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ON THE DEGREES OF GRAPHS WITH $\alpha(G) \leqq 2$

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Certain results are proved concerning the degrees of finite undirected graphs with the stability-number at most 2 (theorems 1, 2, 3). By applying theorems 1 and 2 we obtain the solution of an extremal combinatorial problem proposed by the author in [1] (theorem 4 of this paper). The obtained results generalize respectively modify a special case of a theorem of Turán (see e.g. [2], p. 269). The reader who is interested also in other generalizations or modifications of the Turán's theorem may consult e.g. [3], [4], [5] and [6].

Let n be a given integer, $n \geqq 2$, and put K_n for a complete undirected graph without loops and multiple edges (cf. [2] pp. 5–7) with n vertices $u_1, u_2, \ldots, u_n$. Let us associate with each partial graph (see [2] p. 7) G of K_n its stability-number $\alpha(G)$ (cf. [2], p. 260), and put $d_j(G)$ $(j = 1, 2, \ldots, n)$ for the degree of the vertex u_j in G (cf. [2], p. 6). Further, let us denote by $\mathscr{G}_n$ the family of all partial graphs G of K_n such that $\alpha(G) \leqq 2$.

Theorem 1. *Let $G \in \mathfrak{G}_n$. Then a partition*[1])

$$\{\{u_{i(1)}, \ldots, u_{i(a)}\}, \{u_{i(a+1)}, \ldots, u_{i(n)}\}\}$$

of the vertex-set $\{u_1, u_2, \ldots, u_n\}$ exists such that

$$\text{(i)} \qquad 1 \leqq a \leqq \left[\frac{n}{2}\right]^{2)},$$

$$\text{(ii)} \qquad \min\{d_{i(1)}(G), \ldots, d_{i(a)}(G)\} \geqq a - 1,$$

$$\text{(iii)} \qquad \min\{d_{i(a+1)}(G), \ldots, d_{i(n)}(G)\} \geqq n - a - 1.$$

Proof. Let $G \in \mathfrak{G}_n$. The subgraph (cf. [2], p. 7) of G generated by a nonempty set $V \subset \{u_1, u_2, \ldots, u_n\}$ will be denoted by $G(V)$. We shall distinguish the following

[1]) The term "partition" will denote a disjoint decomposition.

[2]) The symbol $[\xi]$ will denote the integer part of a real number ξ.

two cases:

$$(\alpha) \qquad \min\{d_1(\mathbf{G}), d_2(\mathbf{G}), \ldots, d_n(\mathbf{G})\} > \frac{n}{2} - 1$$

$$(\beta) \qquad \min\{d_1(\mathbf{G}), d_2(\mathbf{G}), \ldots, d_n(\mathbf{G})\} \leqq \frac{n}{2} - 1\,.$$

In the (α) case the assertion of the theorem is obviously fulfilled for $a = [\frac{1}{2}n]$ and for an arbitrary partition having the form $\{\{\mathbf{u}_{i(1)}, \ldots, \mathbf{u}_{i([n/2])}\}, \{\mathbf{u}_{i([n/2]+1)}, \ldots, \mathbf{u}_{i(n)}\}\}$.

Thus, let us consider the (β) case. Choose for $\mathbf{u}_{i(1)}$ any vertex such that

$$d_{i(1)}(\mathbf{G}) = \min\{d_j(\mathbf{G}) \mid j = 1, 2, \ldots, n\}\,,$$

and denote by $\mathbf{u}_{i(2)}, \ldots, \mathbf{u}_{i(a)}$ all the vertices adjacent to $\mathbf{u}_{i(1)}$. Further, let $\mathbf{u}_{i(a+1)}, \ldots$ $\ldots, \mathbf{u}_{i(n)}$ denote all remaining vertices. We shall show that the partition

$$\{\{\mathbf{u}_{i(1)}, \ldots, \mathbf{u}_{i(a)}\}, \{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\}\}$$

constructed in this way, has properties (i)–(iii).

Properties (i) and (ii) follow immediately from our construction and from (β); it remains to verify (iii). Indeed, the subgraph $\mathbf{G}(\{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\})$ must be complete. Since assuming, on the contrary, that non-adjacent vertices $\mathbf{u}_{i(\gamma)}$ and $\mathbf{u}_{i(\delta)}$ exist such that $a < \gamma < \delta \leqq n$ we obtain a stable set (cf. [2], p. 260) $\{\mathbf{u}_{i(1)}, \mathbf{u}_{i(\gamma)}, \mathbf{u}_{i(\delta)}\}$, which contradicts $\alpha(\mathbf{G}) \leqq 2$.

