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ON AN APPLICATION OF THE STONE THEOREM IN THE THEORY OF DIFFERENTIAL EQUATIONS

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In the theory of differential equations one often solves a given problem by approximating it by a similar one with additional properties which allow to solve it and then by making a limit passage. This method sometimes requires the approximation of a continuous function by means of smoother functions, usually by polynomials. However, when some additional properties of the approximating functions are required, the Weierstrass theorem is of no use. Still much can be done by using the more poverful Stone theorem. This theorem is especially useful when the uniform approximation of a continuous function by means of Hölder continuous functions is required. This case is of importance in the theory of differential equations. Of course, the same method can be applied to other classes of continuous functions such as to the class of continuous functions of bounded variation etc.

Let (R, ϱ) be a metric space, $\emptyset \neq M \subset R$ be a set and let $d(M)$ be the diameter of M. A set $A \neq \emptyset$ of real continuous functions on M (in what follows only real continuous functions are considered) will be called *a lattice of continuous functions on M* if with each pair $f, g \in A$ also $\max(f, g) \in A$ and $\min(f, g) \in A$. $H_\alpha(M)$ will mean the set of all Hölder continuous functions on M of exponent α, $0 < \alpha \leqq 1$. Thus $f \in H_\alpha(M)$ iff f is defined on M and there exists a constant $L = L(f) > 0$ such that for any two points $x, y \in M$ the inequality $|f(y) - f(x)| \leqq L[\varrho(x, y)]^\alpha$ is true. The following remarks will be of use.

a) If $d = d(M) < +\infty$, in virtue of the inequality $|h(y)\, g(y) - h(x)\, g(x)| \leqq \leqq |h(y) - h(x)| \,.\, |g(y)| + |g(y) - g(x)| \,.\, |h(x)|$, $x, y \in M$ being arbitrary points, and by the boundedness of $h, g \in H_\alpha(M)$ it follows that $H_\alpha(M)$ forms an algebra of continuous functions on M.

b) Since for each $f \in H_\alpha(M)$ also $|f| \in H_\alpha(M)$, and on basis of the equalities $\max(a, b) = \frac{1}{2}(a + b + |a - b|)$, $\min(a, b) = \frac{1}{2}(a + b - |a - b|)$ which are true for arbitrary real numbers a, b, we get that $H_\alpha(M)$ is a lattice of continuous functions on M.

c) If $d = d(M) < +\infty$, and β satisfies the inequalities $0 < \alpha \leqq \beta \leqq 1$, then from the relation $[\varrho(x, y)]^\beta \leqq [\varrho(x, y)]^\alpha$ and $[\varrho(x, y)]^\beta \leqq d^{\beta-\alpha}[\varrho(x, y)]^\alpha$, respectively, which is valid for $\varrho(x, y) \leqq 1$ and $1 < \varrho(x, y) \leqq d$ (if $d > 1$), respectively, it follows that $H_\beta(M) \subset H_\alpha(M)$.

d) If $f \in H_\alpha(M)$ for $0 < \alpha \leqq 1$, then there exists an extension f_0 of the function f on R with the property $f_0 \in H_\alpha(R)$ ([1], p. 117–118). The proof of that statement is given in the euclidean space, but applies to an arbitrary metric space.

The last remark deals with locally compact separable metric spaces. If (R, ϱ) is such a space, then there exists an increasing sequence $\{M_n\}_{n=1}^{\infty}$ of compact sets in R such that $R = \bigcup_{n=1}^{\infty} M_n$. This follows from the Theorem 3.18.3, [2], p. 62.

Before going to give the application of the Stone theorem, we shall state this theorem in an appropriate formulation. (For the proof, see [3], p. 150–152.)

Stone's theorem. *Let M be a compact set, $f \in C_0(M)$ and let A be a lattice of continuous functions on M with the following property:*

(a) *For every pair $x, y, x \neq y$, of points of M, there exists a function $g \in A$ such that $g(x) = f(x), g(y) = f(y)$.*

Then there exists a sequence $\{f_n\}$ of functions $f_n \in A$ which uniformly converges to f on M.

