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ON A CLASS OF GENERALIZED JACOBI'S
ORTHONORMAL POLYNOMIALS

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1. INTRODUCTION

1.1. Denote by I the closed interval $[-1, 1]$. Let $\alpha > -1$, $\beta > -1$ and let $u(x)$ be a real function integrable and bounded on I . (The integrals in this paper are those of Lebesgue.)

Put

$$(1,1a) \quad J(x) = (1-x)^\alpha (1+x)^\beta$$

and

$$(1,1b) \quad Q(x) = J(x) e^{u(x)}.$$

Let for $n = 0, 1, 2, \dots$

$$(1,1c) \quad Q_n(x) = \sum_{k=0}^n a_k^{(n)} x^{n-k}$$

with

$$(1,1d) \quad a_0^{(n)} > 0$$

be the orthonormal polynomial associated with the function $Q(x)$ on the interval I , i.e.

$$(1,1e) \quad \int_I Q_m(x) Q_n(x) Q(x) dx = \delta_{m,n}.$$

Here

$$m \neq n \Rightarrow \delta_{m,n} = 0, \quad \delta_{m,m} = 1.$$

The function $Q(x)$ is called the weight of the polynomials $Q_n(x)$.

By a well-known theorem there exists for every n one and only one polynomial $Q_n(x)$ satisfying (1,1e) and (1,1d). (See [5] p. 66.)

1,2. Let

$$(1,2a) \quad J_n(x) = \sum_{k=0}^n b_k^{(n)} x^{n-k}$$

where $b_0^{(n)} > 0$ be the orthonormal polynomial associated with the function $J(x)$ defined by (1,1a) on the interval I .

$J_n(x)$ is a special case of the polynomial $Q_n(x)$ for $u(x) \equiv 0$. $J_n(x)$ is the normalized Jacobi's polynomial. (See [1] p. 42.)

Therefore the polynomials $Q_n(x)$ represent a generalization of the Jacobi's polynomials.

1,3. This paper is the first part of a treatise dealing with the above defined generalized Jacobi's polynomials $Q_n(x)$.

The main object of this paper is to establish the differential equation (4,2b) for a certain class of the polynomials $Q_n(x)$. This equation is a very useful tool for solving many problems connected with the polynomials in question.

J. KOROUS has derived a differential equation for a more general class of polynomials. His equation, however, is non-homogeneous. (See [2], [3], [4].)

For a class of the polynomials $Q_n(x)$ we derive the inequality (2,9b) which is an extension of the well-known inequality for Jacobi's polynomials.

We shall also establish a relation between $Q_n(x)$ and $Q'_n(x)$. (See Section 3,4.)

2. SOME PROPERTIES OF THE POLYNOMIALS $Q_n(x)$

2,1. Throughout this paper the following notation is used:

1. $n \geq 0$ is an integer.
2. $\{P\}$ is the degree of the polynomial $P(x)$.
 $\{P\} = -\infty$, if $P(x) \equiv 0$,
 $\{P\} = \pi_n$, if $\{P\} \leq n$.
3. I is the closed interval $[-1, 1]$.
4. c_i ($i = 1, 2, \dots$) are positive constants independent of $x \in I$ and of n .
 $c_i(x)$ ($i = 1, 2, \dots$) is a function of $x \in I$ and n such that $|c_i(x)| < c_i$.

The numbering of c_i a $c_i(x)$ is independent for every section.

2,2. If $P(x) = \sum_{k=0}^r a_k x^k$, $a_r \neq 0$, then by a well-known theorem

$$(2,2a) \quad P(x) = \sum_{k=0}^r p_k Q_k(x),$$

where

$$(2,2b) \quad p_k = \int_I P(x) Q_k(x) Q(x) dx.$$

(See [5] p. 73.)

Hence

$$(2,2c) \quad n > r \Rightarrow \int_I P(x) Q_n(x) Q(x) dx = 0$$

and

$$(2,2d) \quad \int_I P(x) Q_r(x) Q(x) dx = \frac{a_r}{a_0^{(r)}},$$

where $a_0^{(r)} > 0$ is defined by (1,1c).

2,3. Let

$$(2,3a) \quad q_0 = 0, \quad n > 0 \Rightarrow q_n = \frac{a_0^{(n-1)}}{a_0^{(n)}}.$$

By (2,2d) for $n > 0$

$$(2,3b) \quad q_n = \int_I x Q_n(x) Q_{n-1}(x) Q(x) dx$$

and

$$(2,3c) \quad nq_n^{-1} = \int_I Q_n'(x) Q_{n-1}(x) Q(x) dx.$$

Hence for $n = 1, 2, \dots$

$$(2,3d) \quad 0 < q_n < 1.$$

Proof. From (2,3b) we see at once

$$q_n < \int_I |Q_n(x) Q_{n-1}(x)| Q(x) dx \Rightarrow q_n^2 < \int_I Q_n^2(x) Q(x) dx \int_I Q_{n-1}^2(x) Q(x) dx = 1.$$

2,4. The equation

$$(2,4a) \quad q_{n+1} Q_{n+1}(x) + (j_n - x) Q_n(x) + q_n Q_{n-1}(x) = 0,$$

where

$$(2,4b) \quad j_n = \int_I x Q_n^2(x) Q(x) dx \Rightarrow |j_n| < 1$$

is the well-known recurrence formula for orthonormal polynomials. (See [5] p. 77.)