From the completeness of $\mathbf{G}(\{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\})$ (iii) immediately follows. □

The following theorem shows that lowerbounds (ii) and (iii) established in theorem 1 are best possible, and it describes all graphs $\mathbf{G} \in \mathfrak{G}_n$ for which in (ii) and (iii) equality holds.

Theorem 2. *Let* $a \in \{1, 2, \ldots, [\frac{1}{2}n]\}$, and let

$$\{\{\mathbf{u}_{i(1)}, \ldots, \mathbf{u}_{i(a)}\}, \{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\}\}$$

be a partition of the vertex-set of $\mathbf{K}_n$. *Then* $\mathbf{G} \in \mathfrak{G}_n$ *exists such that*

$$(1) \qquad d_j(\mathbf{G}) = a - 1 \quad for \quad j = i(1), i(2), \ldots, i(a)$$

and

$$(2) \qquad d_j(\mathbf{G}) = n - a - 1 \quad for \quad j = i(a+1), \ldots, i(n)\,.$$

This graph is unique if $a < \frac{1}{2}n$, *and it consists of two complete subgraphs as connectivity-components; the first connectivity-component is generated by* $\{\mathbf{u}_{i(1)}, \ldots$ $\ldots, \mathbf{u}_{i(a)}\}$, *the second by* $\{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\}$.

In the case of $a = \frac{1}{2}n$ all the graphs $\mathbf{G} \in \mathfrak{G}_n$ satisfying $d_j(\mathbf{G}) = \frac{1}{2}n - 1$ ($j = 1, 2, \ldots, n$) are just the graphs which consist of two complete connectivity-components having an equal number of vertices.

Proof. Let $\mathbf{G}_0$ be a graph of $\mathfrak{G}_n$ as follows: Vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent in $\mathbf{G}_0$ iff

$$(\{\mathbf{u}, \mathbf{v}\} \subset \{\mathbf{u}_{i(1)}, \ldots, \mathbf{u}_{i(a)}\} \text{ or } \{\mathbf{u}, \mathbf{v}\} \subset \{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\}) .$$

The graph $\mathbf{G}_0$ satisfies conditions (1) and (2), which proves the first part of the theorem.

Conversely, let $\mathbf{G}$ be an arbitrary graph satisfying (1) and (2), and let us choose a vertex $\mathbf{u}_{j(1)}$ such that

$$d_{j(1)}(\mathbf{G}) = \min \{d_j(\mathbf{G}) \mid j = 1, 2, \ldots, n\} .$$

Then $d_{j(1)}(\mathbf{G}) = a - 1$. Further, let $\mathbf{u}_{j(2)}, \ldots, \mathbf{u}_{j(a)}$ be all the vertices adjacent to $\mathbf{u}_{j(1)}$, and denote by $\mathbf{u}_{j(a+1)}, \ldots, \mathbf{u}_{j(n)}$ the remaining vertices. Analogously as in the proof of theorem 1 we obtain

(3) $$\min \{d_{j(1)}(\mathbf{G}), \ldots, d_{j(a)}(\mathbf{G})\} \geqq a - 1 ,$$

(4) $$\min \{d_{j(a+1)}(\mathbf{G}), \ldots, d_{j(n)}(\mathbf{G})\} \geqq n - a - 1 ,$$

and

(5) $$\mathbf{G}(\{\mathbf{u}_{j(a+1)}, \ldots, \mathbf{u}_{j(n)}\})$$

is complete.

By combining (1), (2), (3) and (4) we obtain further

(6) $$d_{j(1)}(\mathbf{G}) = \ldots = d_{j(a)}(\mathbf{G}) = a - 1 ,$$

(7) $$d_{j(a+1)}(\mathbf{G}) = \ldots = d_{j(n)}(\mathbf{G}) = n - a - 1 ,$$

and moreover if $a < \frac{1}{2}n$:

(8) $$\{j(1), j(2), \ldots, j(a)\} = \{i(1), i(2), \ldots, i(a)\}$$
$$\{j(a + 1), \ldots, j(n)\} = \{i(a + 1), \ldots, i(n)\} .$$

Now, it follows from (5), (6) and (7) that no vertex of $\{\mathbf{u}_{j(1)}, \ldots, \mathbf{u}_{j(a)}\}$ is adjacent to a vertex of $\{\mathbf{u}_{j(a+1)}, \ldots, \mathbf{u}_{j(n)}\}$ and hence $\mathbf{G}(\{\mathbf{u}_{j(1)}, \ldots, \mathbf{u}_{j(a)}\})$ must be also complete.

