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Jan Kučera; Ivo Vrkoč

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NOTE ON STABILITY OF A LINEAR HOMOGENEOUS CONTROL SYSTEM

JAN KUČERA, IVO VRKOČ, Praha

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It is shown in [3] that the stability of zero-solution of an equation $\dot{x} = F(u)\,x$, where u ranges a set $\mathscr{U}$ of controls is equivalent to the boundedness of each solution. Now we extend this and some related results on a case when F depends also on time.

In this paper we will be interested in the ordinary, uniform, asymptotic and exponential stability, resp., of a control problem

$$\dot{x} = F(t)\,x\,, \quad F \in \mathscr{F}\,, \tag{1}$$

where $\mathscr{F}$ is a set of locally on $\langle 0, \infty)$ integrable $n \times n$ – matrix-functions.

A function $x(t)$, locally absolutely continuous on $\langle 0, \infty)$, is said to be a solution of (1) if there exists such $F \in \mathscr{F}$ that $x(t)$ solves the equation $\dot{x}(t) = F(t)\,x(t)$ in the sense of Carathéodory (see [1]). We denote such solution by x_F or $x_F(t, t_0, x^0)$ if it is necessary to express that x_F fulfils the initial condition $x_F(t_0, t_0, x^0) = x^0$.

Property (F). We say that a set $\mathscr{F}$ has the property (F) if it is non-empty and for each sequence $F_k \in \mathscr{F}$, $k = 1, 2, \ldots$, and each division $0 = t_0 < t_1 < t_2 < \ldots$, $\mathscr{F}$ contains at least one element F for which $F(t) = F_k(t)$, $t \in \langle t_{k-1}, t_k)$, $k = 1, 2, \ldots$ (We have said "at least" because in the case $\lim_{k\to\infty} t_k < +\infty$ F is not determined uniquely).

Example. Be given $G \subset R^n$, $H : \langle 0, \infty) \times G \to R^{n^2}$. Let a set $\mathscr{U}$ of functions $u : \langle 0, \infty) \to G$ contain with each sequence $u_k \in \mathscr{U}$, $k = 1, 2, \ldots$, and each division $0 = t_0 < t_1 < t_2 < \ldots$ also an element u fulfilling the condition $u(t) = u_k(t)$, $t \in \langle t_{k-1}, t_k)$, $k = 1, 2, \ldots$ Then the set of all functions $H(t, u(t))$, where $u \in \mathscr{U}$, has the property (F). The condition imposed on $\mathscr{U}$ is fulfilled in particular when $\mathscr{U}$ is the set of all functions $\langle 0, \infty) \to G$.

It is suitable for us to say that (1) is stable if it exists such $T \geqq 0$ that for each $t_0 \geqq T$, $\varepsilon > 0$, there exists such $\delta > 0$ that for every $F \in \mathscr{F}$, every $t \geqq t_0$ and every $x^0 \in R^n$, $\|x^0\| \leqq \delta$, an inequality $\|x_F(t, t_0, x^0)\| \leqq \varepsilon$ holds. If $T = 0$ we get the usual definition of stability. The stability is called uniform if $T = 0$ and for each $\varepsilon > 0$ the number δ can be chosen independently of t_0.

We say that (1) is asymptotically stable if it is uniformly stable and

$$\lim_{t \to \infty} (\sup \{\|x_F(t, t_0, x^0)\|; F \in \mathscr{F}\}) = 0$$

for each $t_0 \in \langle 0, \infty)$ and each $x^0 \in R^n$. We say that (1) is exponentially stable if such positive constants C and λ exist that $\|x_F(t, t_0, x^0)\| \leqq C\|x^0\| \exp(-\lambda(t - t_0))$ holds for every $F \in \mathscr{F}$, $x^0 \in R^n$, $t \geqq t_0 \geqq 0$.