2,5. For $x \neq t$

$$(2,5a) \quad \begin{aligned} Q_n(x, t) &= \sum_{k=0}^n Q_k(x) Q_k(t) = \\ &= (x - t)^{-1} q_{n+1} [Q_{n+1}(x) Q_n(t) - Q_n(x) Q_{n+1}(t)]. \end{aligned}$$

Similarly for the polynomials $J_n(x)$ defined by (1,2a)

$$(2,5b) \quad J_n(x, t) = \sum_{k=0}^n J_k(x) J_k(t) = (x - t)^{-1} q_{n+1}^* [J_{n+1}(x) J_n(t) - J_n(x) J_{n+1}(t)],$$

holds where $x \neq t$ and

$$(2,5c) \quad q_{n+1}^* = \frac{b_0^{(n)}}{b_0^{(n+1)}}.$$

(2,5a) is the Christoffel's formula. (See [5] p. 79.)

Applying (2,2b) we can write the formula (2,2a) in the form

$$(2,5d) \quad P(x) = \int_I P(t) Q_r(x, t) Q(t) dt$$

or

$$(2,5e) \quad P(x) = \int_I P(t) J_r(x, t) J(t) dt.$$

2.6. We introduce the following sets of functions:

Let $f_x(t)$ be a real function of t which depends on the parameter $x \in I$ and is defined for all $t \in [-1, 1]$ with the possible exception $t = x$. The functions $f_x(t)$ exist for every value of $x \in I$. Put

$$(2,6a) \quad \gamma = \min(\alpha, \beta).$$

\mathfrak{F}_γ denotes the set of the functions $f_x(t)$ such that for $\gamma \geq -\frac{1}{2}$

$$(2,6b) \quad f_x(t) \in \mathfrak{F}_\gamma \Leftrightarrow \int_I (1 - t^2)^{-1/2} |f_x(t)| dt = c_1(x).$$

Here

$$(2,6c) \quad \int_I (1 - t^2)^{-1/2} |f_x(t)| dt = \lim_{y \rightarrow x^-} \int_{-1}^y (1 - t^2)^{-1/2} |f_x(t)| dt + \\ + \lim_{y \rightarrow x^+} \int_y^1 (1 - t^2)^{-1/2} |f_x(t)| dt.$$

The integrals in (2,6c) are those of Lebesgue.

If $\gamma < -\frac{1}{2}$, then $f_x(t) \in \mathfrak{F}_\gamma$ if and only if there exists a constant $c > 0$ independent of $x \in I$ and $t \in I$ such that

$$(2,6d) \quad |f_x(t)| < c.$$

The inequality (2,6d) implies

$$(2,6c) \quad \overline{\lim}_{t \rightarrow x} |f_x(t)| \leq c$$

for every $x \in I$.

Remark. It is easily seen that $\varphi(t) \in \mathfrak{F}_\gamma$ for $\gamma \geq -\frac{1}{2}$, if $\int_I (1-t^2)^{-1/2} |\varphi(t)| dt < +\infty$ for we may write $f_x(t) = \varphi(t)$ for every $x \in I$.

Similarly $\varphi(t) \in \mathfrak{F}_\gamma$ for $\gamma < -\frac{1}{2}$, if $\varphi(t)$ is bounded on I .

Clearly, if $\gamma_1 < -\frac{1}{2}$ and $\gamma_2 \geq -\frac{1}{2}$, then $\mathfrak{F}_{\gamma_1} \subset \mathfrak{F}_{\gamma_2}$.

2,7. Let $\varphi(t)$ be a real function defined on I . Then we shall use the following notation

$$(2,7a) \quad \Delta_x \varphi(t) = (x-t)^{-1} [\varphi(x) - \varphi(t)].$$

It is easily seen that $\Delta_x \varphi(t) \in \mathfrak{F}_\gamma \Rightarrow \varphi(t) \in \mathfrak{F}_\gamma$.

2,8. In the notation of Sections 1,2 and 2,1,

$$(2,8a) \quad \gamma = \min(\alpha, \beta) \geq -\frac{1}{2} \Rightarrow \sqrt[4]{((1-x^2) J^2(x))} J_n(x) = c_1(x).$$

Proof. See e.g. [2] p. 9. In this paper (2,8a) is proved for $\gamma > \frac{1}{2}$ but a slight modification of the proof establishes (2,8a) also for $\gamma = -\frac{1}{2}$.

