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ON EXPONENTIALLY BOUNDED SOLUTIONS OF LINEAR PARABOLIC DIFFERENCE-DIFFERENTIAL EQUATIONS

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0. INTRODUCTION

The purpose of the present paper is to establish some results concerning the number of linearly independent solutions of the equation

$$\mathscr{L}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}, x, t\right) u = b(x, t)\, u(x, t-1) \tag{1}$$

which are exponentially bounded for $t \to -\infty$. Here $\mathscr{L}$ denotes a parabolic differential operator of the second order, $x \in R^n$. It will be shown that under some assumptions there is only a finite number of linearly independent solutions with this property.

Let us first recall some results by J. KURZWEIL [1] which will be used in our investigation.

If X is a Banach space, W its linear subspace, then the codimension of W with respect to X is denoted by codim $W =$ codim $(W|_X)$. The restriction of a linear operator $Q : X \to X$ onto W is denoted by $Q|_W$.

If $\{k_i\}$, $\{\varrho_i\}$ are sequences of real numbers,

$$k_i \text{ integers}, \quad 0 = k_0 < k_1 < k_2 < \ldots, 0 \leqq \varrho_i \quad \text{for} \quad i = 0, 1, \ldots \tag{2}$$

then $\Omega(\{k_i\}, \{\varrho_i\})$ denotes the set of all linear operators $Q : X \to X$ such that there exists a sequence of linear subspaces $X^{(i)}$ of X with the following property:

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \ldots, \ \operatorname{codim}(X^{(i)}|_X) \leqq k_i \|Q|_{X^{(i)}}\| \leqq \varrho_i .$$

Theorem 0.1. [1; Theorem 2.1.] *A linear operator $Q : X \to X$ is completely continuous if and only if there exist sequences $\{k_i\}$, $\{\varrho_i\}$ satisfying (2) so that $\lim_{i\to\infty} \varrho_i = 0$ and $Q \in \Omega(\{k_i\}, \{\varrho_i\})$.*

If all operators from a set $\mathscr{P}$ of linear operators from X into X belong to $\Omega(\{k_i\}, \{\varrho_i\})$ with the same sequences $\{k_i\}$, $\{\varrho_i\}$, $\varrho_i \to 0$, we shall say that the operators from $\mathscr{P}$ are uniformly completely continuous.

Theorem 0.2. [1; Corollary 3.1.] *Let $Q_j : X \to X$ be linear operators, $j = -1, -2, \ldots$ For $c > 0$ denote by $Z(c)$ the set of such $\{x_j | j = 0, -1, -2, \ldots\}$ that*

$$Q_j x_j = x_{j+1}\,, \quad j = -1, -2, \ldots, \tag{3}$$

$$\limsup_{j \to -\infty} c^j \|x_j\| = +\infty\,. \tag{4}$$

If Q_j are uniformly completely continuous, then $\dim Z(c) < +\infty$ *for any $c > 0$.*

Parallelly to equation (1) we shall consider equations

$$\mathscr{L}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}, x, t\right) u = 0\,, \tag{5}$$

$$\mathscr{L}\left(\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}, x, t\right) u = f(x, t)\,. \tag{6}$$

Throughout the paper, we shall subject the operator $\mathscr{L}$ to the following conditions:

(i) $$\mathscr{L}u = \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial u}{\partial x_i} + a_0(x, t)\, u - \frac{\partial u}{\partial t}\,;$$

(ii) all coefficients of $\mathscr{L}$ are defined for all $(x, t) \in \bar{H}$, $H = D \times (-\infty, +\infty)$ where $D \subset R^n$ is a region and bar denotes the closure, and belong to $H^{\alpha,(1/2)\alpha}$, $0 < \alpha < 1$ (see e.g. [4]), i.e., they are uniformly Hölder continuous in their domain together with their derivatives $D_t^r D_x^s a(x, t)$ (a stands for any one of the coefficients), $2r + s \leqq 2$, with the exponents α, $\frac{1}{2}\alpha$ in the variable x, t, respectively;

(iii) $\mathscr{L}$ is uniformly parabolic, i.e., there are positive constants λ, μ such that

$$\lambda \xi^2 \leqq a_{ij}(x, t)\, \xi_i \xi_j \leqq \mu \xi^2$$

for all $(x, t) \in H$ and all vectors $\xi = (\xi_1, \ldots, \xi_n)$.

