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Svatopluk Fučík

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FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS IN BANACH SPACES AND ITS APPLICATIONS TO DIFFERENTIAL AND INTEGRAL EQUATIONS

SVATOPLUK FUČÍK, Praha

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1. INTRODUCTION

This paper deals with the solution of nonlinear operator equations in Banach spaces and with the nonlinear generalization of the Fredholm alternative. Theorems of the following type are obtained: If T is an operator (generally nonlinear) defined on a real Banach space X with values in a real Banach space Y, then $T(X) = Y$ provided that the equation $Tx = \theta_Y$ has the solution $x = \theta_X$ only and X, Y, T satisfy some additional conditions.

Similar results were obtained by S. I. POCHOŽAJEV [15] for real Banach spaces and for homogeneous operators and, by J. NEČAS [11], for complex Banach spaces and for operators which are "near to homogeneous" ones. M. KUČERA [20] proved a result similar to that in [11] for the real space, his conditions concerning "being near to homogeneity" being stronger than those in [11]. Preceding papers discourse only on the operators the domain of which is a Banach space X, the range being in its dual space X^*. Hence, the integral operators defined on $L_p(\Omega)$ $(p \neq 2)$ with values in $L_p(\Omega)$ are not included in the abstract theory established in [11], [15], [20]. Such a problem is solved in Section 7 on the base of Section 3.

We generalize the preceding results for the case of the operators "near to homogeneous", acting from a real Banach space to another real Banach space. The main result is obtained in the third section of the paper. In Sections 4 and 5 we investigate the notion of the approximation scheme and the A-operator, given in Section 3. These notions are a slight modification of those introduced by W. V. PETRYSHYN [12, 13, 14], S. I. POCHOŽAJEV [15], D. G. DE FIGUEIREDO [5, 6, 7] and F. E. BROWDER - W. V. PETRYSHYN [2].

Section 6 deals with the set of eigenvalues of homogeneous operators. The hypotheses of Theorem 6.1 are very difficult to verify in infinite dimensional Banach space. Theorem 6.2 can be used to "near to linear" operators only.

Finally, in the last section, we apply the abstract Fredholm alternative to the Dirichlet problem for partial differential equations and to some integral equations. In these applications it is necessary to know that the corresponding Banach spaces have a Schauder basis. This concerns particularly the space $\mathring{W}_p^{(k)}(\Omega)$ $(p \neq 2)$. This fact is proved in Section 4 for $\Omega \subset E_1$. Unfortunately, we do not know the corresponding proof in the case of E_n $(n \geqq 2)$. But if this is true, then our main result can be directly applied to more general partial differential equations, such as in [11].

2. TERMINOLOGY, NOTATION AND DEFINITIONS

Let X be a real Banach space with the norm $\|.\|_X$, θ_X its zero element; then X^* denotes the adjoint (dual) space of all bounded linear functionals on X. The pairing between $x^* \in X^*$ and $x \in X$ is denoted by (x^*, x). We shall use the symbols "$\rightarrow$", "$\rhd$" to denote respectively the strong and the weak convergences in X. For a finite dimensional space X, $\dim X$ denotes the dimension of X.

Let M be a subset of X, $\overline{M}$ its closure in X, ∂M its boundary in X. M is said to be compact (weakly compact) if for any sequence $\{x_n\}$, $x_n \in M$ there exists a subsequence $\{x_{n_k}\}$ and an element $x_0 \in X$ such that $x_{n_k} \rightarrow x_0$ $(x_{n_k} \rhd x_0)$ with $k \rightarrow \infty$.

The following assertion will be referred to as Eberlein-Šmuljan Theorem: *A Banach space X is reflexive if and only if every bounded subset of X is weakly compact.*

Let T be a mapping (nonlinear, in general) with the domain $M \subset X$ and the range in the Banach space Y (we write $T: M \rightarrow Y$). Then

(1) T is said to be *continuous on M* if $x_n \rightarrow x_0$ in X implies $Tx_n \rightarrow Tx_0$ in Y for all $x_n, x_0 \in M$.

(2) T is said to be *demicontinuous on M* if $x_n \rightarrow x_0$ in X implies $Tx_n \rhd Tx_0$ in Y for $x_n, x_0 \in M$.

(3) T is said to be *strongly continuous on M* if $x_n \rhd x_0$ in X implies $Tx_n \rightarrow Tx_0$ in Y for $x_n, x_0 \in M$.

(4) T is said to be *weakly continuous on M* if $x_n \rhd x_0$ in X implies $Tx_n \rhd Tx_0$ in Y for $x_n, x_0 \in M$.

(5) T is said to be *strongly closed on M* if $x_n \rhd x_0$ in X and $Tx_n \rightarrow y$ in Y implies $Tx_0 = y$.

(6) T is said to be *completely continuous on M* if T is continuous on M and for each bounded subset $D \subset M$, $T(D)$ is a compact set in Y.

(7) T is said to be *contractive with the constant $\alpha \in \langle 0, 1)$ on M*, if $\|Tx - Ty\|_Y \leqq \leqq \alpha\|x - y\|_X$ for all $x, y \in M$.

(8) $T: X \rightarrow Y$ is said to be *regularly surjective from X onto Y* if $T(X) = Y$ and for any $R > 0$ there exists $r > 0$ such that $\|x\|_X \leqq r$ for all $x \in X$ with $\|Tx\|_Y \leqq R$.

Definition 3.1. Let $K > 0$ be a real number, X and Y Banach spaces, $\{X_n\}$ and $\{Y_n\}$ two sequences of finite dimensional subspaces such that $X_n \subset X$, $Y_n \subset Y$. For each positive integer n let $Q_n : Y \to Y$ be a bounded linear operator from Y onto Y_n, $Q_n^2 = Q_n$ (i.e. linear projection).

We shall say that *the couple* $\langle X, Y \rangle$ *has an approximation scheme* $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ *for the operators from* X *into* Y (briefly speaking, $\langle X, Y \rangle$ has an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$) if the following conditions are satisfied:

(1) $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$,

(2) $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset Y_{n+1} \subset \ldots$,

(3) $\overline{\bigcup_{n=1}^{\infty} X_n} = X$,

(4) $\dim X_n = \dim Y_n$,

(5) $\|Q_n\|_{(Y \to Y)} \leqq K$, where $(Y \to Y)$ is the space of all bounded linear operators from Y into Y,

(6) $Q_n y \to y$ in Y for each $y \in Y$.

Definition 3.2. Let X and Y be two Banach spaces, let $\langle X, Y \rangle$ have an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ and $T : X \to Y$.

(a) *T is said to be an A-operator with respect to a given approximation scheme* $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ (briefly speaking, T is an A-operator) if for any sequence $\{n_j\}$ of positive integers satisfying $n_j \to \infty$ and a bounded sequence $\{x_{n_j}\}$, $x_{n_j} \in X_{n_j}$ such that $Q_{n_j} T x_{n_j} \to y \in Y$ in Y for some $y \in Y$, there exists an infinite subsequence $\{n_{j(k)}\}$ and $x_0 \in X$ such that $x_{n_{j(k)}} \to x_0$ in X and $Tx_0 = y$.