2,9. Let in the notation of Sections 2,7 and 2,6

$$(2,9a) \quad \gamma \geq -\frac{1}{2} \quad \text{and} \quad \Delta_x u(t) \in \mathfrak{F}_\gamma.$$

Then

$$(2,9b) \quad \sqrt[4]{((1-x^2) Q^2(x))} Q_n(x) = c_1(x).$$

Proof. We shall use a method of J. Korous. (See [2] p. 9.) Applying (2,5e) we deduce that

$$(1) \quad Q_n(x) = a_n J_n(x) + R_n(x).$$

Here

$$a_n = \int_I Q_n(t) J_n(t) J(t) dt.$$

Hence

$$(2) \quad a_n^2 \leq \int_I e^{-u(t)} Q_n^2(t) Q(t) dt \int_I J_n^2(t) J(t) dt < c_1.$$

Since

$$J_{n-1}(x, t) = \pi_{n-1}$$

with respect to t ,

(3)

$$R_n(x) = \int_I Q_n(t) J_{n-1}(x, t) J(t) dt = \int_I Q_n(t) J_{n-1}(x, t) [J(t) - e^{-u(x)} Q(t)] dt.$$

Applying (2,5b) we obtain for the integrated function $L_n(t, x)$ in the second integral in (3)

$$(4) \quad |L_n(t, x)| < q_n^* |Q_n(t)| |x - t|^{-1} |1 - \exp[u(t) - u(x)]| \cdot \{|J_n(x) J_{n-1}(t)| + |J_{n-1}(x) J_n(t)|\} J(t).$$

It is easily seen that

$$(5) \quad |1 - \exp[u(t) - u(x)]| < c_2 |x - t| \Delta_x u(t).$$

Let

$$(6) \quad s = \sup_{x \in I} \sqrt[4]{((1 - x^2) J^2(x)) |Q_n(x)|}$$

and $x_0 \in I$ a point in which the above function assumes the value s .

Further let $\delta > 0$ and

$$(7) \quad I_0 = (x_0 - \delta, x_0 + \delta) \cap I.$$

Since $u(x)$ is bounded in I (see Section 1,1) $\Delta_{x_0} u(t)$ is bounded on the interval $I - I_0$.

Making use of (4), (5), (2,8a) and (2,3d) we deduce that

$$(8) \quad \sqrt[4]{((1 - x_0^2) J^2(x_0))} \int_{I - I_0} |L_n(t, x_0)| dt < c_3 \int_I |Q_n(t)| \{|J_{n-1}(t)| + |J_n(t)|\} J(t) dt < c_4 \left[\int_I Q_n^2(t) e^{-u(t)} Q(t) dt \right]^{1/2} \left\{ \int_I [J_{n-1}^2(t) + J_n^2(t)] J(t) dt \right\}^{1/2} < c_5.$$

Further, (3), (4), (5), (2,8a), (2,9a) and (2,6b) yield

$$(9) \quad \sqrt[4]{((1 - x_0^2) J^2(x_0))} \int_{I_0} |L_n(t, x_0)| dt < c_6 s \int_{I_0} (1 - t^2)^{-1/2} |\Delta_x u(t)| dt < \frac{s}{2}$$

if we choose δ in (7) sufficiently small.

It follows from (8) and (9) that

$$(10) \quad \sqrt[4]{((1 - x_0^2) J^2(x_0))} |R_n(x_0)| < c_7 + \frac{s}{2}.$$

(1), (2), (10) and (2,8a) yield

$$s = \sqrt[4]{((1 - x_0^2) J^2(x_0)) |Q_n(x_0)|} \leq \{|a_n| |J_n(x_0)|\} + \\ + |R_n(x_0)| \sqrt[4]{((1 - x_0^2) J^2(x_0))} < c_8 + \frac{s}{2}.$$

Hence

$$(11) \quad s < c_9.$$

As for $x \in I$

$$Q(x) = e^{u(x)} J(x) \leq c_{10} J(x),$$

(2,9b) follows from (11).

3. LEMMAS

3,1. We shall use the following notation:

If $\varphi(t)$ is integrable on I , then for $m = 0, 1, \dots, n = 0, 1, \dots$

$$(3,1a) \quad I_{m,n}[\varphi(t)] = \int_I \varphi(t) Q_m(t) Q_n(t) Q(t) dt.$$

Remark. t on the left-hand side of (3,1a) indicates that the integration variable is t .

3,2. Let in the notation of Section 2,6

$$(3,2a) \quad f_x(t) \in \mathfrak{F}_\gamma.$$

Then for $m \leq n$

$$(3,2b) \quad I_{m,n}[f_x(t)] = c_1(x).$$

Remark 1. The integral (3,2b) is meant in the sense of (2,6e).

Remark 2. If $\varphi(t)$ does not depend on x we put

$$f_x(t) = \varphi(t),$$

so that

$$(3,2c) \quad |I_{m,n}[\varphi(t)]| < c_1$$

provided that $\varphi(t) \in \mathfrak{F}_\gamma$.

Proof. 1. If $\gamma \geq -\frac{1}{2}$, we may apply (2,9b). It is for $t \in (-1, 1)$

$$|Q_m(t) Q_n(t) Q(t)| < c_2(1 - t^2)^{-1/2}$$

so that

$$|I_{m,n}[f_x(t)]| < c_3 \int_I (1-t^2)^{-1/2} |f_x(t)| dt < c_4$$

in virtue of (2,6b).

