1$. If it is less than a and x_2, y_2 belong both to $\langle a, b\rangle$ or both to $\langle b, c\rangle$, we have $x_1 \wedge x_2 = x_1$, $y_1 \wedge y_2 = y_1$, $x_1 \vee x_2 = x_2$,

$y_1 \vee y_2 = y_2$ and again $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. The same follows analogously if $x_1 = y_1 \succ c$ and x_2, y_2 belong either both to $\langle a, b \rangle$, or both to $\langle b, c \rangle$. Finally, if $x_1 = y_1$, $x_2 = y_2$, the proof is easy. We have obtained that ξ is a tolerance compatible with C. We have $(a, b) \in \xi$, $(b, c) \in \xi$, but $(a, c) \notin \xi$ and ξ is not a congruence.

Theorem 4. *There exists a non-complete distributive lattice L such that any tolerance compatible with L is a congruence.*

Proof. Let M be a set of cardinality $\aleph_0$, let L be the lattice of all finite subsets of M ordered by set inclusion. The elements of L will be denoted by capital letters as sets. Let ξ be a tolerance compatible with L, let A, B, C be three elements of L such that $(A, B) \in \xi$, $(B, C) \in \xi$. Let $M_0 = A \cup B \cup C$; it is a finite set. Let L_0 be the lattice of all subsets of M_0; it is a Boolean algebra and a sublattice of L. Let ξ_0 be the restriction of ξ onto L_0. Then ξ_0 is a tolerance compatible with L_0; as L_0 is a Boolean algebra, ξ_0 is a congruence on L_0 and $(A, C) \in \xi_0$. But as $\xi_0 \subset \xi$, we have also $(A, C) \in \xi$. As A, B, C and ξ were chosen quite arbitrarily, any tolerance compatible with L is transitive, therefore it is a congruence. The lattice L is evidently distributive and non-complete.

Theorem 5. *There exists a non-complete distributive lattice in which a tolerance ξ exists which is not a congruence and is compatible with L.*

Proof. We shall construct L. The vertices of L are ordered pairs of integers and $[x_1, y_1] \leqq [x_2, y_2]$ if and only if simultaneously $x_1 \leqq x_2$, $y_1 \leqq y_2$. Evidently

$$[x_1, y_1] \wedge [x_2, y_2] = [\min (x_1, x_2), \min (y_1, y_2)],$$
$$[x_1, y_1] \vee [x_2, y_2] = [\max (x_1, x_2), \max (y_1, y_2)].$$

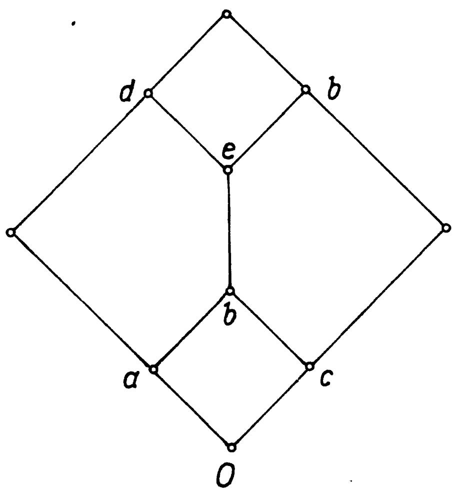
We define ξ so that $([x_1, y_1], [x_2, y_2]) \in \xi$, if and only if simultaneously $|x_1 - x_2| \leqq 1$, $|y_1 - y_2| \leqq 1$. It is evidently a tolerance. Now let $([x_1, y_1], [x_2, y_2]) \in \xi$, $([u_1, v_1], [u_2, v_2]) \in \xi$. We shall prove that then also $([x_1, y_1] \wedge [u_1, v_1], [x_2, y_2] \wedge [u_2, v_2]) \in \xi$, this means $|\min (x_1, u_1) - \min (x_2, u_2)| \leqq 1$ and $|\min (y_1, v_1) - \min (y_2, v_2)| \leqq 1$. If $x_1 \leqq u_1$, $x_2 \leqq u_2$, then $\min (x_1, u_1) = x_1$, $\min (x_2, u_2) = x_2$, and we have $|x_1 - x_2| \leqq 1$, because $([x_1, y_1], [x_2, y_2]) \in \xi$. If $x_1 \geqq u_1$, $x_2 \geqq u_2$, then $\min (x_1, u_1) = u_1$, $\min (x_2, u_2) = u_2$ and the situation is similar. Now let $x_1 \leqq u_1$, $x_2 \geqq u_2$. Then $x_1 - x_2 \leqq x_1 - u_2 \leqq u_1 - u_2$. But $|x_1 - x_2| \leqq 1$, $|u_1 - u_2| \leqq 1$, therefore $x_1 - x_2 \geqq -1$, $u_1 - u_2 \leqq 1$ and thus $-1 \leqq x_1 - u_2 \leqq 1$, which means $|x_1 - u_2| \leqq 1$. Analogously we proceed in the case $x_1 \geqq u_1$, $x_2 \leqq u_2$. We have proved that $|\min (x_1, u_1) - \min (x_2, u_2)| \leqq 1$. The proof of the inequality $|\min (y_1, v_1) - \min (y_2, v_2)| \leqq 1$ is quite analogous. Thus $([x_1, y_1] \wedge [u_1, v_1], [x_2, y_2] \wedge [u_2, v_2]) \in \xi$. Dually we can prove also $([x_1, y_1] \vee [u_1, v_1], [x_2, y_2] \vee [u_2, v_2]) \in \xi$ and therefore ξ is a tolerance compatible with L. We have $([0, 0], [1, 1]) \in \xi$, $([1, 1], [2, 2]) \in \xi$, but $([0, 0], [2, 2]) \notin \xi$ and hence ξ is not transitive.