The set D will be assumed to be bounded and to satisfy the following condition:

(iv) For every point $P \in \partial D$ there exists an n-dimensional neighborhood $V(P)$ such that $V(P) \cap \partial D$ can be represented for some i, $1 \leqq i \leqq n$, in the form

$$x_i = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

where h, $D_x h$, $D_x^2 h$, $D_t h$ are Hölder continuous with exponent α.

(Cf. [3], Definition of property $(\bar{E})$ on p. 64. Since D is independent of t, it has even property $(\bar{\bar{E}})$ from [3], p. 65, i.e., $D_x D_t h$, $D_t^2 h$ exist and are continuous functions.)

It is well known that the solutions of (5) and (6) may be expressed in terms of the Green function or of the fundamental solution. (For definition and properties of Green functions and fundamental solutions see e.g. [3] or [4].)

If $G = G(x, t, \xi, \tau)$ is the Green function of (5), then

$$u(x, t) = \int_D G(x, t, \xi, \tau)\, \varphi(\xi)\, d\xi - \int_\tau^t \int_D G(x, t, \xi, \sigma)\, f(\xi, \sigma)\, d\xi\, d\sigma \tag{7}$$

is the solution of (6) with the initial condition $u(x, \tau) = \varphi(x)$ (φ is a continuous function defined on D with $\varphi = 0$ on ∂D) and zero boundary condition.

Similarly, if Γ is the fundamental solution of (5) and $\mathscr{L}$ satisfies our assumptions with $D = R^n$, then

$$u(x, t) = \int_{R_n} \Gamma(x, t, \xi, \tau)\, \varphi(\xi)\, d\xi - \int_\tau^t \int_{R_n} \Gamma(x, t, \xi, \sigma)\, f(\xi, \sigma)\, d\xi\, d\sigma \tag{8}$$

is a solution of (6), $u(x, \tau) = \varphi(x)$ for $x \in R^n$.

For the Green function G and the fundamental solution Γ the following estimates hold:

$$|D_t^r D_x^s\, G(x, t, \xi, \tau)| \leqq \text{const}\, (t - \tau)^{-\gamma}\, E(x, t, \xi, \tau)\,, \tag{9}$$

$$|D_t^r D_x^s\, \Gamma(x, t, \xi, \tau)| \leqq \text{const}\, (t - \tau)^{-\gamma}\, E(x, t, \xi, \tau)\,, \tag{10}$$

$\gamma = \frac{1}{2}(n + 2r + s)$, $E(x, t, \xi, \tau) = \exp\left[-c|x - \xi|^2/(t - \tau)\right]$ $(c > 0)$ for all non-negative integers r, s such that $2r + s \leqq 2$ and for $t > \tau$. (See e.g. [4; pp. 427, 469].)

1. INITIAL-BOUNDARY VALUE PROBLEM

Let $D \subset R^n$ be a bounded region which satisfies condition (iv) guaranteeing the existence of the Green function for the equation (5) (see e.g. [3; Theorem 16 on p. 82]), $H = D \times (-\infty, +\infty)$. Denote by $C = C(\bar{D} \times \langle -1, 0\rangle)$ the family of all continuous functions $w : \bar{D} \times \langle -1, 0\rangle \to R^n$ which are identically zero on the boundary $\partial D \times \times \langle -1, 0\rangle$. If $w \in C$, denote by $Q_s : C \to C$ the shift operator defined by

$$(Q_s w)(x, t) = u(x, 2s + t + 2)\,, \tag{11}$$

where $u(x, t)$ is the solution of (1) satisfying

$$u(x, 2s + t) = w(x, t) \quad \text{for} \quad (x, t) \in D \times \langle -1, 0\rangle \tag{12}$$

and

$$u(x, t) = 0 \quad \text{for} \quad x \in \partial D\,,\ t \geqq 2s - 1\,. \tag{13}$$

If G is the Green function of the equation (5), we can write the solution u of (1) satisfying (12), (13) in the form

$$u(x, t) = \int_D G(x, t, \xi, 2s)\, w(\xi, 0)\, d\xi + \int_{2s}^{t} \int_D G(x, t, \xi, \tau)\, b(\xi, \tau)\, u(\xi, \tau - 1)\, d\xi\, d\tau\,.$$