Recall that an algebra A of functions on M which separates points on M and vanishes at no point of M has the property (a) ([3], p. 149).

Theorem 1. *Let M be a compact set, α be a real number satisfying the inequalities $0 < \alpha \leqq 1$ and $f \in C_0(M)$. Let further the functions $F_1, F_2 \in H_\alpha(M)$ and such that $F_1(x) \leqq f(x) \leqq F_2(x)$ for each $x \in M$. Then there exists a sequence $f_n \in H_\alpha(M)$, $n = 1, 2, 3, \ldots$ which is uniformly convergent to f on M whereby*

$$F_1(x) \leqq f_n(x) \leqq F_2(x) \tag{1}$$

for each $x \in M$ and all natural n.

Proof. By the remarks a) and b), $H_\alpha(M)$ forms an algebra as well as a lattice of continuous functions on M. Since for $x_1 \in M$ the function $\varrho(x, x_1) \in H_1(M) \subset H_\alpha(M)$ (with regard to the remark c) is such that $\varrho(x_1, x_1) = 0 < \varrho(x_2, x_1)$ for $x_2 \in M$, $x_2 \neq x_1$, and $H_\alpha(M)$ contains a constant function different from 0, the lattice $H_\alpha(M)$ has the property (a). Hence, by the Stone theorem, there exists a sequence $g_n \in H_\alpha(M)$, $n = 1, 2, 3, \ldots$ which is uniformly convergent to f on M.

Let us consider now the functions f_n, $n = 1, 2, 3, \ldots$, defined on M in this way:

$$f_n(x) = \begin{cases} g_n(x) & \text{if} \quad F_1(x) \leqq g_n(x) \leqq F_2(x) \\ F_2(x) & \text{if} \quad F_2(x) < g_n(x) \\ F_1(x) & \text{if} \quad g_n(x) < F_1(x) \end{cases}$$

The sequence f_n already satisfies the inequalities (1) for each $x \in M$ and all natural n, and is uniformly convergent to f on M. Further by the remark b) as well as by the equality $f_n(x) = \max \{F_1(x), \min [g_n(x), F_2(x)]\}$, $x \in M$, we have that all $f_n \in H_\alpha(M)$.

Theorem 2. *Let (R, ϱ) be a locally compact separable metric space and $\{M_m\}_{m=1}^{\infty}$ an arbitrary increasing sequence of compact sets in R such that $R = \bigcup_{m=1}^{\infty} M_m$. Let the number α satisfy the inequalities $0 < \alpha \leqq 1$. Let the function $f \in C_0(R)$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of the functions $f_n \in H_\alpha(R)$ which is uniformly convergent to f on each M_m, $m = 1, 2, 3, \ldots$*

If moreover there exist two functions F_1, F_2 such that $F_1, F_2 \in H_\alpha(R)$ (for each $m = 1, 2, 3, \ldots$, $F_1, F_2 \in H_\alpha(M_m)$) and $F_1(x) \leqq f(x) \leqq F_2(x)$ for every $x \in R$, then the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the inequalities (1) for each $x \in R$ and all natural n (the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the inequalities (1) for each $x \in R$ and all natural n, but instead of $f_n \in H_\alpha(R)$ it is only true that $f_n \in H_\alpha(M_m)$ for each $m = 1, 2, 3, \ldots$).