2. If $\gamma < -\frac{1}{2}$, we have by (2,6d) and (2,6e)

$$\begin{aligned} |I_{m,n}[f_x(t)]| &< c_5 \int_I |Q_m(t) Q_n(t)| Q(t) dt < \\ &< c_6 \left[\int_I Q_m^2(t) Q(t) dt \int_I Q_n^2(t) Q(t) dt \right]^{1/2} = c_6. \end{aligned}$$

3,3. Let $\psi(t)$ be integrable in I . We put

$$(3,3a) \quad A(x, \psi) = q_n^{-1} \int_I \psi(t) Q_n(t) Q_{n-1}(x, t) Q(t) dt,$$

where $Q_{n-1}(x, t)$ is defined by (2,5a).

Further put

$$(3,3b) \quad \begin{aligned} \lambda_1(t) &= 1, \quad \lambda_2(t) = t, \quad \lambda_3(t) = 1 - t^2, \\ \psi_i(t) &= \lambda_i(t) \varphi(t) \quad (i = 1, 2, 3). \end{aligned}$$

Let

$$(3,3c) \quad \Delta_x \varphi(t) \in \mathfrak{F}_\gamma.$$

Then for $i = 1, 2, 3$

$$(3,3d) \quad \begin{aligned} A(x, \psi_i) &= \{\alpha_i(x) + I_{n,n}[\lambda_i(t) \Delta_x \varphi(t)]\} Q_{n-1}(x) + \\ &+ \{\beta_i(x) - I_{n,n-1}[\lambda_i(t) \Delta_x \varphi(t)]\} Q_n(x). \end{aligned}$$

Here

$$(3,3e) \quad \begin{aligned} \alpha_1(x) &= \beta_1(x) = \beta_2(x) = 0, \\ \alpha_2(x) &= \varphi(x), \quad \alpha_3(x) = -(x + j_n) \varphi(x), \quad \beta_3(x) = q_n \varphi(x), \end{aligned}$$

where j_n is defined by (2,4b).

Further

$$(3,3f) \quad A(x, \psi_i) = c_1(x) Q_{n-1}(x) + c_2(x) Q_n(x).$$

Proof. 1. The existence of the integrals on the right-hand side of (3,3d) is made evident by (3,2b). Further, (3,3d) and (3,2b) verify (3,3f) provided that (3,3e) is true.

2. Since

$Q_{n-1}(x, t) = \pi_{n-1}$ with respect to the variable t we have for

$$\psi(t) = 1, \quad \psi^*(t) = \varphi(t) - \varphi(x), \quad \psi^{**}(t) = x - t$$

the equations

$$(1) \quad A(x, \psi) = 0, \quad A(x, \varphi) = A(x, \psi^*)$$

and in virtue of (2,5a),

$$(2) \quad A(x, \psi^{**}) = -Q_{n-1}(x).$$

3. As a consequence of (2,5a), (1) yields

$$(3) \quad A(\psi_1) = A(x, \varphi) = I_{n,n}[\Delta_x \varphi(t)] Q_{n-1}(x) - I_{n,n-1}[\Delta_x \varphi(t)] Q_n(x).$$

4. Since

$$\psi_2(t) = \psi_2(x) + (t - x) \varphi(x) + t[\varphi(t) - \varphi(x)]$$

we deduce from (1) and (2) that (3,3e) holds also for $i = 2$.

5. It can be easily seen that

$$(4) \quad \psi_3(t) = \psi_3(x) + (1 - t^2) [\varphi(t) - \varphi(x)] + (x^2 - t^2) \varphi(x).$$

Hence, if we put $\varphi_1(t) = t$, $\varphi_2(t) = x + t$ then owing to (1),

$$A(x, \psi_3) = I_{n,n}[\lambda_3(t) \Delta_x \varphi(t)] Q_{n-1}(x) - I_{n,n-1}[\lambda_3(t) \Delta_x \varphi(t)] Q_n(x) + [I_{n,n-1}[\varphi_1(t)] Q_n(x) - I_{n,n}[\varphi_2(t)] Q_{n-1}(x)] \varphi(x).$$

From this equation (3,3e) follows for $i = 3$ by applying (2,3b) and (2,4b).

3,4. Let

$$(3,4a) \quad \Delta_x u'(t) \in \mathfrak{F}_\gamma \quad \text{and} \quad \frac{\partial}{\partial x} [\Delta_x u'(t)] \in \mathfrak{F}_\gamma.$$

For $v = 0, 1, \dots, n$

$$(3,4b) \quad \gamma_v = -I_{n,v}[\lambda_3(t) u'(t)],$$

where $\lambda_3(t)$ is defined by (3,3b).

$$(3,4c) \quad [1 + e_n(x)]^{-1} = 1 + (2n)^{-1} \{ \alpha + \beta + 1 + (j_n + x) u'(x) - I_{n,n}[\lambda_3(t) \Delta_x u'(t)] \},$$

where j_n is defined by (2,4b).