Hence according to (11),

$$(14) \qquad (Q_s w)(x, t) = \int_D G(x, 2s + t + 2, \xi, 2s)\, w(\xi, 0)\, d\xi +$$
$$+ \int_{2s}^{2s+t+2} \int_D G(x, 2s + t + 2, \xi, \tau)\, b(\xi, \tau) u(\xi, \tau - 1)\, d\xi\, d\tau =$$
$$= \int_D G(x, 2s + t + 2, \xi, 2s) w(\xi, 0)\, d\xi +$$
$$+ \int_{-1}^{0} \int_D G(x, 2s + t + 2, \xi, 2s + \tau + 1)\, b(\xi, 2s + \tau + 1)\, w(\xi, \tau)\, d\xi\, d\tau +$$
$$+ \int_0^{t+1} G(x, 2s + t + 2), \xi, 2s + \tau + 1)\, b(\xi, 2s + \tau + 1)\, u(\xi, 2s + \tau)\, d\xi\, d\tau\,.$$

Lemma 1.1. *Let $|b(x, t)| \leqq B(t)$ for all $x \in D$ and $t \in R$,*

$$(15) \qquad \int_{-1}^{t+1} (t - \tau + 1)^{-1/2}\, B(2s + \tau + 1)\, d\tau \leqq B\,,$$

$$(16) \qquad \int_{t_1}^{t_2} \int_{-1}^{t_1+1} (t - \tau + 1)^{-1}\, B(2s + \tau + 1)\, d\tau\, dt \leqq |\chi(t_2) - \chi(t_1)|$$

for all $t_1, t_2, t \in \langle -1, 0 \rangle$ and $i = 1, 2, \ldots, n$. Here B is a constant and $\chi : \langle -1, 0 \rangle \to R$ is a continuous nondecreasing function, $\chi(0) = 0$.

Then for all $t_1, t_2 \in \langle -1, 0 \rangle$, $x^1, x^2 \in D$ and $w \in C$, Q_s satisfies

$$(17) \qquad |Q_s w(x^1, t_1) - Q_s w(x^2, t_2)| \leqq \text{const}\, \|w\| \left(\|x^1 - x^2\| + |\chi(t_1) - \chi(t_2)|\right),$$

$s = -1, -2, \ldots$, $\|w\| = \sup |w(x, t)|$ *taken for all $(x, t) \in D \times \langle -1, 0 \rangle$. The constant on the right hand side of* (17) *depends on the coefficients of* (1) *and on n.*

Remark. Conditions (15), (16) are satisfied e.g. if $|B(t)| \leqq$ const for all $t \in R$.

Proof. Denote the right hand side integrals in (14) successively by I_1, I_2, I_3. We estimate these integrals by means of (9), taking into account that for $x^{(j)} = (x_1, \ldots \ldots, x_{i-1}, x_i^j, x_{i+1}, \ldots, x_n)$

$$(18) \qquad G(x^2, t, \xi, \tau) - G(x^1, t, \xi, \tau) = \int_{x^1_i}^{x^2_i} D_{x_i} G(x, t, \xi, \tau)\, dx_i\,,$$

provided the whole integration interval belongs to D.

Given two arbitrary points $x^1, x^2 \in D$, we can pass from the former to the latter along a piecewise linear line formed by segments s_i parallel to coordinate axes which lies wholly in D. Moreover, the assumption (iv) guarantees that, denoting the length of s_i by $l(s_i)$, s_i may be chosen so that

$$\sum_i l(s_i) \leqq \text{const } \|x^1 - x^2\|$$

where the constant depends on D but not on the points x^1, x^2. Hence we can write

$$G(x^2, t, \xi, \tau) - G(x^1, t, \xi, \tau) = \sum_i \int_{s_i} D_x G(x, t, \xi, \tau) \, dx$$

where every s_i is a segment parallel to a coordinate axis, say x_j, and D_x indicates differentiation with respect to x_j.

