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A NOTE ON SYMMETRICALLY CONTINUOUS FUNCTIONS

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Dedicated to the memory of Prof. VOJTĚCH JARNÍK

A function f defined on the real line R is called symmetrically continuous (on R) if for every $x \in R$

$$\lim_{h \to 0} (f(x + h) - f(x - h)) = 0 .$$

H. FRIED [1] proved that every symmetrically continuous function is continuous at every point of a dense subset of R. In the present paper it is proved that such a function must be continuous almost everywhere.

Lemma. *Let $E \subset R$ be a measurable set, let 0 be a point of density of E (see [2]). Then there exists $\varepsilon > 0$ such that for every $x \in (0, \varepsilon)$ there exists $t \in E \cap (\frac{1}{3}x, \frac{1}{2}x)$ such that $2t \in E$, $4t - x \in E$.*

Proof. We denote $|A|$ the measure of A, $2A = \{y \in R;\ y = 2z,\ z \in A\}$, $A - a = \{y \in R;\ y = z - a,\ z \in A\}$. Let ε be such a positive number that for every $h \in (0, \varepsilon)$ it is $|E \cap (0, h)| > \frac{13}{15}h$. Let $x \in (0, \varepsilon)$. We set $E_1 = E \cap (\frac{1}{3}x, \frac{1}{2}x)$, $E_2 = (2E_1) \cap E$, $E_3 = [(2E_2) - x] \cap E$. Now an easy calculation shows $|E_1| > \frac{1}{10}x$, $E_1 \subset (\frac{1}{3}x, \frac{1}{2}x)$, $|2E_1| > \frac{1}{5}x$, $2E_1 \subset (\frac{2}{3}x, x)$, $|E \cap (\frac{2}{3}x, x)| > \frac{1}{5}x$, $|E_2| = |E \cap (2E_1) \cap (\frac{2}{3}x, x)| > \frac{1}{3}x - 2(\frac{1}{3}x - \frac{1}{5}x) = \frac{1}{15}x$, $|2E_2| > \frac{2}{15}x$, $|(2E_2) - x| = |2E_2| > \frac{2}{15}x$, $E_3 \subset (\frac{1}{3}x, x)$, $|E_3| > \frac{2}{3}x - (\frac{2}{3}x - \frac{8}{15}x + \frac{2}{3}x - \frac{2}{15}x) = 0$. Therefore $E_3 \neq \emptyset$. Then there exists $t_3 \in E_3$, $t_3 = 2t_2 - x$, $t_2 \in E_2$, hence $t_2 = 2t$, $t \in E_1$ and t is the required point.

Theorem. *Let f be a symmetrically continuous function. Then f is continuous almost everywhere.*

Proof. We put

$$\operatorname{osc} f(x) = \limsup_{h \to 0+} \{|f(x_1) - f(x_2)|;\ |x_1 - x| < h,\ |x_2 - x| < h\} ,$$

$$\varphi(x) = \min(\operatorname{osc} f(x), 1) .$$

The function f is continuous at $x \in R$ if and only if $\varphi(x) = 0$. According to the Fried's result $\varphi(x) = 0$ at every point of a dense subset of R.

At first we prove that φ is symmetrically continuous. Let $x \in R$. Let ε be an arbitrary positive number, let $\delta > 0$ be such that for every h, $0 < |h| < \delta$ it is $|f(x + h) - f(x - h)| < \frac{1}{2}\varepsilon$. If $0 < |h_0| < \delta$, $K < \operatorname{osc} f(x + h_0)$, then there exist x_n^1, x_n^2 such that

$$x + h_0 = \lim_{n \to +\infty} x_n^1 = \lim_{n \to +\infty} x_n^2\,, \quad |f(x_n^1) - f(x_n^2)| > K\,.$$

We set $y_n^1 = 2x - x_n^1$, $y_n^2 = 2x - x_n^2$. Then $x - h_0 = \lim_{n \to +\infty} y_n^1 = \lim_{n \to +\infty} y_n^2$. For large n it is $|f(x_n^1) - f(y_n^1)| < \frac{1}{2}\varepsilon$, $|f(x_n^2) - f(y_n^2)| < \frac{1}{2}\varepsilon$, and it follows that $\operatorname{osc} f(x - h_0) \geqq K - \varepsilon$. From this fact it is easy to deduce that φ is symmetrically continuous.

Now φ is measurable. Suppose at there exists $\alpha > 0$ such that $\varphi(x) > \alpha$ in a set A of positive measure. Let $P \subset A$ be a perfect set, $|P| > 0$.

