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A NOTE ON THE FUNCTIONS WITH CLOSED GRAPHS

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In the paper we shall consider the functions, whose domain X and range Y are topological spaces and have closed graphs in $X \times Y$ (the product $X \times Y$ with Tychonoff's topology). G_f denotes the graph of a function f. We shall generalize some results of papers [2] and [3]. The relation of the family $B_0(X, Y)$ of all continuous functions from the topological space X to the space Y and of the family $B_1(X, Y)$ of all functions of the first Baire's class to the family $U(X, Y)$ of all functions with closed graphs will be considered. If Z is a topological space then: $G(Z)$ stands for the family of all open sets of the space Z, $F(Z)$ stands for the family of all closed sets of the space Z and $F_\sigma(Z)$ stands for the family of all F_σ sets of the space Z.

Theorem 1. *Let X and Y be topological spaces and $f \in U(X, Y)$. Then the subspace $f(X)$ of the space Y is T_1-space and $f^{-1}(y) \in F(X)$ for each $y \in Y$.*

Proof. The first part of the statement will be proved by contradiction. If $f(X)$ is not T_1-space, there are distinct points $y, y_0 \in f(X)$ such that every neighborhood of the point y_0 contains y. Let $y = f(x)$. The fact that sets of the form $G_1 \times G_2$ $(G_1 \in G(X), G_2 \in G(Y))$ are the base of the topology for $X \times Y$ and that the convergence in the product space is pointwise implies that each constant net $\{(x, y), \alpha \in A\}$ $((x, y) \in G_f)$ converges to the point $(x, y_0) \notin G_f$, hence $f \notin U(X, Y)$.

Let $y \in f(X)$. The set $f^{-1}(y)$ is closed in X if and only if no net in $f^{-1}(y)$ converges to a point $x \notin f^{-1}(y)$ (see [1], p. 66). Let a net $\{x_\alpha, \alpha \in A\}$, $x_\alpha \in f^{-1}(y)$ converge to the point x. Using this net we can construct the convergent net $\{(x_\alpha, y), \alpha \in A\}$, $(x_\alpha, y) \to (x, y)$, which is in the closed set G_f. Hence $(x, y) \in G_f$, $y = f(x)$, $x \in f^{-1}(y)$, therefore $f^{-1}(y) \in F(X)$. If $y \in Y - f(X)$ then $f^{-1}(y) = \emptyset \in F(X)$.

A stronger statement than that of the first part of Theorem 1 on Hausdorff spaces is not possible, because there are functions with closed graphs whose range is not Hausdorff space.

Example 1. $X = Y = N = 1, 2, \ldots$ The topology for X is the discrete topology and the topology for Y is the family of all sets whose complement is finite or the void set. Obviously Y is T_1 – space, but it is not Hausdorff one. We shall show that the identical map has closed graph in $X \times Y$. Let the net $S = \{(x_\alpha, x_\alpha), \alpha \in A\}$ converge to the point (x, y). If $y \neq x$ then the neighborhood $\{x\} \times (Y - \{x\})$ of the point (x, y) contains no point of the net S. Therefore from the convergence $(x_\alpha, x_\alpha) \to \to (x, y)$ there follows $x = y$, $(x, y) \in G_f$.

Theorem 2. *Let X and Y be topological spaces, $f \in U(X, Y)$ and let X be a compact. Then $f(X) \in F(Y)$.*

Proof. Let the net $\{y_\alpha, \alpha \in A\}$, $y_\alpha \in f(X)$ converge to y. Using this net we can construct a net $\{x_\alpha, \alpha \in A\}$ $(x_\alpha \in X)$ such that $y_\alpha = f(x_\alpha)$. Since X is a compact, there is aconvergent subnet $\{x_{n_\beta}, \beta \in B\}$ $(x_{n_\beta} \to x)$ of the net $\{x_\alpha, \alpha \in A\}$. Then $(x_{n_\beta}, y_{n_\beta}) \to \to (x, y) \in G_f$, $y = f(x)$, hence $y \in f(X)$.

It is easy to see that the assumption "X is a compact" is essential. This is shown by the following example.

Example 2. $X = (0,1)$ with the usual topology, $Y = (0, \infty)$ with the usual topology and $f(x) = x$. Obviously $f(X) = (0,1) \notin F((0, \infty))$, but the function f has closed graph. This follows from the next theorem.

Theorem 3. *Let X and Y be topological spaces and let Y be a Hausdorff space. Then $B_0(X, Y) \subset U(X, Y)$.*

Proof. The statement of Theorem 3 is a consequence of the following characterization of continuous functions: for each net $\{x_\alpha, \alpha \in A\}$ in X which converges to a point x, the net $\{f(x_\alpha), \alpha \in A\}$ converges to $f(x)$ (see [1], p. 86).

If a net $\{(x_\alpha, f(x_\alpha)), \alpha \in A\}$ converges to (x, y) then the net $\{x_\alpha, \alpha \in A\}$ converges to x and the net $\{f(x_\alpha), \alpha \in A\}$ converges to y. Since Y is Hausdorff space, each net converges to at most one point (see [1], p. 67) and from the assumption $f \in B_0(X, Y)$ follows $y = f(x)$.

The following example shows that Theorem 3 is false if Y is only T_1 – space and not Hausdorff space.

