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EMBEDDING INFINITE TREES INTO A CUBE OF INFINITE DIMENSION

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I. HAVEL and P. LIEBL [1] have investigated embeddings of finite trees into n-dimensional cube graphs, where n is finite. Here we shall prove a theorem concerning infinite trees. The validity of the Axiom of Choice is assumed. An infinite tree is an infinite connected undirected graph without circuits.

At first we shall define the cube graph of the infinite dimension $\aleph_\alpha$ (α is some ordinal number). Let ω_α be the initial ordinal number [2] of the cardinality $\aleph_\alpha$. The set of edges of the cube graph $Q(\aleph_\alpha)$ of the dimension $\aleph_\alpha$ is the set of all transfinite sequences of the ordinal number ω_α such that in each of them a finite number of elements are equal to 1 and all others are equal to 0. Two sequences $\{u_\iota\}_{\iota<\omega_\alpha}$, $\{v_\iota\}_{\iota<\omega_\alpha}$ are joined by an edge if and only if $u_\iota \neq v_\iota$ for exactly one $\iota < \omega_\alpha$ and $u_\iota = v_\iota$ for all other $\iota < \omega_\alpha$. This definition is analogous to the definition of the cube graph of a finite dimension. The assumption that only a finite number of elements are equal to 1 is made in order that the graph $Q(\aleph_\alpha)$ might be connected. If a sequence with infinitely many ones were admitted, it would not be joined by a path with the sequence consisting only of zeroes and the graph would not be connected. This is an analogy with the definition of the Hilbert cube in the functional analysis.

Before expressing the theorem, some preparatory considerations will be made. Let N be the set of all non-negative integers, P the set of positive integers. We shall define a one-to-one mapping f of $P \times N$ onto N by the following way:

$$f(x, y) = \binom{x + y}{2} + x$$

for $x \in P$, $y \in N$. It can be proved that this mapping is really one-to-one. Now let N_β, where β is some infinite ordinal number, be the set of all ordinal numbers less than β. We shall define a one-to-one mapping F of $P \times N_\beta$ onto N_β. Any ordinal number γ can be expressed uniquely by the following way:

$$\gamma = \lambda(\gamma) + n(\gamma),$$

where $\lambda(\gamma)$ is a limit ordinal number or 0 and $n(\gamma)$ is a finite ordinal number (i.e. a non-negative integer). We put

$$F(x, \eta) = \lambda(\eta) + f(x, n(\eta))$$

for $x \in P$, $\eta \in N_\alpha$. This mapping is also one-to-one.

Now let $\mathfrak{M}(\omega_\alpha, \omega_0)$ be the set of all infinite matrices $\|m_{i\varkappa}\|$, where i runs through all positive integers and $\varkappa$ runs through all ordinal numbers less than ω_α; the elements $m_{i\varkappa}$ are positive integers. Further let $\mathfrak{N}(\omega_\alpha)$ be the set of all transfinite sequences of positive integers $\{n_\varkappa\}_{\varkappa<\omega_\alpha}$. With help of $F(x, \eta)$ we can introduce a one-to-one correspondence between $\mathfrak{M}(\omega_\alpha, \omega_0)$ and $\mathfrak{N}(\omega_\alpha)$. If $\boldsymbol{M} \in \mathfrak{M}(\omega_\alpha, \omega_0)$, $\boldsymbol{M} = \|m_{i\varkappa}\|$, then $\Phi(\boldsymbol{M})$ will be the transfinite sequence $\{n_\mu\}_{\mu<\omega_\alpha}$, where $n_\mu = m_{i\varkappa}$ for $\mu = F(i, \varkappa)$. As F is one-to-one, so is Φ.

We shall explicate intuitively the sense of Φ. For the sake of simplicity we do it only for $\omega_\alpha = \omega_0$. We put an ordering into the set of all symbols m_{ij} $(i \in P, j \in N)$ such that $m_{ij} \prec m_{kl}$ if and only if either $i + j < k + l$, or $i + j = k + l$ and $i < k$. Then for the symbols m_{ij} we substitute their values in the given matrix $\boldsymbol{M}$ and obtain the transfinite sequence $\Phi(\boldsymbol{M})$.

Let $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ be the subset of $\mathfrak{M}(\omega_\alpha, \omega_0)$ consisting of the matrices $\|m_{i\varkappa}\|$ in which $m_{i\varkappa} = 1$ for a finite number of pairs $[i, \varkappa]$ and $m_{i\varkappa} = 0$ for all other pairs $[i, \varkappa]$. Let $\mathfrak{N}_0(\omega_\alpha)$ be the subset of $\mathfrak{N}(\omega_\alpha)$ constisting of the sequences $\{n_\varkappa\}_{\varkappa<\omega_\alpha}$ in which $n_\varkappa = 1$ for a finite number of ordinals $\varkappa$ and $n_\varkappa = 0$ for all other ordinals $\varkappa$. Evidently the restriction of Φ on $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ is a one-to-one mapping of $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ onto $\mathfrak{N}_0(\omega_\alpha)$. If we have two matrices $\boldsymbol{M} = \|m_{i\varkappa}\|$, $\boldsymbol{M}' = \|m'_{i\varkappa}\|$ of $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ and $m_{i\varkappa} \neq m'_{i\varkappa}$ for exactly one pair $[i, \varkappa]$, then in the sequences $\Phi(\boldsymbol{M}) = \{n_\mu\}_{\mu<\omega_\alpha}$, $\Phi(\boldsymbol{M}') = \{n'_\mu\}_{\mu<\omega_\alpha}$ we have $n_\varkappa \neq n'_\varkappa$ exactly for one $\varkappa$.