(b) Let T be an A-operator. T is said to be *an A^*-operator* if the following condition is satisfied: Let $R > 0$, $h \in Y$. If for some $\alpha > 0$ and a sequence $\{k_j\}$ of positive integers satisfying $k_j \to \infty$, $\|Q_{k_j} Tu - tQ_{k_j} h\|_Y \geqq \alpha$ holds for $u \in X_{k_j}$, $\|u\|_X = R$ and any $t \in \langle 0, 1 \rangle$, then there exists $x_0 \in X$, $\|x_0\|_X \leqq R$ such that $Tx_0 = h$.

Lemma 3.1. *Let X and Y be two Banach spaces such that X is a reflexive space and $\langle X, Y \rangle$ has an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$. Let $T : X \to Y$ be an A-operator, $h \in Y$, $R > 0$. Suppose that for all $u \in X$, $\|u\|_X = R$ and any $t \in \langle 0, 1 \rangle$ there is $\|Tu - th\|_Y > 0$.*

Then there exist $\alpha > 0$ and a sequence $\{k_j\}$ of positive integers, $k_j \to \infty$ such that $\|Q_{k_j} Tu - t\, Q_{k_j} h\|_Y \geqq \alpha$ for any k_j, all $u \in X_{k_j}$, $\|u\|_X = R$ and $t \in \langle 0, 1 \rangle$.

Proof. To prove the assertion, let us suppose the contrary. Then $u_{n_j} \in X_{n_j}$, $\|u_{n_j}\|_X = R$ and $t_{n_j} \in \langle 0, 1 \rangle$ with $\|Q_{n_j} Tu_{n_j} - t_{n_j} Q_{n_j} h\|_Y \to 0$ (as $n_j \to \infty$) exist. According to the compactness of $\langle 0, 1 \rangle$ and to Eberlein-Šmuljan Theorem the subsequences $\{t_{n_{j(k)}}\} \subset \{t_{n_j}\}$ and $\{u_{n_{j(k)}}\} \subset \{u_{n_j}\}$ such that $t_{n_{j(k)}} \to t_0$, $u_{n_{j(k)}} \rightharpoonup u_0 \in X$ in X can be chosen. Since $Q_{n_{j(k)}}h \to h$ by Definition 3.2.a) there is $\{u_{n_{j(k')}}\} \subset \{u_{n_{j(k)}}\}$ such that $u_{n_{j(k')}} \to u_0$ in X and $Tu_0 = h$. Thus $\|u_0\|_X = R$, $\|Tu_0 - t_0 h\|_Y = 0$. This is a contradiction with our assumptions.

Theorem 3.1. *Let X and Y be two Banach spaces, let X be a reflexive space and let $\langle X, Y\rangle$ have an approximation scheme. Let $T: X \to Y$ be an A^*-operator satisfying*

$$\lim_{\|u\|_X \to \infty} \|Tu\|_Y = +\infty .$$

Then T is regularly surjective from X onto Y.

Proof. For $h \in Y$ there exists $R > 0$ such that $\|Tu\|_Y > \|h\|_X$ for all $u \in X$, $\|u\|_X = R$. Thus for any $t \in \langle 0, 1\rangle$ and all $u \in X$ it is $\|Tu - th\|_Y \geqq \|Tu\|_Y - t\|h\|_Y > 0$. By Lemma 3.1 and Definition 3.2b there is $x_0 \in X$, $\|x_0\|_X \leqq R$ such that $Tx_0 = h$, hence $T(X) = Y$.

It can be easily shown that T is regularly surjective.

Proposition 3.1. *Let X be a reflexive Banach space, Y a Banach space, let $\langle X, Y\rangle$ have an approximation scheme. Let $T: X \to Y$ be an A-operator and let $S: X \to Y$ be completely continuous.*

Then $T + S$ is an A-operator.

Proof. Let $\{n_j\}$ be a sequence of positive integers, $n_j \to \infty$, $\{x_{n_j}\}$ a bounded sequence with $x_{n_j} \in X_{n_j}$ such that $Q_{n_j}(T + S)\, x_{n_j} \to y \in Y$ in Y. Eberlein-Šmuljan Theorem and the complete continuity of the operator S imply that there is a subsequence $\{x_{n_{j(k)}}\} \subset \{x_{n_j}\}$ such that $x_{n_{j(k)}} \rightharpoonup x_0 \in X$ in X and $Sx_{n_{j(k)}} \to w \in Y$ in Y. The uniform boundedness of $\{Q_n\}$ implies $Q_{n_{j(k)}}Sx_{n_{j(k)}}$ in Y since $\|Q_{n_{j(k)}}Sx_{n_{j(k)}} - w\|_Y \leqq K\|Sx_{n_{j(k)}} - w\|_Y + \|Q_{n_{j(k)}}w - w\|_Y$. Thus $Q_{n_{j(k)}}Tx_{n_{j(k)}} \to y - w$ in Y and by Definition 3.2a there is a subsequence of $\{x_{n_{j(k)}}\}$ (we denote it by $\{x_{n_{j(k)}}\}$ again) such that $x_{n_{j(k)}} \to x_0$ in X, $Tx_0 = y - w$ and $Sx_{n_{j(k)}} \to Sx_0$. This implies $Tx_0 + Sx_0 = y$, and the proof is complete.

Proposition 3.2. *Let X and Y be two Banach spaces, let $\langle X, Y\rangle$ have an approximation scheme. Let $\lambda \neq 0$ be a real number and $T: X \to Y$ an A-operator.*

Then λT is an A-operator.

The proof follows immediately from Definition 3.2.

Lemma 3.2. *Let X and Y be two finite dimensional spaces,* $\dim X = \dim Y$. *Denote* $K_R = \{x;\, x \in X,\, \|x\|_X < R\}$, $S_R = \partial K_R$.

Let $h \in Y$ and $f: \bar{K}_R \to Y$ be a continuous mapping such that $f(-x) = -f(x)$ for arbitrary $x \in \bar{K}_R$ and $\|f(x) - th\|_Y > 0$ for each $t \in \langle 0, 1\rangle$ and all $x \in S_R$.

Then there exists $x_0 \in K_R$ such that $f(x_0) = h$.

Proof. Let E be a linear homeomorphism $Y \to X$. Then for the Brouwer degree d of mappings $Ef - Eh$ and Ef on the set K_R with respect to the point θ_X the relation

$$d[Ef - Eh;\ K_R, \theta_X] = d[Ef; K_R, \theta_X] \neq 0$$

holds. (See [3], [8].) This property of the degree of the mapping $Ef - Eh$ implies the existence of $x_0 \in K_R$ such that $Ef(x_0) = Eh$ and thus $f(x_0) = h$.

Proposition 3.3. *Let X and Y be two Banach spaces, X reflexive, $T: X \to Y$ such that $T(-x) = -T(x)$ for arbitrary $x \in X$ (the so called odd mapping). Let $\langle X, Y\rangle$ have an approximation scheme and let T be a demicontinuous A-operator.*

Then T is an A^*-operator.

Proof. Let $R > 0$ and $h \in Y$. Let for some $\alpha > 0$ and some sequence $\{k_j\}$ of positive integers, $k_j \to \infty$

$$\|Q_{k_j} Tu - t\, Q_{k_j}h\|_Y \geqq \alpha$$

hold for each $t \in \langle 0, 1\rangle$ and all $u \in X_{k_j}$, $\|u\|_X = R$.