Proof. By Theorem 1 for each $m = 1, 2, 3, \ldots$ there exists a function $g_m \in H_\alpha(M_m)$ such that $|g_m(x) - f(x)| \leqq 1/m$, $x \in M_m$. Let g_{m0} be its extension with $g_{m0} \in H_\alpha(R)$ as it is mentioned in the remark d). Then, with respect to the inclusion $M_m \subset M_{m+1}$, $m = 1, 2, 3, \ldots$, the sequence $\{g_{n0}\}$ uniformly converges to f on each M_m, $m = 1, 2, 3, \ldots$

If $F_1, F_2 \in H_\alpha(R)$ (if $F_1, F_2 \in H_\alpha(M_m)$ for each $m = 1, 2, 3, \ldots$) and $F_1(x) \leqq f(x) \leqq F_2(x)$, $x \in R$, we define the functions f_n, $n = 1, 2, 3, \ldots$ on R by the relation $f_n(x) = \max \{F_1(x), \min [g_{n0}(x), F_2(x)]\}$. The sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the inequalities (1) for all $x \in R$ and all natural n, is uniformly convergent to f on each M_m and by the remark b) all $f_n \in H_\alpha(R)$ (all $f_n \in H_\alpha(M_m)$ for $m = 1, 2, 3, \ldots$).

Remarks. 1. By approximating a continuous function by means of polynomials in the euclidean space R^n we obtain Theorems 1 and 2 only in a special case when $F_1 \leqq f - \varepsilon$, $F_2 \geqq f + \varepsilon$, $\varepsilon > 0$.

2. When the Stone theorem is considered in a compact topological space, then Theorem 2 is true for a σ-compact space R.

3. Theorem 2 was applied to the proof of the existence of a generalized solution to the first boundary value problem for a nonlinear parabolic equation [4]. Here another theorem is given by means of which a result in the theory of ordinary differential equations will be improved.

Theorem 3. *Suppose a is a real number, $f = f(x, y, z)$ is continuous on $D = \langle a, +\infty) \times R^2$ and such that*

b) f is nondecreasing in y for fixed x, z, f is nondecreasing in z for fixed x, y,

c) $f(x, 0, 0) \equiv 0$ on $\langle a, +\infty)$.

Then there exists a sequence $\{f_n\}$ *of functions* $f_n \in C_0(D)$ *satisfying the conditions b) and c) as well as*

d) a Lipschitz condition with respect to z on each compact subset of D which uniformly converges to f on each compact subset of D.

Proof. For each natural m, let $M_m = \langle a, a + m\rangle \times \langle -m, m\rangle \times \langle -m, m\rangle$. Let us choose and fix an M_m. Let A be the set of all functions $f \in C_0(D)$ having the properties b), c) and d). $A \neq \emptyset$, since $f_1(x, y, z) \equiv y + z \in A$. When considering the restriction of the functions $g \in A$ on M_m, we shall show that A is a lattice of continuous functions on M_m with the property (a). This will imply that there is a function $f_m \in A$ such that $|f(x, y, z) - f_m(x, y, z)| < 1/m$ for $(x, y, z) \in M_m$. Then $\{f_m\}_{m=1}^{\infty}$ will possess all required properties.

If $g_1 \in A$, $g_2 \in A$, $(x, y_1, z) \in M_m$, $(x, y_2, z) \in M_m$ and $g_i(x, y_k, z) = g_{ik}$, $i, k = 1, 2$, then in the case $g_{11} \leqq g_{21}$, $g_{12} \geqq g_{22}$

$$\min(g_{12}, g_{22}) = g_{22} \geqq g_{21} \geqq \min(g_{11}, g_{21})$$

and

$$\max(g_{11}, g_{21}) = g_{21} \leqq g_{22} \leqq \max(g_{12}, g_{22}) .$$

The same result will be obtained in the other cases. It can be similarly proved that $\min(g_1, g_2)$ as well as $\max(g_1, g_2)$ are nondecreasing in z, too. Hence $\min(g_1, g_2)$, $\max(g_1, g_2)$ possess the property b). The property c) is clearly shared by these two functions. Finally, by using the remark b) we have that A is a lattice of continuous functions on M_m.

In proving that A satisfies the condition (a), the following lemma will be useful.