Theorem 6. *For each cardinal number* $\boldsymbol{n} \geqq 5$ *there exists a modular non-distributive lattice* L, $|L| = n$, *such that any tolerance* ξ *compatible with it is either the identity* (*i.e.*, $(x, y) \in \xi$ *if and only if* $x = y$), *or the universal relation* (*i.e.*, $(x, y) \in \xi$ *for each* x *and* y).

Proof. Let K be a set of cardinality $\boldsymbol{n} - 2$, if $\boldsymbol{n}$ is finite, and of the cardinality $\boldsymbol{n}$, if $\boldsymbol{n}$ is infinite. The set of elements of L consists of the elements a_k $(k \in K)$ and of the elements O, I. We define $O \prec a_k \prec I$ for all $k \in K$ and $a_k \parallel a_l$ for $k \in K$, $l \in K$, $k \neq l$. Let ξ be a tolerance compatible with L and suppose that there exist $x \in L$, $y \in L$ such that $x \neq y$, $(x, y) \in \xi$. As ξ is symmetric, we may suppose without loss of generality that either $x \leqq y$, or $x \parallel y$. According to Theorem 1 it suffices to prove that then $(O, I) \in \xi$. If $x = O$, $y = I$, this is immediate. If $x = a_k$, $y = a_l$ for $k \in K$, $l \in K$, $k \neq l$ then according to Corollary, ξ is the universal relation, because a_l is a complement of a_k. If $x = O$, $y = a_k$ for some $k \in K$, then take some a_l for $l \in K$, $l \neq k$; as $|L| \geqq 5$, such a_l exists. From $(O, a_k) \in \xi$, $(a_l, a_l) \in \xi$ we obtain $(O \vee a_l, a_k \vee a_l) = (a_l, I) \in \xi$. If we take some $m \in K$, $m \neq k$, $m \neq l$, we can prove in the same way that $(a_m, I) \in \xi$. From $(a_l, I) \in \xi$, $(a_m, I) \in \xi$ we obtain $(a_l \wedge a_m, I \wedge I) = (O, I) \in \xi$. In the case $x = a_k$, $y = I$ we proceed dually.

Remark. For $\boldsymbol{n} = 5$ this lattice is actually the "forbidden sublattice" for distributive lattices.

Theorem 7. *There exists a non-modular lattice on which a tolerance compatible with it exists which is not a congruence.*



Proof. The Hasse diagram of such a lattice is in Fig. 1. The tolerance ξ is given so that $(x, y) \in \xi$, if and only if x and y lie simultaneously either in $\langle O, b\rangle$, or in $\langle a, d\rangle$, or in $\langle c, f\rangle$, or in $\langle e, I\rangle$. The reader may verify himself that ξ is compatible with L. The tolerance ξ is not a congruence.

Theorem 8. *Let L be a lattice, L_0 its sublattice, let there exist a homomorphism φ which maps L onto a lattice L_1 and such that $\varphi(x) = \varphi(y)$, if and only if $x \in L_0$, $y \in L_0$. On L_0 let there exist a tolerance ξ_0 compatible with L_0 which is not a congruence. Then there exists a tolerance ξ compatible with L which is not a congruence.*