Lemma 3.2 implies that there is a sequence $\{u_{k_j}\}$, $u_{k_j} \in X_{k_j}$, $\|u_{k_j}\|_X \leqq R$ with $Q_{k_j} Tu_{k_j} = Q_{k_j}h$. According to Eberlein-Šmuljan Theorem we can suppose $u_{k_j} \rhd u_0 \in \in X$ in X. Since $Q_{k_j}h \to h$ in Y we have $Q_{k_j} Tu_{k_j} \to h$ in Y. By Definition 3.2a there is a subsequence $\{u_{k_{j(n)}}\} \subset \{u_{k_j}\}$ such that $u_{k_{j(n)}} \to u_0$ in X, $Tu_0 = h$ and thus T is an A^*-operator.

Corollary 3.1. *Let X and Y be two Banach spaces, X reflexive and let $\langle X, Y\rangle$ have an approximation scheme. Suppose that $T: X \to Y$ is an odd demicontinuous A-operator with*

$$\lim_{\|u\|_X \to \infty} \|Tu\|_Y = +\infty\,.$$

Then T is regularly surjective from X onto Y.

Definition 3.3. Let X and Y be two Banach spaces, $T: X \to Y$, $T_0 : X \to Y$ and $a > 0$ a real number.

(a) T_0 is said to be *a-homogeneous* if $T_0(tu) = t^a\, T_0(u)$ holds for each $t \geqq 0$ and all $u \in X$.

(b) Let T_0 be an a-homogeneous operator. T is said to be *a-quasihomogeneous with respect to T_0* if $t_n \searrow 0$ (i.e. $t_1 \geqq t_2 \geqq \ldots \geqq t_n > 0$ are real numbers and $\lim_{n\to\infty} t_n = 0$), $u_n \rhd u_0$ in X, $t_n^a\, T(u_n/t_n) \to g \in Y$ in Y, then $T_0 u_0 = g$.

(c) T is said to be *a-strongly quasihomogeneous with respect to T_0*, if $t_n \searrow 0$, $u_n \rhd u_0$ in X imply $t_n^a\, T(u_n/t_n) \to T_0 u_0$ in Y.

Proposition 3.4. *Let X and Y be two Banach spaces, $T : X \to Y$, $T_0 : X \to Y$.*

(a) *If T is a-homogeneous and strongly closed, then T is a-quasihomogeneous with respect to T.*

(b) *If T is a-homogeneous and strongly continuous, then T is a-strongly quasihomogeneous with respect to T.*

(c) *If T is a-strongly homogeneous with respect to T_0, then T_0 is a-homogeneous.*

Proposition 3.5. *Let X and Y be two Banach spaces, $S : X \to Y$, $S_0 : X \to Y$. Let S be an a-strongly quasihomogeneous operator with respect to S_0.*

Then S_0 is strongly continuous.

Proof. For $u \in X$ it is $\lim_{t \searrow 0} t^a S(u/t) = S_0 u$ in Y. Suppose that there exists a sequence $\{u_n\}$, $u_n \in X$ and $\varepsilon > 0$ such that $u_n \rhd u_0$ in X and $\|S_0 u_n - S_0 u_0\|_Y \geqq \varepsilon$. For each n there exists t_n, $0 < t_n \leqq 1/n$ such that

$$\left\| S_0 u_n - t_n^a S\left(\frac{u_n}{t_n}\right) \right\|_Y \leqq \frac{\varepsilon}{4}.$$

Then

$$\varepsilon \leqq \|S_0 u_n - S_0 u_0\|_Y \leqq \left\| S_0 u_n - t_n^a S\left(\frac{u_n}{t_n}\right) \right\|_Y + \left\| t_n^a S\left(\frac{u_n}{t_n}\right) - S_0 u_0 \right\|_Y \leqq$$

$$\leqq \frac{\varepsilon}{4} + \left\| S_0 u_0 - t_n^a S\left(\frac{u_n}{t_n}\right) \right\|_Y.$$

Letting n tend to infinity we obtain $\varepsilon \leqq \frac{1}{4}\varepsilon$. This is a contradiction proving the proposition.

Definition 3.4. Let X and Y be two Banach spaces, $T_0 : X \to Y$, $S_0 : X \to Y$ a-homogeneous operators and $\lambda \neq 0$ a real number.

λ is said to be *an eigenvalue for the couple* (T_0, S_0) if there exists $u_0 \in X$, $u_0 \neq \theta_X$ such that $\lambda T_0 u_0 - S_0 u_0 = \theta_Y$.

Lemma 3.3. *Let X and Y be two reflexive Banach spaces $T : X \to Y$, $T_0 : X \to Y$ an a-homogeneous operator, $S : X \to Y$ and $S_0 : X \to Y$. Let T be an a-quasihomogeneous operator with respect to T_0 and let S be an a-strongly quasihomogeneous operator with respect to S_0. Suppose that there exists a constant $c > 0$ such that*

$$\|Tu\|_Y \geqq c\|u\|_X^a$$

holds for each $u \in X$.

Let $\lambda \neq 0$ be a real number. If λ is not an eigenvalue for the couple (T_0, S_0), then

$$\lim_{\|u\|_X \to \infty} \|\lambda Tu - Su\|_Y = \infty .$$

Proof. Let us assume the contrary. Then there exist a sequence $\{u_n\}$, $u_n \in X$ and a real number $K > 0$ so that $\|u_n\|_X \to \infty$ and $\|\lambda T u_n - S u_n\|_Y < K$. Set $v_n = u_n / \|u_n\|_X$. By Eberlein-Šmuljan Theorem we can suppose that $\lambda T u_n - S u_n \rhd g \in Y$ in Y and $v_n \rhd v_0 \in X$ in X. We have

$$\lambda T(\|u_n\|_X v_n) - S(\|u_n\|_X v_n) \rhd g$$

and

$$\lambda \frac{1}{\|u_n\|_X^a} T(\|u_n\|_X v_n) - \frac{1}{\|u_n\|_X^a} S(\|u_n\|_X v_n) \to \theta_Y \quad \text{in} \quad Y.$$

Hence

$$\frac{1}{\|u_n\|_X^a} S(\|u_n\|_X v_n) \to S_0 v_0 \quad \text{in} \quad Y$$

and

$$\lambda \frac{1}{\|u_n\|_X^a} T(\|u_n\|_X v_n) \to S_0 v_0 \quad \text{in} \quad Y.$$

Definition 3.3b implies $\lambda T_0 v_0 = S_0 v_0$ and the proof will be complete if $v_0 \neq \theta_X$. It is clear that

$$\left\| \lambda \frac{1}{\|u_n\|_X^a} T(\|u_n\|_X v_n) \right\|_Y = |\lambda| \frac{1}{\|u_n\|_X^a} \|T u_n\|_Y \geqq c|\lambda| > 0.$$

Hence $S_0 v_0 \neq \theta_Y$ and $v_0 \neq \theta_X$.