$$(3,4d) \quad d_n(x) = \frac{1}{2} [1 + e_n(x)] \{ x - j_n - (2n)^{-1} [(\alpha + \beta + 2) j_n + \alpha - \beta + \gamma_n - 2q_n^2 u'(x) + 2q_n I_{n,n-1}[\lambda_3(t) \Delta_x u'(t)] \}.$$

Then

$$(3,4e) \quad q_n Q_{n-1}(x) = (2n)^{-1} [1 + e_n(x)] (1 - x^2) Q'_n(x) + d_n(x) Q_n(x),$$

$$(3,4f) \quad e_n(x) = n^{-1} c_1(x),$$

$$(3,4g) \quad d_n(x) = c_2(x),$$

$$(3,4h) \quad e'_n(x) = n^{-1} c_3(x).$$

Proof. 1. The existence of integrals on the right-hand sides of (3,4b), (3,4c) and (3,4d) as well as the existence of

$$\frac{d}{dx} I_{n,n}[\lambda_3(t) \Delta_x u'(t)]$$

is a consequence of (3,4a) and (3,2b). Since by (3,4a) $u''(x) \in \mathfrak{F}_v$, $e'_n(x)$ exists in the interval I.

(3,4f), (3,4g) and (3,4h) follow then from (3,2b).

2. It is easily seen that

$$U_n(x) = (1 - x^2) Q'_n(x) + nx Q_n(x) = \pi_n.$$

Hence by (2,2a)

$$(1) \quad U_n(x) = \sum_{v=0}^n \alpha_v Q_v(x),$$

where by (2,2b)

$$(2) \quad \alpha_v = \int_I [(1 - t^2) Q'_n(t) + nt Q_n(t)] Q_v(t) Q(t) dt.$$

Integrating by parts we obtain

$$(3) \quad \alpha_v = - \int_I (1 - t^2) Q_n(t) Q'_v(t) Q(t) dt + I_{nv}[\psi_4(t)],$$

where

$$\psi_4(t) = (\alpha - \beta) + (\alpha + \beta + 2 + n)t - (1 - t^2)u'(t).$$

3. (3) enables us to establish the following results:

$$(4) \quad v < n - 1 \Rightarrow \alpha_v = \gamma_v,$$

where γ_v is defined by (3,4b).

Adding (2) to (3) for $v = n$ we obtain

$$(5) \quad \alpha_n = \frac{1}{2}[(\alpha + \beta + 2n + 2)j_n + \alpha - \beta + \gamma_n].$$

Since

$$x^2 Q'_{n-1}(x) = (n-1)x Q_{n-1}(x) + \pi_{n-1},$$

(3) yields

$$(6) \quad \alpha_{n-1} = (2n + \alpha + \beta + 1) q_n + \gamma_{n-1}.$$

4. By means of (2,5d) we obtain

$$(7) \quad (1-x^2) Q'_n(x) = \alpha_{n-1} Q_{n-1}(x) + (\alpha_n - nx) Q_n(x) - q_n A[x, \lambda_3 u'].$$

(3,4e) is a consequence of (7), (5), (6) and (3,3d).

3.5. Provided that (3,4a) holds,

$$(3,5a) \quad \left| \int_I (1-x^2)^2 Q_n'^2(x) Q(x) dx \right| < c_1 n^2.$$

Proof. (3,5a) is a consequence of (3,4e).

3.6. The following equation holds:

$$(3,6a) \quad K_n = q_n \int_I (x+t) Q_n'(t) Q_{n-1}(t) Q(t) dt = nx + s_1^{(n)}.$$

Here $s_1^{(n)}$ is the sum of the zeros of the polynomial $Q_n(x)$. Since all these zeros are contained in the interval $(-1, 1)$, it is

$$(3,6b) \quad |s_1^{(n)}| < n.$$

Proof. 1.

$$(1) \quad \begin{aligned} x^2 Q'_n(x) &= nx[a_0^{(n)}x^n + a_1^{(n)}x^{n-1}] - \\ &- a_1^{(n)}x^n + \pi_{n-1} = nx[Q_n(x) + \pi_{n-2}] + \\ &+ s_1^{(n)}[a_0^{(n)}x^n + \pi_{n-1}] = nx Q_n(x) + \pi_{n-1} + \\ &+ s_1^{(n)}[Q_n(x) + \pi_{n-1}] = [nx + s_1^{(n)}] Q_n(x) + \pi_{n-1}. \end{aligned}$$

2. Making use of (2,4a) and (1) we deduce that

$$\begin{aligned} K_n &= \int_I (x+t) Q'_n(t) [(t-j_n) Q_n(t) - q_{n+1} Q_{n+1}(t)] Q(t) dt = \\ &= nx + \int_I \{[nt + s_1^{(n)}] Q_n(t) - j_n t Q'_n(t)\} Q_n(t) Q(t) dt = \\ &= n(x + j_n) + s_1^{(n)} - nj_n = nx + s_1^{(n)}. \end{aligned}$$