Now we shall prove the theorem.

Theorem. *Every tree with the vertex set of the cardinality $\aleph_\alpha$ (where α is some ordinal number) can be embedded into the cube graph $Q(\aleph_\alpha)$ of the dimension $\aleph_\alpha$ and cannot be embedded into any cube graph of the dimension less than $\aleph_\alpha$.*

Proof. Let T be a tree, let its vertex set U have the cardinality $\aleph_\alpha$. This means that the degree of any vertex of T is at most $\aleph_\alpha$. Choose an arbitrary vertex $u \in U$. For any non-negative integer n let U_n be the set of all vertices of T whose distance from u is n. As T is a tree, it is connected and any two vertices are joined by a path, whose length is a non-negative integer. Thus $U = \bigcup_{n \in N} U_n$. As in a tree any two vertices are joined by exactly one path, we have $U_m \cap U_n = \emptyset$ for $m \neq n$. And evidently $U_0 = \{u\}$. Now to the vertices of T we shall assign matrices of $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ such that to a vertex of U_n a matrix $\|m_{i\varkappa}\|$ is assigned such that $m_{i\varkappa} = 0$ for all $i > n$. We shall do it by the following way. To the vertex u we assign the zero matrix of $\mathfrak{M}_0(\omega_\alpha, \omega_0)$. Now let $n > 0$ and assume that we have assigned the matrices of $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ to all vertices of U_{n-1}. As T is a tree, any vertex of U_n is joined by an edge exactly with one vertex of U_{n-1}. Thus if $v \in U_{n-1}$, then let $V(v)$ be the set of vertices of U_n joined with v. We have $V(v) \wedge V(w) = \emptyset$ for $v \neq w$, $v \in U_{n-1}$, $w \in U_{n-1}$. The cardinality of $V(v)$ is at most $\aleph_\alpha$. We can well-order the set $V(v)$ so that the corresponding ordinal number is at most ω_α. Thus we have a transfinite sequence $\{w_\iota\}_{\iota<\beta}$ where $\beta \leqq \omega_\alpha$, of all elements of $V(v)$. Let the matrix assigned to v be $\|m_{i\varkappa}\|$; as $v \in U_{n-1}$, we assume that $m_{i\varkappa} = 0$ for $i > n - 1$. To any vertex w_μ of this sequence we assign the matrix $\|m'_{i\varkappa}\| \in \mathfrak{M}_0(\omega_\alpha, \omega_0)$ such that $m'_{n\mu} = 1$, $m'_{n\varkappa} = 0$ for $\varkappa \neq \mu$, $m'_{i\varkappa} = m_{i\varkappa}$ for $i \neq n$.

The matrix assigned to a vertex of U_n for $n > 0$ has some 1 in the n-th row and only zeroes in the further rows; thus no matrix can be assigned to two vertices of different sets U_n. We shall prove by induction according to n that for any two different vertices of U_n the corresponding matrices are different. The set U_0 contains only one vertex, thus it fulfills this assertion trivially. In a set U_n for some $n > 0$ any vertex belongs to exactly one $V(v)$ for $v \in U_{n-1}$. The matrix assigned to a vertex of $V(v)$ has the first $n - 1$ rows equal to these rows in the matrix assigned to v. If $v \in U_{n-1}$, $w \in U_{n-1}$, $v \neq w$, the matrices (according to the induction assumption) assigned to v and w are different; they differ in some element of the first $n - 1$ rows (because the following rows consist only of zeroes). Thus if we take a vertex of $V(v)$ and a vertex of $V(w)$, the assigned matrices to these vertices must differ, too. Finally, if we have two different vertices of the same $V(v)$, the assigned matrices differ in the n-th row. We have proved that the assigning matrices of $\mathfrak{M}_0(\omega_\alpha, \omega_0)$ to the vertices of T is one-to-one.

Further we see that for any pair of vertices joined by an edge the assigned matrices differ exactly in one element.

Now if $u \in U$ and M is the matrix assigned to u, we put

$$\mathfrak{v}(u) = \Phi(M).$$

Thus $\mathfrak{v}(u)$ is some transfinite sequence of the ordinal ω_α and if u, v are two vertices of T joined by an edge, then $\mathfrak{v}(u)$ and $\mathfrak{v}(v)$ differ exactly in one element. If we identify any vertex u of T with the vertex of $Q(\aleph_\alpha)$ corresponding to the sequence $\mathfrak{v}(u)$, we have embedded T into $Q(\aleph_\alpha)$, q.e.d.

It is easy to prove that the cardinality of the vertex set of $Q(\aleph_\alpha)$ for any infinite cardinal number $\aleph_\alpha$ is equal to $\aleph_\alpha$. Thus no graph whose vertex set has the cardinality $\aleph_\alpha$ can be embedded into $Q(\aleph_\beta)$, where $\aleph_\beta < \aleph_\alpha$.

References

[1] *I. Havel* and *P. Liebl:* O vnoření dichotomického stromu do krychle. Čas. pěst. mat. *97* (1972), 201—205.

[2] *W. Sierpiński:* Cardinal and Ordinal Numbers. Warszawa 1958.

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