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ON CONNECTED GRAPHS CONTAINING EXACTLY TWO POINTS OF THE SAME DEGREE

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Following BEHZAD and CHARTRAND [1], we shall say that a graph G with $p \geqq 2$ points is quasiperfect if it contains exactly two points v and w of the same degree. The points v and w will be called the exceptional points of G. (For basic notions of graph theory, see HARARY [2].)

By D_2 we shall denote a line. If p is an integer and $p \geqq 3$, then by D_p we shall denote the complement of a graph obtained from D_{p-1} by adding an isolated point. As it immediately follows from Theorem 2 (and from its proof) in [1], for any integer $p \geqq 2$ it holds that: (i) G is a connected quasiperfect graph with p points if and only if G is isomorphic to D_p; (ii) G is a disconnected quasiperfect graph with p points if and only if G is isomorphic to the complement $\bar{D}_p$ of the graph D_p; (iii) each exceptional point of D_p has degree $[p/2]$. (If x is a real number, then $[x]$ is the greatest integer n such that $n \leqq x$; similarly, $\{x\} = -[-x]$.)

Let p be any integer such that $p \geqq 2$. We shall investigate properties of the graph D_p.

Proposition. *D_p has $[p/2] . \{p/2\}$ lines.*

Theorem 1. *Let t and u be points of D_p having degree d and e, respectively. Then t and u are adjacent if and only if $d + e \geqq p$.*

Proof. The case $p = 2$ is obvious. Assume that $p = n \geqq 3$ and that for $p = n - 1$ the theorem is proved. Let $d \leqq e$.

The case when $e = p - 1$ is obvious. Assume that $e \leqq p - 2$; then t and u lie in D_{p-1}. The points t and u are adjacent in D_p if and only if they are not adjacent in D_{p-1}. The points t and u are not adjacent in D_{p-1} if and only if $(p - 1 - d) + (p - 1 - e) < p - 1$. Hence the theorem follows.

Corollary 1. *Let i be an integer, $1 \leqq i \leqq [p/2]$. By t_i and u_i we denote points of D_p*

with degree i and $p - i$, respectively, such that $t_{[p/2]} \neq u_{[p/2]}$. Then t_i and u_i are adjacent for any i and the set $\{t_1u_1, \ldots, t_{[p/2]}u_{[p/2]}\}$ is a maximum matching of D_p.

Theorem 2. *Let v be an exceptional point of D_p. If $p \geqq 3$, then $D_p - v$ is isomorphic to D_{p-1}.*

Proof. Let u be a point of D_p, $u \neq v$. By d and d' we denote the degree of u in D_p or in $D_p - v$, respectively. From Theorem 1 it follows that if $d < \{p/2\}$, then $d' = d$, and if $d \geqq \{p/2\}$, then $d' = d - 1$. This means that $D_p - v$ contains exactly two points of the same degree. As $\{p/2\} \geqq 2$, $D_p - v$ is connected.

Theorem 3. *Let $p \geqq 3$. The graph G obtained from D_p by identifying its exceptional points is isomorphic to D_{p-1}.*

Proof. Let v and w be the exceptional points of D_p and u be any point of D_p such that $v \neq u \neq w$. From Theorem 1 it follows that u is adjacent to v if and only if u is adjacent to w. This means that G is isomorphic to $D_p - v$. Hence the theorem follows.

Lemma. *Let m be a positive integer. Then D_p contains a subgraph isomorphic to K_m if and only if $m \leqq \{(p + 1)/2\}$.*

Proof. The cases when $p = 2, 3$ are obvious. Let $p = n \geqq 4$ and assume that for $p = n - 2$ the lemma is proved. If from D_p we delete simultaneously the point of degree 1 and the point of degree $p - 1$, we obtain D_{p-2}, which contains a subgraph isomorphic to K_m if and only if $m \leqq \{(p - 1)/2\}$. Obviously, $\{(p - 1)/2\} + 1 = \{(p + 1)/2\}$. Hence the lemma follows.

Corollary 2. *D_p is planar if and only if $p \leqq 7$.*

Theorem 4. *The chromatic number of D_p is $\{(p + 1)/2\}$.*

Proof. The case when $p = 2$ is obvious. Let $p = n > 3$ and assume that for $p = n - 1$ the theorem is proved. From the lemma it follows that $\{(p + 1)/2\} \leqq \chi(D_p)$. It is easy to see that $\chi(\bar{D}_p) = \chi(D_{p-1}) = \{p/2\}$. From one of the inequalities of NORDHAUS and GADDUM [3] it follows that $\chi(D_p) \leqq p + 1 - \chi(\bar{D}_p) = p + 1 - \{p/2\} = \{(p + 1)/2\}$. Hence the theorem follows.

References

[1] *M. Behzad* and *G. Chartrand:* No graph is perfect. Amer. Math. Monthly *74* (1967), 962—963.

[2] *F. Harary:* Graph Theory. Addison-Wesley, Reading 1969.

[3] *E. A. Nordhaus* and *J. W. Gaddum:* On the complementary graphs. Amer. Math. Monthly *63* (1956), 175—177.

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