Consequently,

$$|I_1(x^2, t) - I_1(x^1, t)| \leqq$$

$$\leqq \text{const } \|w\| \sum_i \int_{s_i} \int_D (t + 2)^{-(n+1)/2} E(x, 2s + t + 2, \xi, 2s) \, d\xi \, dx \leqq$$

$$\leqq \text{const } \|w\| \, \|x^1 - x^2\| \, .$$

Taking into account (15), (16) we obtain similarly

$$|I_2(x^2, t) - I_2(x^1, t)| \leqq \text{const } \|w\| \quad \|x^1 - x^2\| \, .$$

The last integral I_3 involves the values of $u(\xi, 2s + \tau)$ for $\tau \in \langle 0, 1 \rangle$. Nevertheless,

$$u(\xi, 2s + \tau) = \int_D G(\xi, 2s + \tau, \eta, 2s) \, w(\eta, 0) \, d\eta +$$

$$+ \int_{2s}^{2s+\tau} \int_D G(\xi, 2s + \tau, \eta, \sigma) \, b(\eta, \sigma) \, u(\eta, \sigma - 1) \, d\eta \, d\sigma \, .$$

Since $\sigma - 1 \in \langle 2s - 1, 2s \rangle$ we can write $u(\eta, \sigma - 1) = w(\eta, \tilde{\sigma})$ where $\tilde{\sigma} = \sigma - 1 - 2s \in \langle -1, 0 \rangle$. By virtue of (9), this implies the estimate

$$|u(\xi, 2s + \tau)| \leqq \text{const } \|w\| \, .$$

Hence we obtain by (15), (16)

$$|I_3(x^2, t) - I_3(x^1, t)| \leqq$$

$$\leqq \text{const } \|w\| \, \|x^2 - x^1\| \int_0^{t+1} (t - \tau + 1)^{-1/2} B(2s + \tau + 1) \, d\tau \leqq$$

$$\leqq \text{const } \|w\| \, \|x^2 - x^1\| \, .$$

Similarly we estimate the differences in t, using

$$(19)\qquad G(x, t_2, \xi, \tau) - G(x, t_1, \xi, \tau) = \int_{t_1}^{t_2} D_t G(x, t, \xi, \tau)\, dt,$$

$$|I_1(x, t_2) - I_1(x, t_1)| \leqq$$

$$\leqq \text{const } \|w\| \int_{t_1}^{t_2} \int_D (t + 2)^{-(n+2)/2} E(x, 2s + t + 2, \xi, 2s)\, d\xi\, dt \leqq$$

$$\leqq \text{const } \|w\|\, |t_2 - t_1|.$$

For the second integral we have

$$|I_2(x, t_2) - I_2(x, t_1)| \leqq \text{const } \|w\| \int_{t_1}^{t_2} \int_{-1}^{0} \int_D (t - \tau + 1)^{-(n+2)/2}\, .$$

$$. E(x, 2s + t + 2, \xi, 2s + \tau + 1) B(2s + \tau + 1)\, d\xi\, d\tau\, dt \leqq$$

$$\leqq \text{const } \|w\|\, |\chi(t_2) - \chi(t_1)|$$

and for the last one, assuming without loss of generality $t_1 < t_2$,

$$|I_3(x, t_2) - I_3(x, t_1)| = \Big| \int_0^{t_1+1} \int_D [G(x, 2s + t_2 + 2, \xi, 2s + \tau + 1) -$$

$$- G(x, 2s + t_1 + 2, \xi, 2s + \tau + 1)]\, b(\xi, 2s + \tau + 1)\, u(\xi, 2s + \tau)\, d\xi\, d\tau +$$

$$+ \int_{t_1+1}^{t_2+1} \int_D G(x, 2s + t_2 + 2, \xi, 2s + \tau + 1)\, b(\xi, 2s + \tau + 1) u(\xi, 2s + \tau)\, d\xi\, d\tau \Big| \leqq$$

$$\leqq \int_{t_1}^{t_2} \int_0^{t_1+1} \int_D \left| \frac{\partial G}{\partial t}(x, 2s + t + 2, \xi, 2s + \tau + 1)\, b(\xi, 2s + \tau + 1)\, u(\xi, 2s + \tau) \right| .$$

$$. d\xi\, d\tau\, dt +$$

$$+ \int_{t_1+1}^{t_2+1} \int_D |G(x, 2s + t_2 + 2, \xi, 2s + \tau + 1)\, b(\xi, 2s + \tau + 1)\, u(\xi, 2s + \tau)|\, d\xi\, d\tau \leqq$$

$$\leqq \text{const } \|w\| \int_{t_1}^{t_2} \int_0^{t_1+1} (t - \tau + 1)^{-1} B(2s + \tau + 1)\, d\tau\, dt +$$

$$+ \text{const } \|w\| \int_{t_1+1}^{t_2+1} B(2s + \tau + 1)\, d\tau \leqq \text{const } \|w\|\, |\chi(t_2) - \chi(t_1)|.$$