Theorem 3.2. *Let X and Y be two reflexive Banach spaces, let $\langle X, Y \rangle$ have an approximation scheme, $T : X \to Y$ let be an odd operator, $T_0 : X \to Y$ an a-homogeneous operator, $S : X \to Y$ an odd completely continuous operator, $S_0 : X \to Y$. Suppose that T is demicontinuous and a-quasihomogeneous with respect to T_0 A-operator, and S is a-strongly quasihomogeneous operator with respect to S_0.*

Suppose that there exists a constant $c > 0$ such that $\|Tu\|_Y \geqq c\|u\|_X^a$ holds for all $u \in X$.

Let $\lambda \neq 0$ be a real number which is not an eigenvalue for the couple (T_0, S_0).

Then the operator $\lambda T - S$ is regularly surjective from X onto Y.

Proof. See Lemma 3.3, Propositions 3.1 and 3.2 and Corollary 3.1.

Theorem 3.3. (This theorem is a generalization of the results in [15] for the case $Y \neq X^*$. The proof is analogous to that in [15].) *Let X and Y be two reflexive Banach spaces, let $\langle X, Y \rangle$ have an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$. Let $T : X \to Y$ be an odd a-homogeneous and continuous A-operator. Let $S : X \to Y$ be an odd completely continuous a-homogeneous operator. Let $\lambda \neq 0$ be a real number such that λ is not an eigenvalue for the couple (T, S). Then the operator $\lambda T - S$ is regularly surjective from X onto Y.*

Proof. It is $\|\lambda Tu - Su\|_Y > 0$ for each $u \in X$, $\|u\|_X = 1$. By Lemma 3.1 there exist $\alpha > 0$ and a sequence $\{k_j\}$ of positive integers, $k_j \to \infty$ such that

$$\|\lambda\, Q_{k_j}Tu - Q_{k_j}Su\|_Y \geqq \alpha$$

holds for each k_j and all $u \in X_{k_j}$, $\|u\|_X = 1$. Let $u_0 \in X$, $\|u_0\|_X = 1$. Then there exist $u_{k_j} \in X_{k_j}$, $u_{k_j} \to u_0$ in X and we have

$$\alpha \leqq \left\|\lambda\, Q_{k_j}T\left(\frac{u_{k_j}}{\|u_{k_j}\|_X}\right) - Q_{k_j}S\left(\frac{u_{k_j}}{\|u_{k_j}\|_X}\right)\right\|_Y \leqq |\lambda| \left\|Q_{k_j}T\left(\frac{u_{k_j}}{\|u_{k_j}\|_X}\right) - Q_{k_j}Tu_0\right\|_Y +$$

$$+ \|\lambda\, Q_{k_j}Tu_0 - Q_{k_j}Su_0\|_Y + \left\|Q_{k_j}S\left(\frac{u_{k_j}}{\|u_{k_j}\|_X}\right) - Q_{k_j}Su_0\right\|_Y \leqq$$

$$\leqq K|\lambda| \left\|T\left(\frac{u_{k_j}}{\|u_{k_j}\|_X}\right) - Tu_0\right\|_Y + K\left\|S\left(\frac{u_{k_j}}{\|u_{k_j}\|_X}\right) - Su_0\right\|_Y + K\|\lambda\, Tu_0 - Su_0\|_Y \to$$

$$\to K\|\lambda\, Tu_0 - Su_0\|_Y$$

for $k_j \to \infty$. Hence $\|\lambda\, Tu_0 - Su_0\|_Y \geqq \alpha/K$ holds for each $u_0 \in X$, $\|u_0\|_X = 1$, and thus, for arbitrary $u \in X$ there is

$$\|\lambda\, Tu - Su\|_Y \geqq \|u\|_X^a \frac{\alpha}{K}.$$

By the previous statement it is $\lim\limits_{\|u\|_X \to \infty} \|\lambda\, Tu - Su\|_Y = \infty$ and according to Corollary 3.1, the proof is complete.

4. APPROXIMATION SCHEME

Proposition 4.1. *Let X be a reflexive Banach space and let $\langle X, X\rangle$ have an approximation scheme $[\{X_n\}, \{X_n\}, \{Q_n\}]_K$ and $Q_{n+1}Q_n = Q_nQ_{n+1}$.*

Then $\langle X, X^*\rangle$ has an approximation scheme.

Proof. For each integer n let $Q_n^* : X^* \to X^*$ be the operator adjoint to Q_n and set $Y_n = Q_n^*(X^*)$. Then $\dim X_n = \dim Y_n$,

$$X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots,$$

$$Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset Y_{n+1} \subset \ldots,$$

and $\|Q_n^*\|_{(X^* \to X^*)} = \|Q_n\|_{(X \to X)} \leqq K$. To show that $[\{X_n\}, \{Y_n\}, \{Q_n^*\}]_K$ is an approximation scheme for $\langle X, X^*\rangle$ we must prove that for each $x^* \in X^*$ it is $Q_n^*x^* \to x^*$ in X^*. It is easy to show that $Q_n^*x^* \rhd x^*$ in X^* (i.e. $\overline{\bigcup_{n=1}^{\infty} Y_n}$ is weakly dense in X^*)

and $Q_n^* x^* \in \overline{\bigcup_{n=1}^{\infty} Y_n}$. The set $\overline{\bigcup_{n=1}^{\infty} Y_n}$ is a convex one and by the well-known theorem $\overline{\bigcup_{n=1}^{\infty} Y_n}$ is weakly closed. Hence $\overline{\bigcup_{n=1}^{\infty} Y_n} = X^*$. For each $x^* \in \bigcup_{n=1}^{\infty} Y_n$ there is an integer n_0 such that $x^* \in Y_n$ for $n \geqq n_0$ and $x^* = Q_{n_0}^* x^* = Q_n^* x^*$. Thus $Q_n^* x^* \to x^*$ in X^* for all $x^* \in \bigcup_{n=1}^{\infty} Y_n$, $\overline{\bigcup_{n=1}^{\infty} Y_n} = X^*$ and $\|Q_n^*\|_{(X^* \to X^*)} \leqq K$ and according to Uniform Boundedness Theorem (see [4]) the proof is complete.

Proposition 4.2. *Let X and Y be two infinite dimensional Banach spaces. Suppose that X is a separable space and $\langle Y, Y \rangle$ has an approximation scheme $[\{Y_n\}, \{Y_n\}, \{Q_n\}]_K$.*
Then $\langle X, Y \rangle$ has an approximation scheme.

(This Proposition shows that whether the couple $\langle X, Y \rangle$ has an approximation scheme depends only on the space Y.)

Proof. Let $x_1, x_2, \ldots$ be a dense sequence in X. Let X_n be the linear hull of $\{x_1, \ldots \ldots, x_n\}$. Then there exists a subsequence $\{X_{k(n)}\}$ such that $[\{X_{k(n)}\}, \{Y_n\}, \{Q_n\}]_K$ is an approximation scheme for the couple $\langle X, Y \rangle$.

Definition 4.1 ([1], [5], [6], [7]). Let $K \geqq 1$. A separable Banach space X is said *to have Property* (π_K) if there exists a sequence of finite dimensional subspaces $X_n \subset X$ such that

(a) $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$,

(b) $\overline{\bigcup_{n=1}^{\infty} X_n} = X$,

(c) each X_n is the range of continuous linear projection $Q_n : X \to X$ with the norm $\leqq K$.