For $x \in R$ we choose $\delta(x) > 0$ such that for $0 < |h| < \delta(x)$ it is $|\varphi(x + h) - \varphi(x - h)| < \frac{1}{6}\alpha$. Let $A_k = \{x \in P,\ \delta(x) > 1/k\}$. From the fact that $P = \bigcup_{k=1}^{\infty} \bar{A}_k$ it follows that there exists k_0 such that $|\bar{A}_{k_0}| > 0$. Let x_0 be a point of density of $P_1 = \bar{A}_{k_0}$. We can suppose that $x_0 = 0$. We choose $0 < \varepsilon < \min(1/k_0, \frac{1}{2}\delta(0))$ according to the lemma (where $E = P_1$). Let $x_1 \in (0, \varepsilon)$ such that $\varphi(x_1) = 0$. Then there exists $t \in P_1 \cap (\frac{1}{3}x_1, \frac{1}{2}x_1)$ such that $s = 2t \in P_1$, $x_2 = 4t - x_1 \in P_1$. We set $d = \frac{1}{2}(x_1 - x_2)$. Let $u \in A_{k_0} \cap (\frac{1}{3}x_1, \frac{1}{2}x_1)$, $|u - t| < \min(\frac{1}{2}\delta(d), \frac{1}{4}\delta(0))$. We put $s_1 = 2u - s$. It is

$$|s_1| = |2u - 2t| < \delta(d)\,, \quad |d - s_1| < |d| + |s_1| < \delta(0)\,,$$

$$|\varphi(s_1 + d) - \varphi(s_1 - d)| \leqq$$
$$\leqq |\varphi(d + s_1) - \varphi(d - s_1)| + |\varphi(d - s_1) - \varphi(-(d - s_1))| < \tfrac{1}{3}\alpha\,,$$

$$|x_1 - u| < \frac{1}{k_0} < \delta(u)\,, \quad |x_2 - u| < \frac{1}{k_0} < \delta(u)\,,$$

$$s_1 - d = u - (x_1 - u)\,, \quad s_1 + d = u - (x_2 - u)$$

$$|\varphi(x_1) - \varphi(s_1 - d)| = |\varphi(u + (x_1 - u)) - \varphi(u - (x_1 - u))| < \tfrac{1}{6}\alpha$$

$$|\varphi(x_2) - \varphi(s_1 + d)| = |\varphi(u + (x_2 - u)) - \varphi(u - (x_2 - u))| < \tfrac{1}{6}\alpha$$

$$|\varphi(x_1) - \varphi(x_2)) \leqq |\varphi(x_1) - \varphi(s_1 - d)| + |\varphi(s_1 - d) - \varphi(s_1 + d)| +$$
$$+ |\varphi(s_1 + d) - \varphi(x_2)| < \tfrac{2}{3}\alpha\,.$$

But $\varphi(x_1) = 0$, $\varphi(x_2) > \alpha$ which is a contractidion. Hence it follows that $\varphi(x) = 0$ a.e., therefore f is continuous almost everywhere.

The following example shows that the set of points at which a symmetrically continuous function is not continuous can be uncountable.

Example. It is well known that there exists such a trigonometrical series

$$\sum_{n=1}^{\infty} \varrho_n \cos(nx - \alpha_n) \quad \text{that} \quad \sum_{n=1}^{\infty} |\varrho_n \cos(nx - \alpha_n)| = +\infty \quad \text{a.e.}$$

and

$$\sum_{n=1}^{\infty} |\varrho_n \cos(nx - \alpha_n)| < +\infty$$

at every point of an uncountable set. We set

$$f(x) = \left(1 + \sum_{n=1}^{\infty} |\varrho_n \cos(nx - \alpha_n)|\right)^{-1} \quad \text{if} \quad \sum_{n=1}^{\infty} |\varrho_n \cos(nx - \alpha_n)| < +\infty\,,$$

$$f(x) = 0 \qquad \text{if} \quad \sum_{n=1}^{\infty} |\varrho_n \cos(nx - \alpha_n)| = +\infty\,.$$

If $f(x_0) = 0$, then f is continuous at x_0. If $f(x_0) > 0$, then we use the following inequalities

$$\big||\varrho_n \cos[n(x_0 + h) - \alpha_n]| - |\varrho_n \cos[n(x_0 - h) - \alpha_n]|\big| \leqq 2|\varrho_n \cos(nx_0 - \alpha_n)|$$

$$\big||\varrho_n \cos[n(x_0 + h) - \alpha_n]| - |\varrho_n \cos[n(x_0 - h) - \alpha_n]|\big| \leqq 2|\varrho_n|\,|\sin nh|\,.$$

From the first formula it follows that if $f(x_0 + h) = 0$ then $f(x_0 - h) = 0$. If $f(x_0 + h) > 0$ then for every N

$$|f(x_0 + h) - f(x_0 - h)| \leqq \sum_{n=1}^{\infty} \big||\varrho_n \cos[n(x_0 + h) - \alpha_n]| - |\varrho_n \cos[n(x_0 - h) - \alpha_n]|\big| \leqq$$

$$\leqq \sum_{n=1}^{N} 2|\varrho_n|\,|\sin nh| + \sum_{n=N+1}^{\infty} |\varrho_n \cos(nx_0 - \alpha_n)|\,.$$

It follows easily that f is symmetrically continuous. Obviously f is not continuous at every point where it is positive.

References

[1] *H. Fried:* Über die symmetrische Stetigkeit von Funktionen, Fund. Math. *29* (1937), 134—137.

[2] *S. Saks:* Theory of the Integral, Warszawa 1937.

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