Theorem 1. *Let a set $\mathscr{F}$ have property* (F). *Let each matrix function $F \in \mathscr{F}$ be locally integrable on $\langle 0, \infty)$. Let each solution $x_F(t, 0, x^0)$, where $F \in \mathscr{F}$, $x^0 \in R^n$, of* (1) *be bounded on $\langle 0, \infty)$.*

Then (1) *is stable.*

Proof. Put

$$L(t) = \{y \in R^n; \sup \{\|x_F(\tau, t, y)\|; \tau \geqq t, F \in \mathscr{F}\} < \infty\} \quad \text{for} \quad t \geqq 0 .$$

Then L has the following properties:

1) *Evidently, $L(t)$ is a linear space for each $t \geqq 0$.*

2) dim $L(t)$ *is a nondecreasing function on $\langle 0, \infty)$.*

In fact, let $0 \leqq t_1 \leqq t_2$. Take linearly independent points $x^1, x^2, \ldots, x^k \in L(t_1)$ and an arbitrary $F \in \mathscr{F}$. Then the points $x_F(t_2, t_1, x^i) \in L(t_2)$, $i = 1, 2, \ldots, k$, are also linearly independent.

3) *If such $T \geqq 0$ exists that $L(T) = R^n$, then* (1) *is stable.*

Actually, choose $t_0 \geqq T$ and $\varepsilon > 0$. Then according to property 2) we have $L(t_0) = R^n$. Let us now take an orthonormal basis $e^1, e^2, \ldots, e^n$ in R^n, denote $s_i = \sup \{\|x_F(t, t_0, e^i)\|; t \geqq t_0, F \in \mathscr{F}\}$, $i = 1, 2, \ldots, n$, and choose $\delta > 0$ so that $\delta \sum_{i=1}^{n} s_i \leqq \varepsilon$. Then for every $F \in \mathscr{F}$, $t \geqq t_0$, $x^0 \in R^n$, $\|x^0\| \leqq \delta$, we have $\|x_F(t, t_0, x^0)\| \leqq \delta \sum_{i=1}^{n} \|x_F(t, t_0, e^i)\| \leqq \delta \sum_{i=1}^{n} s_i \leqq \varepsilon$.

4) *Denote* $d = \max \{\dim L(t);\ t \geqq 0\}$. *If* $\inf \{t; \dim L(t) = d\} < \tau_1 \leqq \tau_2$ *and* $y \notin L(\tau_1)$ *then* $x_F(\tau_2, \tau_1, y) \notin L(\tau_2)$ *for each* $F \in \mathscr{F}$.

To prove it, take an arbitrary $F \in \mathscr{F}$ and choose $y^1, \ldots, y^d \in L(\tau_1)$ linearly independent. Then $x_F(\tau_2, \tau_1, y^i)$, $i = 1, 2, \ldots, d$, are also linearly independent. If $x_F(\tau_2, \tau_1, y) \in L(\tau_2)$ we could write $x_F(\tau_2, \tau_1, y) = \sum_{i=1}^{d} \alpha_i x_F(\tau_2, \tau_1, y^i)$. This would imply $x_F(\tau, \tau_1, y) = \sum_{i=1}^{d} \alpha_i x_F(\tau, \tau_1, y^i)$ and especially

$$y - \sum_{i=1}^{d} \alpha_i y^i = x_F(\tau_1, \tau_1, y) - \sum_{i=1}^{d} \alpha_i x_F(\tau_1, \tau_1, y^i) = 0 .$$

As $L(\tau_1)$ is a linear space it contradicts to the assumption $y \notin L(\tau_1)$.