Definition 4.2 ([4]). A separable Banach space X is said *to have Schauder basis* $\{e_n\}$, $e_n \in X$ if for each $x \in X$ there exists a unique sequence $\{a_1, a_2, \ldots\}$ of real numbers such that $\sum_{i=1}^{n} a_i x_i \to x$ in X (with $n \to \infty$).

Proposition 4.3. *A Banach space with a Schauder basis has Property (π_K) for some K.*

Proof. See [7].

Proposition 4.4. *Let X be a Banach space with Property (π_K).*
Then $\langle X, X \rangle$ has an approximation scheme.

Proposition 4.5. *Let X be an infinite dimensional Banach space with a Schauder basis. Then the couple $\langle X, X\rangle$ has an approximation scheme.*

Moreover, if Y is a separable infinite dimensional Banach space, then $\langle Y, X\rangle$ has an approximation scheme.

If X is a reflexive Banach space with a Schauder basis, then $\langle X, X^*\rangle$ has an approximation scheme.

Proposition 4.6. *Let X be a Banach space with Property (π_1), such that* $\dim X_n = n$. *Then X has a Schauder basis.*

Proof. See [10].

Remark 4.1. A separable Hilbert space, $C[0, 1]$, $L_p[0, 1]$, $C^k([0, 1])$, $C^1([0, 1]^N)$ (see [17]) all have an approximation scheme (they have a Schauder basis).

Remark 4.2. Let Ω be a bounded open subset of the Euclidean N-space E_N. Then $L_p(\Omega)$ is linearly homeomorphic to $L_p[0, \text{meas } \Omega]$, where meas Ω is the N-dimensional Lebesgue measure of Ω (see [9, Chapter II, § 12]). Hence $L_p(\Omega)$ has a Schauder basis.

Remark 4.3. Let X be a Banach space with Schauder basis $\{e_1, e_2, \ldots\}$. For $x \in X$ there exists a unique sequence $\{a_n\}$ of real numbers such that $x = \sum_{i=1}^{\infty} a_i e_i$. Set $a_i = \alpha_i(x)$. Then $\alpha_i \in X^*$.

Definition 4.3. Let $I = (0, 1)$, $k \geqq 1$ integer and $p \geqq 1$ real number. *By Sobolev space* $W_p^{(k)}(I)$ we mean the set of all functions f such that f and its derivatives $f^{(i)}$ up to the order $k - 1$ are absolutely continuous functions in I and the derivative of the order k (which exists almost everywhere) belongs to $L_p(I)$. The norm in $W_p^{(k)}(I)$ is

$$\|f\|_{W_p^{(k)}(I)} = \left(\sum_{i=0}^{k} \|f^{(i)}\|_{L_p(I)}^p\right)^{1/p}.$$

We set

$$\mathring{W}_p^{(k)}(I) =$$
$$= \{f; f \in W_p^{(k)}(I), f(0) = f'(0) = \ldots = f^{(k-1)}(0) = f(1) = \ldots = f^{(k-1)}(1) = 0\}.$$

Proposition 4.7. $W_p^{(k)}(I)$ *has a Schauder basis.*

Proof. We prove the proposition by induction with respect to k. Suppose that $\{f_n^k\}$ is a Schauder basis in $W_p^{(k)}(I)$ and $\{\alpha_n^k\}$ is a sequence of continuous linear functionals such that for each $f \in W_p^{(k)}(I)$ there is $f = \sum_{n=1}^{\infty} \alpha_n^k(f) f_n^k$ (see Remark 4.3). Set

$$f_1^{k+1}(x) \equiv 1\,, \qquad \alpha_1^{k+1}(f) = f(0)$$

$$f_n^{k+1}(x) = \int_0^x f_{n-1}^k(t)\,\mathrm{d}t\,, \quad \alpha_n^{k+1}(f) = \alpha_{n-1}^k(f')$$

for $n \geqq 2$, $x \in I$, $f \in W_p^{(k+1)}(I)$. Then $f_n^{k+1} \in W_p^{(k+1)}(I)$. For arbitrary $f \in W_p^{(k+1)}(I)$, $l \geqq 1$ we have

$$\|f - \sum_{n=1}^{l} \alpha_n^{k+1}(f) f_n^{k+1}\|_{W_p^{(k+1)}(I)}^p = \int_0^1 \left| f(x) - f(0) - \sum_{n=1}^{l-1} \alpha_n^k(f') \int_0^x f_n^k(t)\, \mathrm{d}t \right|^p \mathrm{d}x +$$

$$+ \|f' - \sum_{n=1}^{l-1} \alpha_n^k(f') f_n^k\|_{W_p^{(k)}(I)}^p \leqq 2\|f' - \sum_{n=1}^{l-1} \alpha_n^k(f') f_n^k\|_{W_p^{(k)}(I)}^p$$

and hence

$$\lim_{l \to \infty} \|f - \sum_{n=1}^{l} \alpha_n^{k+1}(f) f_n^{k+1}\|_{W_p^{(k+1)}(I)} = 0 .$$

Let

$$\lim_{l \to \infty} \|\sum_{n=1}^{l} c_n f_n^{k+1}\|_{W_p^{(k+1)}(I)} = 0$$

for some sequence $\{c_n\}$ of real numbers, i.e.

$$0 = \lim_{l \to \infty} \int_0^1 \left| c_1 + \sum_{n=2}^{l} c_n \int_0^x f_{n-1}^k(t)\, \mathrm{d}t \right|^p \mathrm{d}x$$

and

$$0 = \lim_{l \to \infty} \|\sum_{n=2}^{l} c_n f_{n-1}^k\|_{W_p^{(k)}(I)} .$$

Since $\{f_n^k\}$ is a Schauder basis in $W_p^{(k)}(I)$ we have $c_n = 0$ for $n \geqq 2$ and $\lim_{l \to \infty} \int_0^1 |c_1|^p \,\mathrm{d}x = 0$, i.e. $c_n = 0$ for each positive integer n. We obtain that the sequence $\{f_n^{k+1}\}$ is a Schauder basis in $W_p^{(k+1)}(I)$ and, since for $k = 0$ the space $L_p(I) = W_p^{(0)}(I)$ has a Schauder basis (see Remark 4.1), we proved our assertion.

Proposition 4.8. *$\mathring{W}_p^{(1)}(I)$ has a Schauder basis.*

Proof. Let us construct the basis $\{f_n^1\}$ in $W_p^{(1)}(I)$ from the basis $\{f_n^0\}$ in $L_p(I)$ as in Proposition 4.7 where $\{f_n^0\}$ is a Haar orthogonal system in $L_p(I)$. Set $\mathring{f}_n^1 = f_{n+1}^1$, $\mathring{\alpha}_n^1(f) = \alpha_{n+1}^1(f)$ for each positive integer n and all $f \in \mathring{W}_p^{(1)}(I)$. Then $\mathring{f}_n^1 \in \mathring{W}_p^{(1)}(I)$, $\{\mathring{f}_n^1\}$ is a Schauder basis in $\mathring{W}_p^{(1)}(I)$ and $\{\mathring{\alpha}_n^1\}$ are functionals coresponding to $\{\mathring{f}_n^1\}$.