Lemma 1. *Given three points* (y_i, z_i), $i = 0, 1, 2$, *on the plane and a function* $f_0 = f_0(y, z)$ *which is the restrinction to these points of a function satisfying the condition b) from the last theorem, there exists an extension* g_0 *of* f_0 *on the whole plane which is continuous, fulfils the condition b) and satisfies a Lipschitz condition with respect to z on the entire plane.*

Proof. The straight lines $y = y_i$, $z = z_i$, $i = 0, 1, 2$, divide the plane into a system of rectangles, half-stripes and quadrants. f_0 can be extended first to all vertices of those sets and then to the entire plane, by using the linear inter- and extrapolation in such a way that g_0 possesses all the properties stated in the lemma.

Now, consider two different points $(x_i, y_i, z_i) \in M_m$, $i = 1, 2$. When $x_1 \neq x_2$, by Lemma 1, there exist two functions $g_i = g_i(x, y, z)$ defined on the plane $x = x_i$, $i = 1, 2$, such that $g_i(x_i, 0, 0) = 0$, $g_i(x_i, y_i, z_i) = f(x_i, y_i, z_i)$, having the property b) and d). The functions g_i can be extended into a function $g \in C_0(D)$ possessing the required properties b), c) and d) and satisfying

$$g(x_i, y_i, z_i) = f(x_i, y_i, z_i) , \quad i = 1, 2 . \tag{2}$$

If $x_1 = x_2$, again by Lemma 1, there exists a function g_1 with the properties mentioned in Lemma 1 such that $g_1(x_1, y_i, z_i) = f(x_1, y_i, z_i)$, $i = 1, 2$, and $g_1(x_1, 0, 0) = = 0$. Then $g(x, y, z) = g_1(x_1, y, z)$ for $x \in \langle a, \infty)$, y, z arbitrary shows all the desired properties. Thus $g \in A$ and satisfies the conditions (2).

By means of Theorem 3 a theorem by L. K. Jackson [5], p. 342, will be improved. Preserving the notations from Theorem 3, the statement of the Jackson's theorem given here as Lemma 2 is as follows:

Lemma 2. *Let $f = f(x, y, z)$ be continuous on D, satisfy the conditions b) and c). Assume either that f satisfies d) or is such that the solutions of initial-value problems for $y'' = f(x, y, y')$ are unique.*

Then for any real A the boundary-value problem

$$y'' = f(x, y, y')\,, \quad y(a) = A \tag{3}$$

has a unique bounded solution on $\langle a, \infty)$.

From the proof of that theorem it follows that this unique bounded solution y satisfies the inequalities: If $A \geqq 0$ $(A < 0)$, then $0 \leqq y(x) \leqq A$, $y'(x) \leqq 0$, for every $x \geqq a$ (then $A \leqq y(x) \leqq 0$, $y'(x) \geqq 0$, for $x \geqq a$). The bounds for y' from the other side are given by

Lemma 3. *Suppose that f satisfies all conditions from Lemma 2, for every $x \geqq a$, $y \geqq 0$, $S(x, y) = \{(s, t) : x \leqq s \leqq x + 1,\ 0 \leqq t \leqq y(1 - s + x)\}$ for every $x \geqq a$, $y < 0$, $S(x, y) = \{(s, t) : x \leqq s \leqq x + 1,\ y(1 - s + x) \leqq t \leqq 0\}$, $M(x, y) = \max_{(s,t)\in S(x,y)} f(s, t, 0)$, $m(x, y) = \min_{(s,t)\in S(x,y)} f(s, t, 0)$. Then the unique bounded solution y of the problem (3) satisfies for $x \geqq a$ the inequality*

$$y' \geqq -y - M(x, y) \tag{4}$$

when $A \geqq 0$ and the inequality

$$y' \leqq -y - m(x, y)$$

if $A < 0$.