Putting all these estimates together we obtain (17), which completes the proof of Lemma 1.1.

Lemma 1.2. *There exist constants μ, ν such that the operators Q_s defined by* (11), $s = -1, -2, \ldots$ *belong to* $\Omega(\{k_i\}, \{\varrho_i\})$ *with*

$$(20)\qquad \varrho_0 = \mu,\ \varrho_1 = \nu,\ \varrho_i = \nu/2^{i-2} \quad \textit{for} \quad i = 2, 3, \ldots,$$
$$k_0 = 0,\ k_1 = 1,\ k_i = (2^{i-2} + 1)^{n+1} \quad \textit{for} \quad i = 2, 3, \ldots$$

Proof. Let us omit the subscript s. (14) together with (15), (16) implies $|Qz(x, t)| \leqq \leqq \text{const} \|z\|$ for all $x \in D$, $t \in \langle -1, 0 \rangle$ and $z \in C$. Hence $\|Q\| \leqq \text{const} = \mu$.

Fix a point $x^0 \in D$ and denote $d_k = \sup_{x,y \in D} |x_k - y_k|$ $(x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n))$. The set of points

$$x = (x_1^0 + j_1 d_1/2^{i-2},\ x_2^0 + j_2 d_2/2^{i-2}, \ldots, x_n^0 + j_n d_n/2^{i-2})$$

with j_k integers forms a rectangular net. Choose one point $x^{j_1,\ldots,j_n} \in D$ in every (n-dimensional) rectangle of this net which has non-empty intersection with D and denote the set of these points by $\Theta^{(i)}$ (for fixed i). (Since D is a bounded set, $\Theta^{(i)}$ is finite and the number of its points is not greater than $(2^{i-2} + 1)^n$.) Further, let $\vartheta : \langle 0, \beta \rangle \to \langle -1, 0 \rangle$ be such that $\chi \circ \vartheta(\zeta) = \zeta$ for all $\zeta \in \langle 0, \beta \rangle$.

Denote

$$X^{(0)} = C,\ X^{(1)} = \{z \in C \mid Qz(x^0, 0) = 0\},$$

$$X^{(i)} = \{z \in C \mid Qz(x, \vartheta(j_0\beta/2^{i-2})) = 0 \quad \text{for} \quad x \in \Theta^{(i)},\ \ j_0 = 0, 1, \ldots, 2^{i-2}\}.$$

Then evidently $\operatorname{codim} X^{(i)}|_X \leqq k_i$, $k_0 = 0$, $k_1 = 1$, $k_i = (2^{i-2} + 1)^{n+1}$ for $i = 2, 3, \ldots$

Let $z \in X^{(i)}$, $i \geqq 2$. For any $(x, t) \in D \times \langle -1, 0 \rangle$ there exist integers $\varkappa_0, \varkappa_1, \ldots, \varkappa_n$ such that

$$|t - \vartheta(\varkappa_0\beta/2^{i-2})| \leqq \beta/2^{i-2}, \quad |x_k - x_k^{\varkappa_1,\ldots,\varkappa_n}| \leqq d_k/2^{i-2}.$$

We have according to Lemma 1.1

$$|Qz(x, t) - Qz(x^{\varkappa_1,\ldots,\varkappa_n},\ \vartheta(\varkappa_0\beta/2^{i-2})| \leqq$$
$$\leqq \text{const} \|z\| (|x - x^{\varkappa_1,\ldots,\varkappa_n}| + |\chi(t)(-\chi(\vartheta(\varkappa_0\beta/2^{i-2}))|).$$