To conclude the proof of Theorem 1 assume that (1) is not stable. Then according to property 3) $L(t) \neq R^n$ for every $t \geqq 0$. Let $t_0 > \inf\{t; \dim L(t) = d\}$, where $d = \max\{\dim L(t); t \geqq 0\}$. We can take $x^0 \notin L(t_0)$ and choose $t_1 > t_0$, $F_1 \in \mathscr{F}$ so that $\|x_{F_1}(t_1, t_0, x^0)\| > 1$. According to property 4) $x^1 = x_{F_1}(t_1, t_0, x^0) \notin L(t_1)$. Having already chosen $x^k \notin L(t_k)$ we can find such $t_{k+1} > t_k$, $F_{k+1} \in \mathscr{F}$ that $x^{k+1} = x_{F_{k+1}}(t_{k+1}, t_k, x^k) \notin L(t_{k+1})$ and $\|x^{k+1}\| > k + 1$, $k = 1, 2, \ldots$

Now take such $F \in \mathscr{F}$ that $F(t) = F_k(t)$ for $t \in \langle t_{k-1}, t_k)$, $k = 1, 2, \ldots$ Then the solution $x_F(t, t_0, x^0)$ is not bounded and the proof is complete.

Remark. Assume moreover that $m(t) = \sup\{\|F(t)\|; F \in \mathscr{F}\}$, $t \geqq 0$, is locally integrable on $\langle 0, \infty)$. Then we can put $T = 0$ in our notion of stability. It follows immediately from the inequality $\|x_F(t, t_0, x^0)\| \leqq \|x^0\| \exp \int_{t_0}^t m(t)\, dt$.

Theorem 2. *Let* 1) *the assumptions of Theorem* 1 *be fulfilled.*

2) $m(t) = \sup\{\|F(t)\|; F \in \mathscr{F}\}$, $t \geqq 0$, *be locally integrable on* $\langle 0, \infty)$.

3) $\sup\{\|x_F(t, t_0, x^0)\|; t \geqq t_0 \geqq 0, \|x^0\| \leqq 1\} < +\infty$ *for each* $F \in \mathscr{F}$.

Then (1) *is uniformly stable.*

Proof. If (1) is not uniformly stable then there exist such $\varepsilon > 0$, $x^k \in R^n$, $F_k \in \mathscr{F}$, $0 \leqq t_k < \tau_k$, that $\|x^k\| \to 0$, and $\|x_{F_k}(\tau_k, t_k, x^k)\| > \varepsilon$, $k = 1, 2, \ldots$ Assume that $\sup t_k = s < +\infty$. Put $M = \exp \int_0^s m(t)\, dt$. Then $\|x_{F_k}(0, t_k, x^k)\| \leqq M\|x^k\|$. According to assumption 2 and the proved stability of (1) we can put $T = 0$ in the definition of stability. There exists such $\delta > 0$ that $\|y\| \leqq \delta$ implies $\|x_{F_k}(t, 0, y)\| \leqq \varepsilon$ for every $t \geqq 0$. If we put $y^k = x_{F_k}(0, t_k, x^k)$ then $\|y^k\| \leqq M\|x^k\| \to 0$ with $k \to +\infty$. Take k_0 so that for $k > k_0$ the inequality $M\|x^k\| < \delta$ holds. Thus for $k > k_0$ we have got a contradiction $\varepsilon < \|x_{F_k}(\tau_k, t_k, x^k)\| = \|x_{F_k}(\tau_k, 0, y^k)\| \leqq \varepsilon$. Hence $\sup\{t_k; k = 1, 2, \ldots\} = +\infty$ and we can assume $t_1 < \tau_1 < t_2 < \tau_2 < \ldots$

Now take such $F \in \mathscr{F}$ for which $F(t) = F_k(t)$, where $t \in \langle t_k, t_{k+1})$, $k = 1, 2, \ldots$ Then evidently

$$\|x_F(\tau_k, t_k, \|x^k\|^{-1} x^k)\| = \|x^k\|^{-1} \|x_{F_k}(\tau_k, t_k, x^k)\| > \|x^k\|^{-1} \varepsilon \to \infty$$

which violates assumption 3 and Theorem 2 is proved.

Theorem 3. *Let the assumption of Theorem* 2 *be fulfilled and moreover*

$$\lim_{t \to \infty} \|x_F(t, t_0, x^0)\| = 0$$

for each $F \in \mathscr{F}$, $t_0 \geqq 0$, $x^0 \in R^n$.