Proof. Consider only the case $A \geqq 0$. The remaining case can be dealt with similarly. Let x, $a \leqq x < \infty$, be arbitrary but fixed. By the properties b) and c) of f, $M(x, y) \geqq 0$. Two cases may happen: 1. $y'(x) \geqq -y(x)$. Then (4) is true at x. 2. $y'(x) < -y(x)$. Then $y(x) > 0$ since otherwise $y(x + \delta) < 0$ for a $\delta > 0$. Let H be the hypotenuse of $S(x, y)$. y must cross H at a point b, $x < b \leqq x + 1$. By the mean value theorem there is a point c, $x < c < b$, such that $y'(c) = -y(x)$. Then there exists c_1, $x < c_1 \leqq c$, with $y'(s) < -y(x)$ for $x \leqq s < c_1$, $y'(c_1) = -y(x)$

and $(s, y(s)) \in S(x, y)$, $x \leqq s \leqq c_1$. This implies

$$y'(c_1) = -y(x) = y'(x) + \int_x^{c_1} f(s, y(s), y'(s))\, ds \leqq y'(x) + \int_x^{c_1} f(s; y(s), 0)\, ds \leqq$$
$$\leqq y'(x) + M(x, y)\,.$$

Again (4) is true.

Applying the last theorem and lemmas the following theorem will be proved. The notations will be the same as in Theorem 3.

Theorem 4. *Suppose $f = f(x, y, z)$ is continuous on D and fulfils the conditions b) and c). Then for every real A the boundary-value problem* (3) *has at least one bounded solution on $\langle a, \infty)$.*

Proof. The case $A = 0$ is trivial. The case $A < 0$ can be dealt with similarly to the case $A > 0$, therefore only the last case will be considered.

By Theorem 3, there exists a sequence $\{f_n\}$ of functions $f_n \in C_0(D)$ satisfying the conditions $b)$, $c)$ and $d)$ which is uniformly convergent to f on each compact subset of D. Lemma 2 then assures that the problem

$$y'' = f_n(x, y, y')\,, \quad y(a) = A \tag{5}$$

has a unique bounded solution y_n on $\langle a, \infty)$. Following the remark after Lemma 2, and by Lemma 3, all y_n satisfy the inequalities:

$$0 \leqq y_n(x) \leqq A\,, \quad -A - M(x, A) \leqq -y_n(x) - M(x, y_n(x)) \leqq \tag{6}$$
$$\leqq y_n'(x) \leqq 0\,, \quad x \geqq a\,.$$

Choose an arbitrary positive integer m. By the continuity of f_n the inequalities (6) imply that $\{y_n''\}$ is uniformly bounded on $\langle a, a + m\rangle$ and so, both sequences $\{y_n\}$, $\{y_n'\}$ are uniformly bounded and equicontinuous on $\langle a, a + m\rangle$. Therefore there is a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ which is uniformly convergent on $\langle a, a + m\rangle$ together with $\{y_{n_k}'\}_{k=1}^{\infty}$ to a function y, and its derivative, respectively. Making use of (5) we have that $\{y_{n_k}''\}_{k=1}^{\infty}$ is uniformly convergent to y'' on the same interval and that y satisfies $y'' = f(x, y, y')$ on $\langle a, a + m\rangle$. In this way for every $m = 1, 2, 3, \ldots$ a sequence $\{y_{m,n}\}_{n=1}^{\infty}$ can be constructed such that:

1. $\{y_{1,n}\}$ is a subsequence of $\{y_n\}$;
2. $\{y_{m+1,n}\}$ is a subsequence of $\{y_{m,n}\}$ for every $m = 1, 2, 3, \ldots$;
3. The sequences $\{y_{m,n}\}$, $\{y_{m,n}'\}$, $\{y_{m,n}''\}$ are uniformly convergent on $\langle a, a + m\rangle$ to a function $\bar{y}_m$ and $\bar{y}_m'$, $\bar{y}_m''$ respectively, whereby $\bar{y}_m''(x) = f(x, \bar{y}_m(x), \bar{y}_m'(x))$ for every $x \in \langle a, a + m\rangle$. By 2., $\bar{y}_{m+1}(x) = \bar{y}_m(x)$ for every $x \in \langle a, a + m\rangle$ and so there is a function y on $\langle a, \infty)$ such that $y(x) = \bar{y}_m(x)$ on $\langle a, a + m\rangle$. From 1. and 3. it follows that y is a bounded solution of (3) on $\langle a, \infty)$.

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