Proof. Let ξ consist of all pairs of elements which are in ξ_0 and of all pairs of equal elements of L. We shall prove that ξ is compatible with L. Let $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$. If all elements x_1, y_1, x_2, y_2 belong to L_0, then $(x_1, y_1) \in \xi_0$, $(x_2, y_2) \in \xi_0$. The elements $x_1 \wedge x_2$, $x_1 \vee x_2$, $y_1 \wedge y_2$, $y_1 \vee y_2$ belong to L_0 and $(x_1 \wedge x_2, y_1 \wedge {}_2) \in \in \xi_0 \subset \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi_0 \subset \xi$. Now let $x_1 \in L_0$, $x_2 \in L_0$, $x_2 = y_2 \notin L_0$. If $x_2 \leqq x_1$, then $\varphi(x_2) \leqq \varphi(x_1) = \varphi(y_1)$ and therefore $y_2 = x_2 \leqq y_1$. We have $(x_1 \wedge \wedge x_2, y_1 \wedge y_2) = (x_2, y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) = (x_1, y_1) \in \xi$. In the case $x_2 \geqq \geqq x_1$ we proceed dually. If $x_1 \parallel x_2$, we have $\varphi(x_1) \parallel \varphi(x_2)$, because evidently $\varphi(x_1) \neq \varphi(x_2)$. But $\varphi(x_1) = \varphi(y_1)$, therefore $\varphi(x_2) \parallel \varphi(y_1)$ in L_1 and $x_2 \parallel y_1$ in L. In L_1 we have $\varphi(x_2) \wedge \varphi(x_1) = \varphi(x_2) \wedge \varphi(y_1) \neq \varphi(x_1)$, therefore $x_2 \wedge y_1 \notin L_0$, $x_2 \wedge y_1 \notin L_0$. But as $\varphi(x_2 \wedge x_1) = \varphi(x_2) \wedge \varphi(x_1) = \varphi(x_2) \wedge \varphi(y_1) = \varphi(x_2 \wedge y_1)$, the elements $x_2 \wedge x_1$, $x_2 \wedge y_1$ must be equal (they are not in L_0 and their images in φ are equal). Thus $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$. For joins we proceed dually. Finally, if $x_1 = y_1$, $x_2 = y_2$, the proof is easy. We have proved that ξ is a tolerance compatible with L. Now if ξ_0 is not transitive, also ξ is not transitive, because ξ contains no pair of elements of L_0 which are not in ξ_0.

In the end we shall prove a theorem concerning tolerance relations on arbitrary algebraic structures.

Theorem 9. *Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be an algebraic structure. The tolerances compatible with $\mathbf{A}$ form a lattice $LT(\mathbf{A})$ with respect to the set inclusion. In general, this lattice is not a sublattice (in the algebraic sense) of the lattice of all tolerances on A.*

Proof. As shown in [5], the intersection of two tolerances compatible with A is a tolerance compatible with $\mathbf{A}$. Thus in $LT(\mathbf{A})$ we put $\xi_1 \wedge \xi_2 = \xi_1 \cap \xi_2$ for any two tolerances ξ_1, ξ_2 which are compatible with $\mathbf{A}$. Now consider the set of all tolerances which are compatible with $\mathbf{A}$ and which contain $\xi_1 \cup \xi_2$. This set is non--empty, because it contains the universal relation on A. It is closed under intersection, the intersection of all tolerances of this set being a tolerance compatible with $\mathbf{A}$ and containing $\xi_1 \cup \xi_2$. This tolerance will be denoted by $\xi_1 \vee \xi_2$ and it will be the join of ξ_1 and ξ_2 in $LT(\mathbf{A})$, because it is contained in all tolerances compatible with $\mathbf{A}$ which contain $\xi_1 \cup \xi_2$.

In general $\xi_1 \vee \xi_2$ need not be equal to $\xi_1 \cup \xi_2$. For example, let $\mathbf{A}$ be the lattice whose elements are a, b, O, I and in which $O \prec a \prec I$, $O \prec b \prec I$, $a \parallel b$. Let $\xi_1 = \{(O, O), (O, a), (a, O), (a, a), (b, b), (b, I), (I, b), (I, I)\}$, $\xi_2 = \{(O, O), (O, b), (a, a), (a, I), (b, O), (b, b), (I, a), (I, I)\}$. These tolerances are compatible with $\mathbf{A}$; the proof is left to the reader. The tolerance $\xi_1 \vee \xi_2$ is the universal relation, because

$(O, I) \in \xi_1 \vee \xi_j$; we obtain this from $(O, a) \in \xi_1 \subset \xi_1 \vee \xi_2$, $(O, b) \in \xi_2 \subset \xi_1 \vee \xi_2$ taking joins. But the set union $\xi_1 \cup \xi_2$ does not contain (O, I). Therefore $LT(\mathbf{A})$ is not a sublattice in the algebraic sense of the lattice of all tolerances on A.

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