Thus $\mathbf{G}$ consists of two complete subgraphs

$$\mathbf{G}(\{\mathbf{u}_{j(1)}, \ldots, \mathbf{u}_{j(a)}\}) \quad \text{and} \quad \mathbf{G}(\{\mathbf{u}_{j(a+1)}, \ldots, \mathbf{u}_{j(n)}\})$$

as connectivity-components, and moreover (8) holds if $a < \frac{1}{2}n$. □

Let us denote by Δ_n the family of all n-dimensional vectors $(\delta_1, \delta_2, \ldots, \delta_n)$ such that $\delta_j = d_j(\mathbf{G})$ $(j = 1, 2, \ldots, n)$ holds for some $\mathbf{G} \in \mathfrak{G}_n$. Let $\mathcal{R}$ denote the partial-order relation on Δ_n as follows:

$$(\delta_1, \ldots, \delta_n) \mathcal{R} (\delta'_1, \ldots, \delta'_n) \quad \text{iff} \quad \delta_j \leqq \delta'_j \quad (j = 1, \ldots, n).$$

Vector $(\delta_1^*, \ldots, \delta_n^*) \in \Delta_n$ will be called a *minimal element* of Δ_n iff for any $(\delta_1, \ldots, \delta_n) \in \in \Delta_n$

$$(\delta_1, \ldots, \delta_n) \mathcal{R} (\delta_1^*, \ldots, \delta_n^*) \Rightarrow (\delta_1, \ldots, \delta_n) = (\delta_1^*, \ldots, \delta_n^*).$$

By using theorems 1 and 2 we obtain easily the following characterization of minimal elements of Δ_n.

Corollary. *Vector $(\delta_1, \delta_2, \ldots, \delta_n) \in \Delta_n$ is a minimal element of Δ_n iff there exists a partition*

$$\{\{i(1), \ldots, i(a)\}, \{i(a+1), \ldots, i(n)\}\} \quad \text{of} \quad \{1, 2, \ldots, n\}$$

such that $1 \leqq a \leqq \frac{1}{2}n$,

$$\delta_j = a - 1 \qquad \text{for} \quad j = i(1), \ldots, i(a),$$
$$\delta_j = n - a - 1 \quad \text{for} \quad j = i(a+1), \ldots, i(n). \;\square$$

In the next theorem, another extremal property of the degrees of $\mathbf{G} \in \mathfrak{G}_n$ is investigated.

Theorem 3. *Let $\mathbf{G} \in \mathfrak{G}_n$. Then*

$$\sum_{j=1}^{n} \left(d_j(\mathbf{G}) - \frac{3n-4}{4} \right)^2 \leqq \frac{n^3}{16}. \tag{9}$$

Further, the equality

$$\sum_{j=1}^{n} \left(d_j(\mathbf{G}) - \frac{3n-4}{4} \right)^2 = \frac{n^3}{16}$$

holds iff either $\mathbf{G} = \mathbf{K}_n$ or $\mathbf{G}$ consists of two complete subgraphs as connectivity-components.

Remark 1. The first part of this theorem follows also from [1].

Remark 2. The assertion may be formulated in a geometrical fashion as follows: The vector $(d_1(\mathbf{G}), d_2(\mathbf{G}), \ldots, d_n(\mathbf{G}))$ is contained for any $\mathbf{G} \in \mathfrak{G}_n$ in the ball defined by (9). Moreover, the theorem describes all graphs $\mathbf{G} \in \mathfrak{G}_n$ the vector of degrees of those belongs to the boundary of the ball (9).