5. *A*-OPERATORS

Definition 5.1. A Banach space X is said to be *strictly convex* if for each $x, y \in X$, $x \neq y$, $\|x\|_X = \|y\|_X = 1$ and all $t \in (0, 1)$ there is $\|tx + (1 - t)\, y\|_X < 1$.

Definition 5.2. A Banach space X is said to have *Property* (H) if X is strictly convex and if $x_n \rightharpoonup x_0$ in X and $\|x_n\|_X \to \|x_0\|_X$ implies $x_n \to x_0$ in X.

Remark 5.1. $L_p(\Omega)$, $l_p(p > 1)$, Hilbert spaces all have Property (H).

Proposition 5.1. *Let X be a reflexive Banach space, Y a Banach space, $T: X \to Y$, $S: X \to Y$, $f: X \to E_1$, $\Phi: X \to Y^*$. Let $\langle X, Y \rangle$ have an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$. Let S be a completely continuous operator, let f be a weakly upper semi-continuous functional (i.e. $x_n \rightharpoonup x_0$ in X implies $\limsup f(x_n) \leqq f(x_0)$), $f(\theta_X) = 0$ and let Φ be a weakly continuous operator, $\Phi(\theta_X) = \theta_{Y^*}$.*

Suppose that γ, φ are continuous real valued strictly increasing functions on $\langle 0, \infty)$ such that $\gamma(0) = 0$.

Let $\mu: (0, \infty) \times X \to (0, \infty)$ and suppose that $Q_n^* \Phi(x) = \Phi(x)$ for each positive integer n and all $x \in X_n$.

Then T is an A-operator provided that one of the following conditions is satisfied:

(5.1) *T is continuous and*

$$(\Phi(x - y), Tx - Ty) + f(x - y) \geqq \gamma(\|x - y\|_X)$$

for each $x, y \in X$.

(5.2) *T is continuous and*

$$(\Phi(x - y), Tx - Ty) + (\Phi(x - y), Sx - Sy) + f(x - y) \geqq \gamma(\|x - y\|_X)$$

for each $x, y \in X$.

(5.3) *T is demicontinuous, $\overline{\Phi(X)} = Y^*$, $\Phi(tw) = \mu(t, w)\,\Phi(w)$ for $t > 0$ and all $w \in X$ and*

$$(\Phi(x - y), Tx - Ty) \geqq \gamma(\|x - y\|_X)$$

for each $x, y \in X$.

(5.4) *T is demicontinuous, Φ is the same as in (5.3) and*

$$(\Phi(x - y), Tx - Ty) + (\Phi(x - y), Sx - Sy) \geqq \gamma(\|x - y\|_X)$$

for each $x, y \in X$.

(5.5) *X has Property (H), T is demicontinuous, Φ is the same as in (5.3) and*

$$(\Phi(x - y), Tx - Ty) \geqq (\varphi(\|x\|_X) - \varphi(\|y\|_X))(\|x\|_X - \|y\|_X)$$

for each $x, y \in X$.

(5.6) *X has Property (H), T is demicontinuous, Φ is the same as in (5.3) and*

$$(\Phi(x - y), Tx - Ty) + (\Phi(x - y), Sx - Sy) \geqq$$
$$\geqq (\varphi(\|x\|_X) - \varphi(\|y\|_X))(\|x\|_X - \|y\|_X)$$

for each $x, y \in X$.

Proof. Let $x_{n_j} \in X_{n_j}$, $x_{n_j} \rhd x_0$ in X, $Q_{n_j}Tx_{n_j} \to y$ in Y. Then for $x \in X_{n_j}$ we have

$$(\Phi(x_{n_j} - x), Q_{n_j}Tx_{n_j} - Q_{n_j}Tx) = (\Phi(x_{n_j} - x), Tx_{n_j} - Tx).$$

Let condition (5.1) be satisfied. Then for $x \in X_e$, $n_j \geqq l$ there is

$$\gamma(\|x_{n_j} - x\|_X) \leqq (\Phi(x_{n_j} - x), Q_{n_j}Tx_{n_j} - Q_{n_j}Tx) + f(x_{n_j} - x)$$

and

$$\limsup_{n_j \to \infty} \gamma(\|x_{n_j} - x\|_X) \leqq (\Phi(x_0 - x), y - Tx) + f(x_0 - x).$$

The last inequality holds for each $x \in X$. Set $x = x_0$. We obtain $0 \leqq \limsup_{n_j \to \infty} \gamma(\|x_{n_j} - - x_0\|_X) \leqq 0$ and

$$\|Q_{n_j}Tx_{n_j} - Tx_0\|_Y \leqq K\|Tx_{n_j} - Tx_0\|_Y + \|Q_{n_j}Tx_0 - Tx_0\|_Y.$$

Thus $Tx_0 = y$ and $x_{n_j} \to x_0$ in X.

Let condition (5.3) be satisfied. We obtain $x_{n_j} \to x_0$ in X and $0 \leqq (\Phi(x_0 - x), y - - Tx)$ for each $x \in X$. Set $x_t = x_0 - tw$ for $t > 0$ and $w \in X$. Then

$$0 \leqq (\Phi(tw), y - T(x_0 - tw)) = \mu(t, w))(\Phi(w), y - T(x_0 - tw)),$$
$$0 \leqq (\Phi(w), y - T(x_0 - tw)).$$

Letting t tend to zero we obtain

$$0 \leqq (\Phi(w), y - Tx_0)$$

for each $w \in X$ and $\overline{\Phi(X)} = Y^*$ implies $y = Tx_0$.

Let condition (5.5) be satisfied. Then $\|x_{n_j}\|_X \to \|x_0\|_X$ and $x_{n_j} \rhd x_0$ in X. Hence $x_{n_j} \to x_0$ in X and $0 \leqq (\Phi(x_0 - x), y - Tx)$ for each $x \in X$ and similarly as in the previous part one obtains $y = Tx_0$.

Let condition (5.2) or (5.4) or (5.6) be satisfied. Then the assertion is a consequence of Proposition 3.1 and condition (5.1) or (5.3) or (5.5) respectively.

Remark 5.2. Let X be a reflexive Banach space and let the couple $\langle X, X^* \rangle$ have an approximation scheme. We identify X with X^{**} and set $Y = X^*$ and Φ the identity operator on X. Then Φ satisfies the assumptions of Proposition 5.1.

Definition 5.3. a) *A gauge function* is a real-valued continuous function μ defined on the interval $\langle 0, \infty)$ such that

(1) $\mu(0) = 0$,

(2) $\lim_{t \to \infty} \mu(t) = \infty$,

(3) μ is strictly increasing.

b) *The duality mapping in X with a gauge function μ* is a mapping J from X into the set 2^{X^*} of all subsets of X^* such that

$$Jx = \begin{cases} \{\theta_{X^*}\}\,, & x = \theta_X\,, \\ \{x^*, x^* \in X^*, (x^*, x) = \|x\|_X \|x^*\|_{X^*}, \|x^*\|_{X^*} = \mu(\|x\|_X)\}\,, & x \neq \theta_X\,. \end{cases}$$

For next two remarks see [1], [5], [6] and [7].