Hence

$$|Qz(x, t)| \leqq \text{const} \|z\| \left[\sum_{k=1}^{n} (d_k/2^{i-2}) + \beta/2^{i-2}\right]$$

as $Qz(\tilde{x}, \tilde{t}) = 0$ by virtue of $z \in X^{(i)}$. Consequently,

$$\|Qz\| \leqq \text{const} \frac{1}{2^{i-2}} \left(\sum_{k=1}^{n} d_k + \beta\right) = \frac{\nu}{2^{i-2}}$$

where the constant ν depends on the coefficients of the equation (1) and on the region D. Hence $\varrho_i = \nu/2^{i-2}$ for $i \geqq 2$. Since the cases $i = 0, 1$ are easy, we may consider the proof of Lemma 1.2 complete.

Theorems 0.1 and 0.2 enable us to establish

Theorem 1.1. *Denote by $\mathscr{X}(c)$ the set of all solutions $u(x, t)$ of* (1), (13) *which fulfil $u(x, t) = w(x, t)$ for $(x, t) \in D \times \langle -1, 0 \rangle$, $w \in C$, such that*

(21) $$\limsup_{t \to -\infty} e^{ct} |u(x, t)| < +\infty.$$

Then for every real c,

$$\dim \mathscr{Z}(c) < +\infty \,. \tag{22}$$

Proof. Theorems 0.1 and 0.2 concern the set $Z(c)$. Nevertheless, it is sufficient to define for a solution $u : R^- \to R$ of (1), (13) functions $u_s : D \times \langle -1, 0 \rangle \to R^1$, $u_s(x, t) = u(x, t + 2s)$ and

$$W(u) = \{u_s \mid s = -1, -2, \ldots\}$$

and it is clear immediately that the restriction of W (the set of all $W(u)$) to $\mathscr{Z}(c)$ is a bijection of $\mathscr{Z}(c)$ onto $Z(e^c)$ for any real c. (Cf. [2; Definition 3.2 and Lemma 4.1].)

2. SOLUTIONS PERIODIC IN x

The results of Section 1 may be modified to the initial value problem with the coefficients of $\mathscr{L}$ and the initial function w periodic in x. Formulas (8), (10) used instead of (7), (9) enable us to establish a lemma formally identical with Lemma 1.1. The space $C = C(R^n \times \langle -1, 0 \rangle)$ denotes then the set of all functions $w : R^n \times \langle -1, 0 \rangle \to R$ which are continuous and periodic in x with a given period (more precisely: periodic in x_i, $i = 1, 2, \ldots, n$ with the periods P_i respectively). Defining the operators Q_s by (11) again, Lemma 1.2 holds without any change. Hence we assert

Theorem 2.1. *Assume that the coefficients of $\mathscr{L}$ are periodic in x_i with given periods P_i $(t = 1, 2, \ldots, n)$. Denote by $\mathscr{Z}(c)$ the set of all solutions u of (1) which are periodic in x_i with the same periods P_i $(i = 1, 2, \ldots, n)$ and fulfil (21).*

Then for every real c, (22) holds.

3. ESTIMATES OF $\dim \mathscr{Z}(c)$

In this section we shall establish an estimate of $\dim \mathscr{Z}(c)$ which follows from Theorem 3.2 [1]. Let us briefly recall the notation of [1] which is adopted in the sequel. (Cf. [1; Definitions 1.2 and 2.2].)

Let m be a positive integer, $M(m)$ the set of real $m \times m$ matrices (a_{ij}), $|a_{ij}| \leqq 1$ for $i, j = 1, 2, \ldots, m$. We denote

$$g(m) = \sup \det (a_{ij}) \quad \text{over all} \quad (a_{ij}) \in M(m) \,.$$

Let m, p be positive integers, $\{k_i\}$, $\{\varrho_i\}$ sequences of real numbers satisfying (2). Find the integer $s \geqq 0$ such that

$$m = pk_s + z \,, \quad 0 < z \leqq p(k_{s+1} - k_s) \,. \tag{23}$$

Denote (empty product being equal to one by definition)

$$\Xi(m, p) = g(m) \prod_{i=0}^{s-1} \varrho_i^{p^2(k_{i+1}-k_i)} \varrho_s^{pz} . \tag{24}$$

Note that the estimate $g(m) \leqq m^{m/2}$ holds (see [1; p. 368]).