Proof of the theorem 3. Let us put E for the set of all edges of G. We shall say that $e \in E$ is *incident* to a triplet (u_i, u_j, u_k) where $1 \leqq i < j < k \leqq n$ if e links some pair of vertices u_i, u_j, u_k. Let us denote by $T(e)$ the set of all such triplets which are incident to e, and put

$$T = \bigcup_{e \in E} T(e). \tag{10}$$

It follows from $\alpha(G) \leqq 2$ that T contains all triplets, and hence

$$\operatorname{card}(T) = \frac{n(n-1)(n-2)}{6}. \tag{11}$$

On the other hand, it follows from (10) that

$$\operatorname{card}(T) = \sum_{r=1}^{\operatorname{card}(E)} (-1)^{r-1} \sum^{(r)} \operatorname{card}(T(e_1) \cap \ldots \cap T(e_r)) \text{ [3]} \tag{12}$$

where $\sum^{(r)} \operatorname{card}(T(e_1) \cap \ldots \cap T(e_r))$ denotes the sum of $\operatorname{card}(T(e_1) \cap \ldots \cap T(e_r))$ over the family of all r-element subsets $\{e_1, \ldots, e_r\}$ of E. From (12) it follows immediately

$$\operatorname{card}(T) = \sum_{r=1}^{3} (-1)^{r-1} \sum^{(r)} \operatorname{card}(T(e_1) \cap \ldots \cap T(e_r)) \tag{13}$$

since the remaining summands on the right-hand side of (12) are zero. (This follows from the fact that at most three distinct edges can be incident to a common triplet.)

Now we shall express the first and the second summand on the right-hand side of (13) by using $d_j(G)$, and estimate the third:

$$\sum^{(1)} \operatorname{card}(T(e_1)) = (n-2) \operatorname{card}(E) = \frac{n-2}{2} \sum_{j=1}^{n} d_j(G) \tag{14}$$

(since each edge is incident to exactly $n - 2$ triplets),

$$\sum^{(2)} \operatorname{card}(T(e_1) \cap T(e_2)) = \tfrac{1}{2} \sum_{j=1}^{n} d_j(G)(d_j(G) - 1), \tag{15}$$

(since any two distinct edges are incident to a common triplet iff they have one common vertex), and

$$\sum^{(3)} \operatorname{card}(T(e_1) \cap T(e_2) \cap T(e_3)) \leqq \tag{16}$$
$$\leqq \tfrac{1}{3} \sum^{(2)} \operatorname{card}(T(e_1) \cap T(e_2)) = \tfrac{1}{6} \sum_{j=1}^{n} d_j(G)(d_j(G) - 1),$$

(since to each three edges incident to a common triplet three pairs of edges correspond occuring in $\sum^{(2)} \operatorname{card}(T(e_1) \cap T(e_2))$.

[3]) According to the so called "principle of inclusion and exclusion".

By combining (11), (13), (14), (15) and (16) we obtain the inequality

$$\frac{n-2}{2}\sum_{j=1}^{n} d_j(\mathbf{G}) - \tfrac{1}{3}\sum_{j=1}^{n} d_j(\mathbf{G})\,(d_j(\mathbf{G})-1) \geqq \frac{n(n-1)(n-2)}{6}$$

which yields

$$\sum_{j=1}^{n}\left(d_j(\mathbf{G})^2 - \frac{3n-4}{2}\,d_j(\mathbf{G})\right) \leqq -\frac{n(n-1)(n-2)}{2},$$

and

$$\sum_{j=1}^{n}\left(d_j(\mathbf{G}) - \frac{3n-4}{4}\right)^2 \leqq \frac{n^3}{16},$$

proving the first part of the theorem.

Now, let for $\mathbf{G} \in \mathfrak{G}_n$ the equality in (9) hold. Then

$$3\sum^{(3)} \operatorname{card}\left(\mathrm{T}(\mathbf{e}_1) \cap \mathrm{T}(\mathbf{e}_2) \cap \mathrm{T}(\mathbf{e}_3)\right) = \sum^{(2)} \operatorname{card}\left(\mathrm{T}(\mathbf{e}_1) \cap \mathrm{T}(\mathbf{e}_2)\right),$$

and hence for any vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ of $\mathbf{G}$ if $\mathbf{u}$ is adjacent both to $\mathbf{v}$ and to $\mathbf{w}$ then also $\mathbf{v}$ and $\mathbf{w}$ are adjacent, i.e. the subgraph $\mathbf{G}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$ is complete. We conclude immediately from this fact that each connectivity-component of $\mathbf{G}$ is a complete subgraph. Further $\alpha(\mathbf{G}) \leqq 2$ yields that $\mathbf{G}$ has no more than two connectivity-components, and hence our assertion.