Remark 5.3. a) The set Jx is non-empty.

b) Let X be a Banach space with a strictly convex dual space X^*. Let J be the duality mapping in X with the gauge function μ. Then the set Jx consists of precisely one point.

c) Let X be a Banach space with a strictly convex dual space X^*. Let $J : X \to X^*$ be the duality mapping with the gauge function μ and $t > 0$.Then $J(tw) = \beta(t, w)\,.\,Jw$ where β is a positive function on $(0, \infty) \times X$.

d) Let X^* be a strictly convex Banach space, $J : X \to X^*$ the duality mapping in X with the gauge function μ and $[\{X_n\}, \{X_n\}, \{Q_n\}]_K$ an approximation scheme for $\langle X, X\rangle$. Then $Q_n^* \, Jx = Jx$ for each $x \in X_n$ and all positive integers n.

Remark 5.4. Let X be a Banach space with a strictly convex dual space X^*, $J : X \to$ $\to X^*$ the weakly continuous duality mapping in X with the gauge function μ (for example, there exist a gauge function μ and the duality mapping J which is weakly continuous in the spaces l_p, $1 < p < \infty$). Set $Y = X$ and $\Phi = J$. Then Φ satisfies the assumptions of Proposition 5.1.

Proposition 5.2. *Let X be a Banach space, $[\{X_n\}, \{X_n\}, \{Q_n\}]_K$ an approximation scheme of $\langle X, X\rangle$, $T: X \to X$, $T = I - S$ where I is the identity operator and S is a contraction mapping with the constant $\alpha \in \langle 0, 1)$. Let $\alpha K < 1$.*

Then T is an A-operator.

Proof. According to Banach Contraction Mapping Fixed Point Theorem there exists one and only one $x_0 \in X$ for each $y \in Y$ such that $Tx_0 = y$.

Let $R > 0$, $x_{n_j} \in X_{n_j}$, $\|x_{n_j}\|_X \leqq R$, $Q_{n_j}Tx_{n_j} \to y = Tx_0$ in X. Then

$$\begin{aligned}(1 - \alpha K)\,\|x_{n_j} - Q_{n_j}x_0\|_X &\leqq \|x_{n_j} - Q_{n_j}x_0\|_X - \|Q_{n_j}Sx_{n_j} - Q_{n_j}SQ_{n_j}x_0\|_X \leqq \\ &\leqq \|Q_{n_j}Tx_{n_j} - Q_{n_j}TQ_{n_j}x_0\|_X \leqq \\ &\leqq \|Q_{n_j}Tx_{n_j} - y\|_X + \|Tx_0 - Q_{n_j}TQ_{n_j}x_0\|_X \leqq \\ &\leqq \|Q_{n_j}Tx_{n_j} - y\|_X + K\|Tx_0 - TQ_{n_j}x_0\|_X + \|Q_{n_j}Tx_0 - Tx_0\|_X \to 0\,.\end{aligned}$$

Thus $x_{n_j} - Q_{n_j}x_0 \to \theta_X$ in X and $x_{n_j} \to x_0$ in X.

Corollary 5.1. *Let X, S, K, α satisfy the assumptions of Proposition 5.2. If X is a reflexive Banach space and $U : X \to X$ is a completely continuous operator, then $T = I - S - U$ is an A-operator.*

Proof. See Propositions 5.1 and 3.1.

6. THE SET OF EIGENVALUES

Definition 6.1. Let X be a Banach space, $T_0 : X \to X^*$, $S_0 : X \to X^*$ two potential operators (i.e. there exist functionals f, g such that $T_0 = \operatorname{grad} f$, $S_0 = \operatorname{grad} g$ in the Gâteaux sense – see [18]). Let $f(x) = 0$ iff $x = \theta_X$ and set $\varphi(x) = g(x)/f(x)$ for $x \neq \theta_X$.

$u_0 \in X$, $u_0 \neq \theta_X$ is said to be *an R-eigenvector of* (T_0, S_0) if $D\varphi(u_0, h) = 0$ for each $h \in X$. ($D\varphi(u_0, h)$ is the linear differential Gâteaux at the point u_0). $\lambda_0 = \varphi(u_0)$ is said to be *an R-eigenvalue.*

Proposition 6.1. *Let X be a Banach space, $T_0 : X \to X^*$ and $S_0 : X \to X^*$ two a-homogeneous potential operators. Suppose that there exists a constant $c > 0$ such that*

$$(T_0 x, x) \geqq c\|x\|_X^{a+1}$$

for each $x \in X$.

Then every eigenvalue of the couple (T_0, S_0) is an R-eigenvalue.

Proof. There is

$$f(x) = (T_0 x, x) \cdot \frac{1}{a+1}, \quad g(x) = (S_0 x, x) \cdot \frac{1}{a+1}.$$

Let $\lambda_0 \neq 0$ be an eigenvalue of (T_0, S_0), i.e. there exists $u_0 \neq \theta_X$ such that $\lambda_0 T_0 u_0 - S_0 u_0 = \theta_{X^*}$.

Thus

$$\lambda_0 (T_0 u_0, h) = (S_0 u_0, h)$$

for each $h \in X$ so that

$$\lambda_0 = \frac{(S_0 u_0, u_0)}{(T_0 u_0, u_0)}.$$

Hence

$$\frac{(S_0 u_0, u_0)(T_0 u_0, h) - (T_0 u_0, u_0)(S_0 u_0, h)}{(T_0 u_0, u_0)^2} = 0$$

for each $h \in X$, i.e. $D\,\varphi(u_0, h) = 0$ where

$$\varphi(u) = \frac{(S_0 u, u)}{(T_0 u, u)} = \frac{g(u)}{f(u)}.$$

Lemma 6.1. *Let X be a separable and reflexive Banach space, $G \subset X$ an open subset, $f : G \to E_1$ a functional of the class C^m (i.e. there exists the Fréchet derivative $D^j f(x)$ up to the order m which is continuous* – see [18]). *Let the following conditions be satisfied*:

(6.1) $\sup_{x \in G} \dim \operatorname{Ker} D^2 f(x) = l < \infty$

where $\operatorname{Ker} D^2 f(x) = \{h;\ h \in X, (D^2 f(x)\, h, w) = 0 \text{ for each } w \in X\}$,

$$m \geqq \max(l, 2)\,, \tag{6.2}$$

(6.3) $D^2 f(x)(X)$ *is closed subset of the space* $(X \to X^*)$ *for each* $x \in G$.

Set $M = \{x;\ x \in G,\ Df(x, h) = 0 \text{ for each } h \in X\}$.

Then $\operatorname{meas} f(M) = 0$. (For proof see [16)].

Proposition 6.2. *Let X be a reflexive and separable Banach space, $T_0 : X \to X^*$, $S_0 : X \to X^*$ be two potential operators. Let the functional φ (see Definition 6.1) satisfy the assumptions of Lemma 6.1.*

Then the set of R-eigenvalues of the couple (T_0, S_0) has the Lebesgue measure zero.

Theorem 6.1. *Let X be a reflexive Banach space such that $\langle X, X^* \rangle$ has an approximation scheme. Let $T : X \to X^*$ be an odd A-operator, $T_0 : X \to X^*$ an a-homogeneous operator, $S : X \to X^*$ an odd completely continuous operator and $S_0 : X \to X^*$. Suppose that T is an a-quasihomogeneous operator with respect to T_0 and S is an a-strongly quasihomogeneous operator with respect to S_0. Suppose that there exists a constant $c > 0$ such that*

$$\|Tu\|_{X^*} \geqq c\|u\|_X^a$$

and

$$(T_0 u, u) \geqq c\|u\|_X^{a+1}$$

for each $u \in X$.