3.7. Let

$$(3,7a) \quad \Delta_x u'(t) \in \mathfrak{F}_\gamma.$$

Then

$$(3,7b) \quad L_n = q_n \int_I (1 - t^2) \Delta_x u'(t) Q_n'(t) Q_{n-1}(t) Q(t) dt = n c_1(x).$$

Proof. 1. Let $\gamma \geq -\frac{1}{2}$. Then in virtue of (3,4e), (3,4f) and (3,4g),

$$|(1 - x^2) Q_n'(x)| < c_1 n [|Q_{n-1}(x)| + |Q_n(x)|].$$

Hence by (2,9b)

$$(1) \quad (1 - x^2) |Q_n'(x)| \sqrt{[Q(x)]} < c_2 n (1 - x^2)^{-1/4}.$$

From (3,2b) and (1) it follows that

$$|L_n| < c_3 n \int_I (1 - t^2)^{-1/2} |\Delta_x u'(t)| dt < c_4 n.$$

2. Let $\gamma < -\frac{1}{2}$. Then in virtue of (3,5a) and of (2,6d),

$$|L_n| < c_5 \left[\int_I (1 - t^2)^2 Q_n'^2(t) Q(t) dt \int_I Q_{n-1}^2(t) Q(t) dt \right]^{1/2} < c_6 n.$$

3.8. Let

$$(3,8a) \quad \Delta_x u'(t) \in \mathfrak{F}_\gamma \quad \text{and} \quad (1 - t^2) \frac{\partial}{\partial t} \Delta_x u'(t) \in \mathfrak{F}_\gamma.$$

For the sake of brevity, put

$$(3,8b) \quad \begin{aligned} \varphi_1(t) &= Q^{-1}(t) \frac{d}{dt} [(1 - t^2) \Delta_x u'(t) Q(t)] = \\ &= (1 - t^2) \frac{\partial}{\partial t} \Delta_x u'(t) - [(\alpha + \beta + 2)t + \alpha - \beta + (1 - t^2) u'(t)] \Delta_x u'(t). \end{aligned}$$

Then in the notation (3,7b)

$$(3,8c) \quad \begin{aligned} B(x) &= \int_I (1 - t^2) u'(t) Q_n'(t) Q_{n-1}(x, t) Q(t) dt = (1 - x^2) u'(x) Q_n'(x) - \\ &\quad - \{n u'(x) + \frac{1}{2} J_{n,n}[\varphi_1(t)]\} q_n Q_{n-1}(x) + \{[nx + s_1^{(n)}] u'(x) - L_n\} Q_n(x). \end{aligned}$$

Here $s_1^{(n)}$ is defined in Section 3,6.

Proof. 1. As a consequence of (3,8a) there exists $u''(x)$ in the interval I . Therefore $u'(x)$ is continuous on I and consequently

$$(1) \quad u'(t) \Delta_x u'(t) \in \mathfrak{F}_\gamma.$$

By (1) combined with (3,2b) the existence of $I_{n,n}[\varphi_1(t)]$ is made evident.

2. Clearly

$$(2) \quad (1 - t^2) u'(t) = (1 - x^2) u'(x) + (x^2 - t^2) u'(x) + (1 - t^2) [u'(t) - u'(x)].$$

Making use of (2,5a), (3,6a) and (3,7b) we may write

$$B(x) = (1 - x^2) u'(x) Q_n'(x) + [K_n u'(x) - L_n] Q_n(x) - q_n Q_{n-1}(x) \int_I [t u'(x) - (1 - t^2) \Delta_x u'(t)] Q_n(t) Q_n'(t) Q(t) dt.$$

Integrating by parts, we deduce that the last integral is equal to

$$(3) \quad nx u'(x) + \frac{1}{2} I_{n,n}[\varphi_1(t)].$$

(3) and (3,6a) complete the proof.

4. THE DIFFERENTIAL EQUATION OF THE POLYNOMIAL $Q_n(x)$

4.1. 1. If $\Delta_x u'(t) \in \mathfrak{F}_\gamma$ and $u''(x)$ exists in the interval $[-1, 1]$, then

$$(4,1a) \quad \Delta_x u'^2(t) \in \mathfrak{F}_\gamma.$$

2. If $(\partial/\partial x) [\Delta_x u'(t)] \in \mathfrak{F}_\gamma$ and $u'''(x)$ exists in the interval $[-1, 1]$, then

$$(4,1b) \quad \frac{\partial}{\partial x} [\Delta_x u'^2(t)] \in \mathfrak{F}_\gamma.$$

Proof. 1. Clearly

$$\Delta_x u'^2(t) = [u'(t) + u'(x)] \Delta_x u'(t) \in \mathfrak{F}_\gamma$$

as $u'(t)$ is continuous on I in virtue of the existence of $u''(x)$.

2. It is easily seen that

$$(1) \quad \frac{\partial}{\partial x} [\Delta_x u'^2(t)] = u''(x) \Delta_x u'(t) + [u'(t) + u'(x)] \frac{\partial}{\partial x} [\Delta_x u'(t)].$$

There exists ξ between x and t such that

$$\Delta_x u'(t) = u''(\xi).$$

In virtue of the existence of $u'''(x)$, the function $u''(x)$ is continuous on I and consequently

$$(2) \quad |\Delta_x u'(t)| < c_1 \Rightarrow \Delta_x u'(t) \in \mathfrak{F}_\gamma.$$

From (1) and (2) it follows that (4,1b) is true.