In order to complete the proof we must show that the equality holds in (9) if $\mathbf{G} = \mathbf{K}_n$ or $\mathbf{G}$ consists of two complete connectivity-components. Indeed, it follows from our assumption that an integer $a \in \{0, 1, \ldots, [\frac{1}{2}n]\}$ and a partition $\{\{i(1), \ldots \ldots, i(a)\}, \{i(a+1), \ldots, i(n)\}\}$ of $\{1, 2, \ldots, n\}$ exist such that

$$d_j(\mathbf{G}) = \begin{cases} a-1 & \text{for } j = i(1), \ldots, i(a), \\ n-a-1 & \text{for } j = i(a+1), \ldots, i(n). \end{cases}$$

(If $a = 0$ then $d_j(\mathbf{G}) = n - 1$; this case corresponds to $\mathbf{G} = \mathbf{K}_n$.) Then

$$\sum_{j=1}^{n}\left(d_j(\mathbf{G}) - \frac{3n-4}{4}\right)^2 =$$

$$= a\left(a-1-\frac{3n-4}{4}\right)^2 + (n-a)\left(n-a-1-\frac{3n-4}{4}\right)^2 = \frac{n^3}{16},$$

which completes the proof. □

Theorems 1 and 2 will be now applied for the solution of an open problem from [1]. Let be given nonnegative numbers $c_1, c_2, \ldots, c_n$ assigned respectively to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

(The numbers $c_1, c_2, \ldots, c_n$ will be considered as weights of vertices.) Put

$$\tau_n(c_1, \ldots, c_n) = \min \{ \sum_{j=1}^{n} c_j d_j(\mathbf{G}) \mid \mathbf{G} \in \mathfrak{G}_n \} .$$

The problem of determing $\tau_n(c_1, \ldots, c_n)$ was proposed in [1][4]). The following theorem solves this problem, and moreover, in the case of $c_j > 0$ ($j = 1, 2, \ldots, n$) it describes all extremal graphs $\mathbf{G} \in \mathfrak{G}_n$.

Theorem 4. a) *It holds that*

$$\tau_n(c_1, \ldots, c_n) = \min ((a-1) \sum_{j=1}^{a} c_{i(j)} + (n-a-1) \sum_{j=a+1}^{n} c_{i(j)})$$

where the minimum is taken over the family of all partitions $\{\{\mathbf{u}_{i(1)}, \ldots, \mathbf{u}_{i(a)}\}, \{\mathbf{u}_{i(a+1)}, \ldots, \mathbf{u}_{i(n)}\}\}$ *of* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *such that* $1 \leqq a \leqq \frac{1}{2}n$.

b) *Let* $\min \{c_j \mid j = 1, 2, \ldots, n\} > 0$, *and*

$$\sum_{j=1}^{n} c_j \, d_j(\mathbf{G}) = \tau_n(c_1, c_2, \ldots, c_n) .$$

Then $\mathbf{G}$ *is a graph consisting of two complete subgraphs as connectivity-components.*

Proof. The a) part follows by combining theorem 1 and the fact that the function "$(\delta_1, \ldots, \delta_n) \to c_1\delta_1 + \ldots + c_n\delta_n$" is isotonic on Δ_n.

If $\min \{c_j \mid j = 1, 2, \ldots, n\} > 0$ then the considered function is even strictly isotonic. Thus it follows from

$$\sum_{j=1}^{n} c_j \, d_j(\mathbf{G}) = \tau_n(c_1, \ldots, c_n) \quad \text{that} \quad (d_1(\mathbf{G}), \ldots, d_n(\mathbf{G}))$$

is a minimal element in Δ_n, and hence $\mathbf{G}$ is a graph described in the theorem 2, which completes the proof. □

Remark. It follows from this theorem that for determining all solutions of the considered problem if $c_j > 0$ ($j = 1, \ldots, n$) it is necessary and sufficient to determine all partitions $\{\{u_{i(1)}, \ldots, u_{i(a)}\}, \{u_{i(a+1)}, \ldots, u_{i(n)}\}\}$ such that $1 \leqq a \leqq \frac{1}{2}n$ and

$$(a-1) \sum_{j=1}^{a} c_{i(j)} + (n-a-1) \sum_{j=a+1}^{n} c_{i(j)} = \min = \tau_n(c_1, \ldots, c_n) .$$

[4]) This problem was proposed by the author also at the "Conference on Graph Theory and Combinatorial Analysis" held at Štiřín (May 1972).