Let $T_0 = \operatorname{grad} f$, $S_0 = \operatorname{grad} g$ *and set* $\varphi(u) = g(u)/f(u)$ *for* $u \neq \theta_X$. *Suppose that the functional φ satisfies the assumptions of Lemma* 6.1 *on some neighborhood of the unit sphere in X.*

Then there exists a set $N \subset E_1$, $\operatorname{meas} N = 0$ *such that* $(\lambda T - S) X = X^*$ *for each* $\lambda \in E_1 - N$.

Lemma 6.2. *Let X and Y be two Banach spaces, $T_0 : X \to Y$, $S_0 : X \to Y$ be linear operators such that T_0 is continuous, S_0 is completely continuous, $T_0 X = Y$. Suppose* *that there exists a constant $c > 0$ such that*

$$\|T_0 x\|_Y \geqq c\|x\|_X$$

for each $x \in X$.

Then the set of eigenvalues for the couple (T_0, S_0) is at most denumerable and if it has a limit point λ, then $\lambda = 0$.

Proof. For the problem $\lambda I - T_0^{-1}S_0$ we have well-known theorems (see [4]) about the set of eigenvalues. λ is an eigenvalue for (T_0, S_0) iff λ is an eigenvalue for $(I, T_0^{-1}S_0)$ (I is the identity operator in X).

Theorem 6.2. *Let X and Y be two reflexive Banach spaces such that $\langle X, Y\rangle$ has an approximation scheme. Let $T: X \to Y$ be a demicontinuous and odd A-operator, $S: X \to Y$ be a completely continuous and odd operator, $S_0 : X \to Y$ and $T_0 : X \to Y$ linear operators. Suppose that T is an 1-quasihomogeneous operator with respect to T_0 and S is an 1-strongly quasihomogeneous operator with respect to S_0. Suppose that there exists a constant $c > 0$ such that*

$$\|Tu\|_Y \geqq c\|u\|_X$$

and

$$\|T_0 u\|_Y \geqq c\|u\|_X$$

for each $u \in X$.

Let $T_0 X = Y$.

Then there exists a set $N \subset E_1$, N is at most denumerable and if N has a limit point λ, then $\lambda = 0$ and N is such that $(\lambda T - S) X = Y$ for each $\lambda \in E_1 - N$.

7. APPLICATIONS

a) Let Ω be a bounded domain in E_N and $\mathring{W}_2^{(1)}(\Omega)$ the Sobolev space (for definition see [19, Chapter 1]). The space $\mathring{W}_2^{(1)}(\Omega)$ is a Hilbert separable space. Denote Δ the Laplace operator. We seek the weak solution of the Dirichlet problem

$$-\lambda\Delta u - u\frac{|u|^s}{1 + |u|^s} = f \quad (s > 0)$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

for $f \in (\mathring{W}_2^{(1)}(\Omega))^*$, i.e. we seek $u \in \mathring{W}_2^{(1)}(\Omega)$ such that for each $v \in \mathring{W}_2^{(1)}(\Omega)$ the identity

$$\lambda \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx - \int_\Omega \frac{|u|^s}{1 + |u|^s} uv\,dx = \int fv\,dx$$

holds. This equation has a solution for each $f \in (\mathring{W}_2^{(1)}(\Omega))^*$ if the equation

$$\lambda \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx - \int_\Omega uv \, dx = 0$$

(for all $v \in \mathring{W}_2^{(1)}(\Omega)$) has zero solution only (see Theorem 3.2), i.e. for $\lambda \neq 1/\lambda_k$ where $\{\lambda_k\}$ is the spectrum of the Dirichlet problem for the equation

$$-\Delta u - \lambda u = 0 .$$

b) Let Ω be an open bounded subset of E_N. It is known (see [7]) that $\langle L_p(\Omega), L_p(\Omega)\rangle$ has an approximation scheme with $K = 1$.

Let $p > 1$ and let hu be Němyckij's operator (for the definition and properties see [18]) generated by the function $f(x)\,(u^3/(1 + u^2))$ where $f \in L_\infty(\Omega)$. Let $A \in (L_p(\Omega) \to L_p(\Omega))$ with the norm $\|A\|_{(L_p \to L_p)}$. Suppose that there exists a constant m such that $|f(x)| \leqq m$ almost everywhere in Ω and

$$\alpha = \|A\|_{(L_p \to L_p)} \, . \, m \, . \, \tfrac{9}{8} < 1 \, .$$

Then the operator $U = Ahu$ is a contraction with a constant $\alpha < 1$ and moreover, U is 1-quasihomogeneous with respect to $U_0 u = Ah_0 u$ where $h_0 u$ is Němyckij's operator generated by the function $f(x)\,u$. By Propositions 5.2, 3.2 and 3.3 the operator $T = I - U$ is an A^*-operator which is 1-quasihomogeneous with respect to the operator $T_0 = I - U_0$.

Let $K(x, y)$, $L(x, y)$ be continuous functions on $\bar{\Omega} \times \bar{\Omega}$ and $s \geqq 0$. Set

$$Su = \frac{\left| \int_\Omega L(x, y) \, u(y) \, dy \right|^s}{1 + \left| \int_\Omega L(x, y) \, u(y) \, dy \right|^s} \int_\Omega K(x, y) \, u(y) \, dy \, .$$

The operator S is strongly continuous and 1-strongly quasihomogeneous with respect to the operator

$$S_0 u = \int_\Omega K(x, y) \, u(y) \, dy \, .$$

By Theorem 3.2 the equation

$$\lambda(u(x) - Ahu) - Su = F(x) \tag{1}$$

has a solution $u \in L_p(\Omega)$ for arbitrary $F \in L_p(\Omega)$ provided the equation

$$\lambda(u(x) - Ah_0 u) - S_0 u = 0 \tag{2}$$

has the trivial solution only.

According to Theorem 6.2 there exists a set $N \subset E_1$, N being at most denumerable and if λ is a limit point of N, then $\lambda = 0$ and N is such that (1) has a solution $u \in \in L_p(\Omega)$ for each $F \in L_p(\Omega)$ and all $\lambda \in E_1 - N$.

REMARKS

1. Preliminary communication was published in Comment. Math. Univ. Carolinae *11*, 1970, 271–284.

2. W. V. Petryshyn (Arch. Rat. Mech. Anal. *30*, 1968, 270–284 and same Arch. *33*, 1969, 331–338) solved this problem for the linear operators using similar methods.

3. When the preliminary communication had been published the author obtained a reprint of the paper by W. V. Petryshyn: Nonlinear Equations involving Noncompact Operators, Proc. Symp. Pure Math., Nonlinear Functional Analysis, Vol. XVIII, Part 1, 1970, Providence, Rhode Island, pp. 206–233. W. V. Petryshyn dealt with the same problem and his Theorem 1.4 on the p. 216 is essentially the same as Theorem 3.3 in this paper.

4. Author is very much indebted to the reviewer for his advice and comments.

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Author's address: Praha 8 - Karlín, Sokolovská 83 (Matematicko-fyzikální fakulta KU).