Theorem 3.1. [1; Theorem 3.2.] *Let* $Q_s \in \Omega(\{k_i\}, \{\varrho_i\})$ *for* $s = -1, -2, \ldots, \varrho_i \to 0$. *Let* m, p *be positive integers,* $c > 0$,

$$\Xi(m, p)^{1/mp} < \frac{1}{c} .$$

Then $\dim Z(c) < m$.

The following lemma is a modification of [2; Lemma 3.1].

Lemma 3.1. *Let* m, p *be positive integers, let* k_i, ϱ_i *be defined by* (20). *Then*

(i) $\Xi(m, p)^{1/mp} = g(m)^{1/mp} \mu$ *provided* $k_0 < \frac{m}{p} \leqq k_1$;

(ii) $\Xi(m\ p)^{1/mp} = g(m)^{1/mp} \mu^{p/m} \nu^{(m-p)/m}$ *provided* $k_1 < \frac{m}{p} \leqq k_2$;

(iii) $\Xi(m, p)^{1/mp} = g(m)^{1/mp} \nu \left(\frac{\mu}{\nu}\right)^{p/m} 2^{-r+3+(p/m)\Sigma_{i=3}^{r-1} k_i}$

provided $k_{r-1} < \frac{m}{p} \leqq k_r$, $r \geqq 3$.

Proof. If $k_0 < m/p \leqq k_1$ then according to (23) $s = 0$, $z = m$ and (i) follows immediately. Similarly, if $k_1 < m/p \leqq k_2$ then $s = 1$, $z = m - p$ and (ii) follows. Finally, if $k_{r-1} < m/p \leqq k_r$ and $r \geqq 3$, then

$$\Xi(m, p) = g(m)\, \mu^{p^2} \nu^{p^2(k_2-k_1)} \prod_{i=2}^{r-2} \left(\frac{\nu}{2^{i-2}}\right)^{p^2(k_{i+1}-k_i)} \left(\frac{\nu}{2^{r-3}}\right)^{p(m-pk_{r-1})} =$$

$$= g(m)\, \mu^{p^2} \nu^{p^2(\sum_{i=1}^{r-2}(k_{i+1}-k_i)-k_{r-1})}\, 2^{-p^2[\sum_{i=2}^{r-2}(i-2)(k_{i+1}-k_i)-(r-3)k_{r-1}]} \left(\frac{\nu}{2^{r-3}}\right)^{mp} =$$

$$= g(m) \left(\frac{\mu}{\nu}\right)^{p^2} \left(\frac{\nu}{2^{r-3}}\right)^{mp} 2^{p^2 \sum_{i=3}^{r-1} k_i}$$

which proves the lemma.

The next step is to estimate $\sum_{i=3}^{r-1} k_i$ for $k_i = (2^{i-2} + 1)^{n+1}$. By elementary calculation,

$$\sum_{i=3}^{r-1} k_i = \sum_{i=1}^{r-3} (2^i + 1)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \sum_{i=1}^{r-3} 2^{ik} =$$

$$= r - 3 + \sum_{k=1}^{n+1} \binom{n+1}{k} 2^k \frac{2^{(r-3)k} - 1}{2^k - 1} \leqq r - 3 + 2 \sum_{k=1}^{n+1} \binom{n+1}{k} (2^{(r-3)k} - 1) =$$

$$= r - 3 + 2(2^{r-3} + 1)^{n+1} - 2(2^{n+1} \quad 1) = r - 1 + 2(2^{r-3} + 1)^{n+1} - 2^{n+2} .$$

This estimate together with $g(m) \leqq m^{m/2}$ used in the assertion (iii) of Lemma 3.1 yields

$$\Xi(m, p)^{1/mp} \leqq m^{1/2p} \nu \left(\frac{\mu}{\nu}\right)^{p/m} 2^{-r+3+(p/m)[r-1+2(2^{r-3}+1)^{n+1}-2^{n+2}]}.$$