If we put $c_1 = c_2 = \ldots = c_n = \frac{1}{2}$ in the previous theorem and observe that $\sum_{j=1}^{n} c_j\, d_j(\mathbf{G})$ equals the number of edges of $\mathbf{G}$ (i.e. card (E) according to the notation used in the proof of theorem 3) in this case we obtain easily the following assertion that coincides with a special case of the Turán theorem.

Corollary 1. *If* $\mathbf{G} \in \mathfrak{G}_n$ *then the number of all edges of* $\mathbf{G}$ *is at least* $[\frac{1}{4}(n-1)^2]$. *Moreover,* $\text{card}(\mathbf{E}) = [\frac{1}{4}(n-1)^2]$ *iff* $\mathbf{G}$ *consists of two complete connectivity-components.* □

In order to simplify the computation of $\tau_n(c_1, \ldots, c_n)$ the following simple fact may be useful.

Corollary 2. *If* $c_1 \geqq c_2 \geqq \ldots \geqq c_n \geqq 0$ [5]) *then* $\tau_n(c_1, \ldots, c_n) = \min \{(a-1) \,.$ $\,. \sum_{j=1}^{a} c_j + (n-a-1) \sum_{j=a+1}^{n} c_j \mid a = 1, 2, \ldots, [\frac{1}{2}n]\}$.

Proof. In view of theorem 4, it is sufficient to prove that

$$(a-1)\sum_{j=1}^{a} c_{i(j)} + (n-a-1)\sum_{j=a+1}^{n} c_{i(j)} \geqq$$
$$\geqq (a-1)\sum_{j=1}^{a} c_j + (n-a-1)\sum_{j=a+1}^{n} c_j$$

holds for any partition $\{\{i(1), \ldots, i(a)\}, \{i(a+1), \ldots, i(n)\}\}$ of $\{1, 2, \ldots, n\}$ such that $1 \leqq a \leqq \frac{1}{2}n$. Indeed

$$(a-1)\sum_{j=1}^{a} c_{i(j)} + (n-a-1)\sum_{j=a+1}^{n} c_{i(j)} =$$
$$= (a-1)\sum_{j=1}^{n} c_{i(j)} + (n-2a)\sum_{j=a+1}^{n} c_{i(j)} = (a-1)\sum_{j=1}^{n} c_j + (n-2a)\sum_{j=a+1}^{n} c_{i(j)} \geqq$$
$$\geqq (a-1)\sum_{j=1}^{n} c_j + (n-2a)\sum_{j=a+1}^{n} c_j = (a-1)\sum_{j=1}^{a} c_j + (n-a-1)\sum_{j=a+1}^{n} c_j\,,$$

which completes the proof. □

The results of this paper can be also formulated in a different form. Let $\mathfrak{G}_n^*$ denote the family of all partial graphs $\mathbf{G}$ of K_n that do not contain "triangles", i.e. complete subgraphs with three vertices. Let us consider the bijection $\Phi : \mathfrak{G}_n \leftrightarrow \mathfrak{G}_n^*$ such that $\Phi(\mathbf{G})$ is the complementary graph of $\mathbf{G} \in \mathfrak{G}_n$. By using the mapping Φ we can state theorem 1 in the following equivalent fashion:

[5]) This can be guaranteed by an appropriate numbering of vertices.

"*Let* $\mathbf{G} \in \mathfrak{G}_n^*$. *Then a partition*

$$\{\{u_{i(1)}, \ldots, u_{i(a)}\}, \{u_{i(a+1)}, \ldots, u_{i(n)}\}\} \quad \text{of} \quad \{u_1, \ldots, u_n\}$$

exists such that

(j) $$1 \leqq a \leqq \left[\frac{n}{2}\right]$$

(jj) $$\max \{d_{i(1)}(\mathbf{G}), \ldots, d_{i(a)}(\mathbf{G})\} \leqq n - a$$

(jjj) $$\max \{d_{i(a+1)}(\mathbf{G}), \ldots, d_{i(n)}(\mathbf{G})\} \leqq a \,."$$

Also the remaining assertions of this paper may be formulated in a "complementary" fashion.

In the conclusion we present the following problem: Theorems 1 and 3 are certain necessary conditions for n given nonnegative integers to be representable as degrees of a certain graph $\mathbf{G} \in \mathfrak{G}_n$. We find it interesting to look for some necessary and sufficient conditions.

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