Then (1) *is asymptotically stable.*

Proof. According to Theorem 2 system (1) is uniformly stable, i.e.

$$B = \sup\{\|x_F(t, t_0, x^0)\|; F \in \mathscr{F}, t \geqq t_0 \geqq 0, \|x^0\| \leqq 1\} < +\infty\,.$$

Fix an $\varepsilon > 0$ and for each $t \geqq 0$ denote

$$L(t) = \{x \in R^n;\ \limsup_{\tau \to \infty} \sup \{\|x_F(\tau, t, x)\|;\ F \in \mathscr{F}\} \leqq \varepsilon\}\,.$$

Then L has the following properties:

1. *If $x \in L(t)$ then $x_F(\tau, t, x) \in L(\tau)$ for each $F \in \mathscr{F}$ and each $\tau \geqq t$.*

2. *There exists such $\delta > 0$ that for each $t \geqq 0$ and each $x \in R^n$, $\|x\| < \delta$, we have $x \in L(t)$.*

These two properties follow immediately from the definition of L and from the uniform stability of (1).

3. *$L(t)$ is closed for each $t \geqq 0$.*

Actually, let $x^k \in L(t)$, $x^k \to x^0$. Due to uniform stability of (1) for each $\eta > 0$ there exists such $\mu > 0$ that $\|y - z\| < \mu$ implies $\|x_F(\tau, t, y) - x_F(\tau, t, z)\| = \|x_F(\tau, t, y - z)\| < \eta$, where $\tau \geqq t$, $F \in \mathscr{F}$. Further, there exists such integer k_0 that $\|x^k - x^0\| < \mu$ for every $k > k_0$. Hence for $k > k_0$ we can write

$$\limsup_{\tau \to \infty} \sup \{\|x_F(\tau, t, x^0)\|;\ F \in \mathscr{F}\} \leqq$$
$$\leqq \limsup_{\tau \to \infty} \sup \{\|x_F(\tau, t, x^0) - x_F(\tau, t, x^k)\|;\ F \in \mathscr{F}\} +$$
$$+ \limsup_{\tau \to \infty} \sup \{\|x_F(\tau, t, x^k)\|;\ F \in \mathscr{F}\} \leqq \eta + \varepsilon\,.$$

As η was an arbitrary number $x^0 \in L(t)$ holds.

4. *Let $K \subset R^n$ be compact. Then for each $t \geqq 0$ and each $\eta > 0$ there exists such $T(t, \eta) \geqq 0$ that $\|x_F(\tau, t, x)\| < \varepsilon + \eta$ holds for every $\tau \geqq t + T(t, \eta)$, $F \in \mathscr{F}$ and $x \in K \cap L(t)$.*

To prove it, take $t \geqq 0$, $\eta > 0$, and put $\mu = (2B)^{-1}\,\eta$. Denote $S_\mu = \{z \in R^n;\ \inf\{\|z - x\|;\ x \in L(t)\} < \mu\}$. It can be shown, similarly as in Property 3, that for each $z \in S_\mu$ an inequality $\limsup_{\tau \to \infty} \sup \{\|x_F(\tau, t, z)\|;\ F \in \mathscr{F}\} \leqq \varepsilon + \frac{1}{2}\eta$ holds.

Let $\{G_\alpha;\ \alpha \in A\}$ be such system of open sets $G_\alpha \subset R^n$ that $K \cap L(t) \subset \bigcup_{\alpha \in A} G_\alpha$ and for each $\alpha \in A$ there exist $x^0, x^1, \ldots, x^n \in S_\mu$ such that $G_\alpha = \{x \in R^n;\ x = \sum_{i=0}^{n} \lambda_i x^i,\ \sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} |\lambda_i| < 1\}$. As $K \cap L(t)$ is compact it exists such finite subset $A_0 \subset A$ that $K \cap L(t) \subset \bigcup_{\alpha \in A_0} G_\alpha$.