4.2. Let

$$(4,2a) \quad (1 - t^2) \Delta_x u''(t), (1 - t^2) \frac{\partial}{\partial t} \Delta_x u'(t) \quad \text{and} \quad \Delta_x u'(t)$$

be elements of \mathfrak{F}_γ .

Then

$$(4,2b) \quad Q^{-1}(x) \frac{d}{dx} [(1 - x^2) Q'_n(x) Q(x)] + (1 - x^2) b_n(x) Q'_n(x) + [\lambda_n^2 + a_n(x)] Q_n(x) = 0.$$

Here

$$(4,2c) \quad \lambda_n = \sqrt{[n(n + \alpha + \beta + 1)]},$$

$$(4,2d) \quad a_n(x) = n c_1(x),$$

$$(4,2e) \quad b_n(x) = n^{-1} c_2(x).$$

If (4,2a) is true and, moreover, the functions

$$(4,2f) \quad (1 - t^2) \frac{\partial}{\partial x} [\Delta_x u''(t)], (1 - t^2) \frac{\partial^2}{\partial x \partial t} [\Delta_x u'(t)], \frac{\partial}{\partial x} \Delta_x u'(t)$$

are elements of \mathfrak{F}_γ , then $b'_n(x)$ exists in the interval $[-1, 1]$ and

$$(4,2g) \quad b'_n(x) = n^{-1} c_3(x).$$

Proof. 1. It is easily seen that

$$D_n(x) = Q^{-1}(x) \frac{d}{dx} [(1 - x^2) Q'_n(x) Q(x)] - (1 - x^2) u'(x) Q'_n(x) + \lambda_n^2 Q_n(x) = \pi_{n-1}.$$

By (2,2a)

$$(1) \quad D_n(x) = \sum_{v=0}^{n-1} \beta_v Q_v(x).$$

Making use of (2,2b) and integrating by parts we obtain

$$\begin{aligned} \beta_v + \int_I (1-x^2) u'(x) Q_n'(x) Q_v(x) Q(x) dx &= \\ &= - \int_I (1-x^2) Q_n'(x) Q_v'(x) Q(x) dx = \int_I Q_n(x) d[(1-x^2) Q_v'(x) Q(x)] = \\ &= \int_I (1-x^2) u'(x) Q_n(x) Q_v'(x) Q(x) dx. \end{aligned}$$

Hence

$$\begin{aligned} (2) \quad \beta_v &= \int_I (1-x^2) u'(x) Q_n^2(x) Q(x) d \left[\frac{Q_v(x)}{Q_n(x)} \right] = \\ &= -2 \int_I (1-x^2) u'(x) Q_n'(x) Q_v(x) Q(x) dx - \int_I Q_n(x) Q_v(x) d[(1-x^2) u'(x) Q(x)]. \end{aligned}$$

For the sake of brevity, put

$$\begin{aligned} \psi_0(t) &= Q^{-1}(t) \frac{d}{dt} [(1-t^2) u'(t) Q(t)] = \\ &= (1-t^2) [u''(t) + u'^2(t)] - [(\alpha + \beta + 2)t + \alpha - \beta] u'(t). \end{aligned}$$

From (1), (2), (2,5d), (3,3a) and (3,8c) it follows that

$$(3) \quad D_n(x) = -q_n A(x, \psi_0) - 2B(x).$$

2. For the sake of brevity, put

$$(4) \quad \varphi_2(t) = (1-t^2) \Delta_x [u''(t) + u'^2(t)] - [(\alpha + \beta + 2)t + (\alpha - \beta)] \Delta_x u'(t)$$

and let $\varphi_1(t)$ be defined by (3,8b).

Making use of (3,3d) and (3,8c) we obtain from (3)

$$(5) \quad \begin{aligned} D_n(x) &= -2(1-x^2) u'(x) Q_n'(x) + 2n u'(x) q_n Q_{n-1}(x) + \\ &+ q_n \varrho_1(x) Q_{n-1}(x) + \varrho_2(x) Q_n(x). \end{aligned}$$

Here

$$(6) \quad \varrho_1(x) = (x + j_n) [u''(x) + u'^2(x)] + (\alpha + \beta + 2) u'(x) + I_{n,n} [\varphi_1(t) - \varphi_2(t)]$$

and

$$(7) \quad \varrho_2(x) \approx -q_n^2 [u''(x) + u'^2(x)] - 2[nx + s_1^{(n)}] u'(x) + q_n I_{n,n-1} [\varphi_2(t)] + 2L_n,$$

where $s_1^{(n)}$ is defined in Section 3,6 and L_n by (3,7a).

The existence of the integrals in (6) and (7) is guaranteed by (4,2a) combined with (4,1a).