Since $r \geqq 3$, it is $p(2^{r-3} + 1)^{n+1} < m \leqq p(2^{r-2} + 1)^{n+1}$; hence $2pm^{-1}(2^{r-3} + 1)^{n+1} < 2$ and we can estimate the last exponent as follows:

$$-r + 3 + \frac{p}{m}[r - 1 + 2(2^{r-3} + 1)^{n+1} - 2^{n+2}] \leqq$$

$$\leqq -r + 5 + \frac{p}{m}[r - 1 - 2^{n+2}] \leqq -r + 5 + \frac{p}{m}(r - 5) = (r - 5)\left(\frac{p}{m} - 1\right).$$

On the other hand, $2^{r-2} \geqq (m/p)^{1/(n+1)} - 1$. Since $p < m$, we obtain

$$\Xi(m,p)^{1/mp} \leqq m^{1/2p} \nu \left(\frac{\mu}{\nu}\right)^{p/m} \left\{\frac{1}{8}\left[\left(\frac{m}{p}\right)^{1/(n+1)} - 1\right]\right\}^{(p/m)-1} = \tag{25}$$

$$= 8m^{1/2p} \nu \left(\frac{\mu}{\nu}\right)^{p/m} \left(\frac{p}{m}\right)^{1/(n+1)} \Psi\left(\frac{m}{p}\right),$$

$$\Psi(\zeta) = \zeta^{1/(n+1)}(\zeta^{1/(n+1)} - 1)^{-1} [\tfrac{1}{8}(\zeta^{1/(n+1)} - 1)]^{1/\zeta}.$$

Now we are able to formulate the desired estimate as

Theorem 3.3. *Let μ, ν and k_i, ϱ_i have the meaning from Lemma 1.2, formula (20), let $\mu/\nu \leqq E$.*

Then to every $\varepsilon > 0$ there exists $\lambda(\varepsilon, E)$ so that the following assertion holds: If $Q_j \in \Omega(\{k_i\}, \{\varrho_i\})$, $j = -1, -2, \ldots$ then

$$\dim Z(c) < \tfrac{1}{2}(1 + \varepsilon)(8e\nu c)^{n+1} \ln (\nu c)^{n+1}$$

provided $c\nu \geqq \lambda(\varepsilon, E)$.

Proof. Put $m = [\frac{1}{2}(1 + \varepsilon)(8e\nu c)^{n+1} \ln (\nu c)^{n+1}]$, $p = [\frac{1}{2} \ln (\nu c)^{n+1}]$, the brackets denoting the whole part of a number. Then for $c\nu$ sufficiently large it is $m > 2^{n+1}p$ and hence, substituting the above values of m, p into (25), we obtain

$$\Xi(m, p)^{1/mp} \leqq \Psi^*\left(\frac{m}{p}\right) 8\nu e(1 + \varepsilon)^{-1/(n+1)} (8\nu ec)^{-1} \leqq \Psi^*\left(\frac{m}{p}\right)\frac{1}{c}(1 + \varepsilon)^{-1/(n+1)}$$

where $\Psi^*(m/p) \to 1$ if $m/p \to +\infty$. Hence there exists $\lambda = \lambda(\varepsilon, E)$ such that

$$\Xi(m, p)^{1/mp} < \frac{1}{c}$$

provided $cv \geqq \lambda(\varepsilon, E)$ and Theorem 3.1 implies the inequality for $\dim Z(c)$ which completes the proof.

Taking into account the existence of a bijection of $\mathscr{X}(\gamma)$ onto $Z(e^\gamma)$ where $\mathscr{X}(\gamma)$ is either the set from Theorem 1.1 or from Theorem 2.1, we conclude

Theorem 3.2. *Let $\mathscr{X}(c)$ have the meaning from Theorem 1.1. Then to every $\varepsilon > 0$ and $E > 0$ there exists $\lambda(\varepsilon, E)$ so that*

$$\dim \mathscr{X}(c) < \tfrac{1}{2}(1 + \varepsilon)(8e^{1+c}v)^{n+1}(c + \ln v)(n + 1) \tag{26}$$

provided $e^c v \geqq \lambda(\varepsilon, E)$, $\mu/v \leqq E$.

Similarly we obtain

Theorem 3.3. *Let $\mathscr{X}(c)$ have the meaning from Theorem 2.1. Then to every $\varepsilon > 0$ and $E > 0$ there exists $\lambda(\varepsilon, E)$ so that (26) holds provided $e^c v \geqq \lambda(\varepsilon, E)$, $\mu/v \leqq E$.*

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