Take $\alpha \in A_0$ and the corresponding points $x^0, \ldots, x^n \in S_\mu$. For each x^i, $i = 0, \ldots, n$, there exists such T_i that $\sup\{\|x_F(\tau, t, x^i)\|;\ F \in \mathscr{F}\} < \varepsilon + \eta$ for every $\tau \geqq t + T_i$. Denote $T_\alpha = \max\{T_i;\ i = 0, 1, \ldots, n\}$. Then for every $x \in G_\alpha$ and every $F \in \mathscr{F}$ we have $\|x_F(\tau, t, x)\| = \|x_F(\tau, t, \sum_{i=0}^{n} \lambda_i x^i)\| \leqq \sum_{i=0}^{n} \lambda_i \|x_F(\tau, t, x^i)\| < \varepsilon + \eta$. Thus $T(t, \eta) = \max\{T_\alpha;\ \alpha \in A_0\}$ has obviously the required property.

5. *Let such* $x \in R^n$, $t, \vartheta \geqq 0$ *exist that for every* $F \in \mathscr{F}$ *we have* $x_F(t + \vartheta, t, x) \in \in L(t + \vartheta)$. *Then* $x \in L(t)$.

In fact, let K be the closure of the set $\{y \in R^n$; *there exists* $F \in \mathscr{F}$ *so that* $y = x_F(t + \vartheta, t, x)\}$. Then, according to property 3, $K \subset L(t + \vartheta)$. As $\sup \{\|y\|; y \in K\} \leqq B\|x\|$ holds the set K is compact.

Take an arbitrary $\eta > 0$. Then, according to property 4, there exists such $T(t + \vartheta, \eta)$ that

$$\sup \{\|x_F(\tau, t + \vartheta, y)\|;\ F \in \mathscr{F},\ y \in K,\ \tau > (t + \vartheta) + T(t + \vartheta, \eta)\} < \varepsilon + \eta\,.$$

Hence $\sup \{\|x_F(\tau, t, x)\|;\ F \in \mathscr{F},\ \tau > (t + \vartheta) + T(t + \vartheta, \eta)\} < \varepsilon + \eta$ and as η is arbitrary $x \in L(t)$ holds

6. $L(t) = R^n$ *for each* $t \geqq 0$.

Proof. If it is not true then there are such $t_0 \geqq 0$, $x^0 \in R^n$ that $x^0 \notin L(t_0)$. According to property 5 such $F_1 \in \mathscr{F}$ exists that $x^1 = x_{F_1}(t_0 + 1, t_0, x^0) \notin L(t_0 + 1)$. Hence, it exists such $F_2 \in \mathscr{F}$ that $x^2 = x_{F_2}(t_0 + 2, t_0 + 1, x^1) \notin L(t_0 + 2)$.

By the mathematical induction we can construct sequences $x^k \in R^n - L(t_0 + k)$ and $F_k \in \mathscr{F}$, $k = 1, 2, \ldots$, for which $x^{k+1} = x_{F_{k+1}}(t_0 + k + 1,\ t_0 + k, x^k)$, $k = 1, 2, \ldots$ If we now take such $F \in \mathscr{F}$ that $F(t) = F_k(t)$ for $t \in \langle t_0 + k - 1, t_0 + k)$, $k = 1, 2, \ldots$, then for each integer $k \geqq 0$ we have $x^k = x_F(t_0 + k, t_0, x^0) \notin L(t_0 + k)$. According to property 2 we have $\limsup_{\tau \to \infty} \|x_F(\tau, t_0, x^0)\| \geqq \delta$ which violates assumptions of the theorem.