3. Replacing $q_n Q_{n-1}(x)$ by the right-hand side of (3,4e) we may write (5) in the form

$$(8) \quad D_n(x) = -(1-x^2) [u'(x) + b_n(x)] Q_n'(x) - a_n(x) Q_n(x).$$

Here

$$(9) \quad b_n(x) = -\frac{1}{2n} [1 + e_n(x)] q_1(x) - e_n(x) u'(x)$$

and

$$(10) \quad a_n(x) = -q_2(x) - [2n u'(x) + q_1(x)] d_n(x).$$

4. (4,2d) and (4,2e) may be derived from (6), (7), (8), (9) and (10) by employing (3,4f) and (3,4g).

If (4,2f) is true, then (4,1b) holds and $b_n'(x)$ exists. (4,2g) is then deduced similarly as (4,2e) if we take (3,4h) into consideration.

4,3. Sufficient conditions for (4,2a) and (4,2f).

I. Let $\gamma \geq -\frac{1}{2}$.

1. (4,2a) holds, if there exists $\varepsilon > 0$ such that

$$(4,3a) \quad x \in I, \quad t \in I \Rightarrow |u''(x) - u''(t)| < c_1 |x - t|^\varepsilon.$$

2. (4,2f) holds, if

$$(4,3b) \quad x \in I, \quad t \in I \Rightarrow |u'''(x) - u'''(t)| < c_2 |x - t|^\varepsilon.$$

II. If $\gamma < -\frac{1}{2}$, the assertion is true if we put in (4,3a) and (4,3b) $\varepsilon = 1$.

Proof. I. 1. Let (4,3a) be satisfied. Then $u''(x)$ is continuous on the interval I and consequently

$$(1) \quad |\Delta_x u'(t)| \leq \sup_{x \in I} |u''(x)| \Rightarrow \Delta_x u'(t) \in \mathfrak{F}_\gamma.$$

Further

$$(2) \quad |\Delta_x u''(t)| < c_3 |x - t|^{-1-\varepsilon} \Rightarrow \int_{-1}^1 (1-t^2)^{3/2} \{|\Delta_x u''(t)|\} dt < \\ < \frac{1}{\varepsilon} c_4 [(x+1)^\varepsilon + (1-x)^\varepsilon] < c_5.$$

There exists τ between the numbers x and t such that

$$\frac{\partial}{\partial t} \Delta_x u'(t) = \frac{(t-x)u''(t) - [u'(t) - u'(x)]}{(t-x)^2} = (t-x)^{-1} [u''(t) - u''(\tau)].$$

By (4,3a)

$$\left| \frac{\partial}{\partial t} \Delta_x u'(t) \right| < c_6 |t-x|^{-1} |t-\tau|^\varepsilon < c_7 |t-x|^{-1+\varepsilon}$$

Hence

$$(3) \quad \int_I (1-t^2)^{3/2} \left| \frac{\partial}{\partial t} \Delta_x u'(t) \right| dt < c_8.$$

(1), (2) and (3) show that $\Delta_x u'(t)$, $(1-t^2) \Delta_x u''(t)$ and $(1-t^2) (\partial/\partial t) \Delta_x u'(t)$ are elements of \mathfrak{F}_y

2. Let (4,3b) be satisfied. Then there exists ξ_i ($i = 1, 2, \dots$) between x and t such that

$$\frac{\partial}{\partial x} [\Delta_x u'(t)] = \frac{(x-t)u''(x) - [u'(x) - u'(t)]}{(x-t)^2} = \frac{1}{2} u'''(\xi_1).$$

Hence

$$(4) \quad \left| \frac{\partial}{\partial x} \Delta_x u'(t) \right| \leq \frac{1}{2} \sup_{x \in I} |u'''(x)|.$$

Further

$$(5) \quad \left| \frac{\partial}{\partial x} [\Delta_x u''(t)] \right| = \left| \frac{u'''(x) - u'''(\xi_2)}{x-t} \right| < < c_9 |x-t|^{-1} |x-\xi_2|^\varepsilon < c_{10} |x-t|^{-1+\varepsilon}$$

and

$$(6) \quad \left| \frac{\partial^2}{\partial x \partial t} [\Delta_x u'(t)] \right| = \left| \frac{u'''(\xi_4) - u'''(\xi_3)}{x-t} \right| < < c_{11} |x-t|^{-1} |\xi_4 - \xi_3|^\varepsilon < c_{12} |x-t|^{-1+\varepsilon}.$$

From (4), (5) and (6) we may derive that

$$\frac{\partial}{\partial x} \Delta_x u'(t), \quad (1-t^2) \frac{\partial}{\partial x} \Delta_x u''(t) \quad \text{and} \quad (1-t^2) \frac{\partial^2}{\partial x \partial t} \Delta_x u'(t)$$

are elements of \mathfrak{F}_y .

II. The proof for $\gamma < -\frac{1}{2}$ may be carried out by a similar argument.

Remark. It is obvious that the above condition may be replaced by a more general one. For instance $|x - t|^s$ in (4,3a) and (4,3b) may be replaced by $|\lg^{-1-\epsilon}|x - t||$.

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