To bring the proof of Theorem 3 to the end take $\varepsilon > 0$. Then the mapping $L: \langle 0, \infty) \to R^n$ is defined. Take $t_0 \in \langle 0, \infty)$, $x^0 \in R^n$. According to property 6 we have $x^0 \in L(t_0)$. If we put $\eta = \varepsilon$ in property 4 then there exists such $T(t_0, \varepsilon, x^0) \geqq 0$ that $\|x_F(\tau, t_0, x^0)\| < 2\varepsilon$ holds for every $\tau \geqq t_0 + T(t_0, \varepsilon, x^0)$ and every $F \in \mathscr{F}$. The proof is complete.

Theorem 4. *Let the assumptions of Theorem* 2 *be fulfilled and for each fixed* $F \in \mathscr{F}$ *the linear system* $\dot{x} = Fx$ *be exponentially stable, i.e. there are such positive constants* C_F, λ_F *that* $\|x_F(t, t_0, x^0)\| \leqq C_F\|x^0\| \exp(-\lambda_F(t - t_0))$ *holds for every* $t \geqq t_0 \geqq 0$, $x^0 \in R^n$.

Then (1) *is exponentially stable.*

Proof. As (1) is a homogeneous (in x) system it follows from the uniform stability of (1) the equivalence of the above mentioned definition of exponential stability with the following one: System (1) is exponentially stable if for each $\varepsilon > 0$ there is such $T > 0$ that for every $t \geqq 0$, $x \in R^n$, $F \in \mathscr{F}$ and $\tau > t + T$ we have $\|x_F(\tau, t, x)\| \leqq \varepsilon\|x\|$. Henceforth, if (1) is not exponentially stable then there exist $\varepsilon > 0$, $t_k > t_{0k} + k \geqq k$, $F_k \in \mathscr{F}$, $x^k \in R^n$ such that $\|x_{F_k}(t_k, t_{0k}, x^k)\| > \varepsilon\|x^k\|$, $k = 1, 2, \ldots$

If $\sup \{t_{0k};\ k = 1, 2, \ldots\} < +\infty$ then we can assume that $t_{0k} \to t_0 \neq +\infty$. It exists such integer k_0 that for $k > k_0$ we have $\exp \left|\int_{t_{0k}}^{t_0} m(t)\,\mathrm{d}t\right| < 2$, $|t_{0k} - t_0| < 1$.

As the assumptions of Theorem 3 are fulfilled there exists, according to property 4 in the proof of Theorem 3, such $T > 0$ that $\sup \{\|x_F(t, t_0, x)\|; t > t_0 + T, F \in \mathscr{F}\} \leqq \leqq \frac{1}{3}\varepsilon\|x\|$. Hence for $k > 1 + \max(T, k_0)$ we have $\varepsilon\|x^k\| \leqq \|x_{F_k}(t_k, t_{0k}, x^k)\| = = \|x_{F_k}(t_k, t_0, x_{F_k}(t_0, t_{0k}, x^k))\| \leqq \frac{1}{3}\varepsilon\|x_{F_k}(t_0, t_{0k}, x^k)\| \leqq \frac{2}{3}\varepsilon\|x^k\|$. This contradiction proves that $\sup\{t_{0k}; k = 1, 2, \ldots\} = +\infty$.

Using subsequences we can now assume that $t_{01} < t_1 < t_{02} < t_2 < \ldots$ Take such $F \in \mathscr{F}$ that $F(t) = F_k(t)$ for $t \in \langle t_{0k}, t_{0,k+1})$. Then $\dot{x} = F(t)\, x$ is exponentially stable and we have $\varepsilon\|x^k\| < \|x_{F_k}(t_k, t_{0k}, x^k)\| = \|x_F(t_k, t_{0k}, x^k)\| \leqq C_F\|x^k\| \exp(-\lambda_F(t_k - - t_{0k})) \leqq C_F\|x^k\| \exp(-\lambda_F k)$, $k = 1, 2, \ldots$ We have obtained again a contradiction and Theorem 4 is proved.

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Authors' address: Praha 1, Žitná 25 (Matematický ústav ČSAV).