Example 3. Let X and Y be the same space – the space Y from example 1. The identical map has the required properties. Obviously the identical map is a continuous function, but its graph is not closed, because the sequence $\{(n, n), n \in N\}$ converges to every point $(x, y) \in N \times N$. In fact, in every neighborhood of a point (x, y) there is an element of the base of the topology for $N \times N$ of the form $G_1 \times G_2$, $G_1, G_2 \in \in G(N)$ and $(x, y) \in G_1 \times G_2$. If $m = \max\{n : n \notin G_1, \text{ or } n \notin G_2\}$, then $(n, n) \in \in G_1 \times G_2$ for each $n > m$.

The following theorem generalizes Theorem 1 of paper [2].

Theorem 4. *Let X and Y be topological spaces, $f \in U(X, Y)$ and let Y be a compact. Then $f \in B_0(X, Y)$.*

Proof. Let f satisfy the assumptions of the theorem. We shall show that $F \in F(Y)$ implies $f^{-1}(F) \in F(X)$. It is sufficient to consider the case $f^{-1}(F) \neq \emptyset$. Let a net $\{x_\alpha, \alpha \in A\}$, $x_\alpha \in f^{-1}(F)$ converge to a point x. Then the net $\{y_\alpha, \alpha \in A\}$, $y_\alpha = f(x_\alpha)$ is in the compact F and there is a convergent subnet $\{y_{n_\beta}, \beta \in B\}$, $y_{n_\beta} \to y \in F$, of the net $\{y_\alpha, \alpha \in A\}$. The net $\{(x_{n_\beta}, y_{n_\beta}), \beta \in B\}$, $y_{n_\beta} = f(x_{n_\beta})$ converges to a point $(x, y) \in G_f$, hence $y = f(x)$. Therefore $x \in f^{-1}(F)$ and $f^{-1}(F) \in F(X)$.

The following theorem generalizes Theorem 1′ of paper [3].

Theorem 5. *Let X and Y be topological spaces, $Y = \bigcup_{k=1}^{\infty} Y_k$, let Y_k be a compact, $Y_k \in F(Y)$ $(k = 1, 2, \ldots)$ and $G(Y) \subset F_\sigma(Y)$. Then $U(X, Y) \subset B_1(X, Y)$.*

Proof. Let $f \in U(X, Y)$. We shall show that $f^{-1}(G) \in F_\sigma(X)$ for each set $G \in G(Y)$. If $G \in G(Y)$ then from the assumption of the theorem follows $G = \bigcup_{k=1}^{\infty} F_k$ $(F_k \in F(Y))$ and $M = G_f \cap (X \times G) = G_f \cap (X \times \bigcup_{k=1}^{\infty} F_k) = G_f \cap \bigcup_{k=1}^{\infty} (X \times F_k) \in F_\sigma(X \times Y)$. Therefore $M = \bigcup_{n=1}^{\infty} M_n$, $M_n \in F(X \times Y)$. Let $R_k = X \times Y_k$ $(k = 1, 2, \ldots)$. If we put $M_{nk} = M_n \cap R_k$ then $M_n = \bigcup_{k=1}^{\infty} M_{nk}$ and $M_{nk} \in F(X \times Y)$ $(n, k = 1, 2, \ldots)$. Let

$$E_{nk} = \{x : \sum_{y \in Y} (x, y) \in M_{nk}\}, \quad E = \{x : \sum_{y \in Y} (x, y) \in M\}$$

Then $E = \bigcup_{n,k=1}^{\infty} E_{nk}$ and $E = f^{-1}(G)$. It is sufficient to show that the sets E_{nk} $(n, k = 1, 2, \ldots)$ are closed.

Let a net $\{x_\alpha, \alpha \in A\}$, $x_\alpha \in E_{nk}$ converge to x. From the definition of E_{nk} there follows that there is a net $\{(x_\alpha, y_\alpha), \alpha \in A\}$ such that $(x_\alpha, y_\alpha) \in M_{nk}$. We can construct a net $\{y_\alpha, \alpha \in A\}$ which is in the compact Y_k. There exists its convergent subnet $\{y_{n_\beta}, \beta \in B\}$, $y_{n_\beta} \to y$. The net $\{(x_{n_\beta}, y_{n_\beta}), \beta \in B\}$ is in $M_{nk} \in F(X \times Y)$ and converges to $(x, y) \in M_{nk}$. Hence $x \in E_{nk}$.

M. Sekanina in his review of paper [2] has put the question whether it is possible to generalize Theorem 9 of paper [2] to normal spaces.

Theorem 6. *Let X be a normal topological space and let $f \in U(X, E_1)$ (E_1 – real numbers). Then there is a sequence of functions $f_n \in B_0(X, E_1)$ $(n = 1, 2, \ldots)$ such that $|f_n(x)| \leqq n$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \in X$.*

Proof. Let $F_n = G_f \cap (X \times \langle -n, n \rangle)$ $(n = 1, 2, \ldots)$. The set F_n is closed and its projection X_n to the set X $(X_n = \{x : \sum_{y \in E_1} (x, y) \in F_n\})$ is closed, too. If $X_n \neq \emptyset$, then a function $g_n = f \mid X_n$ is a continuous function on X_n according to Theorem 4 and $|g_n(x)| \leqq n$ $(x \in X_n)$. X is a normal space and (according to Tietze's theorem, see [4], p. 134) there is a continuous extension f_n of the function g_n on the space X such that $|f_n(x)| \leqq n$ for each $x \in X$. If $X_n = \emptyset$ we put $f_n(x) \equiv 0$.

The equality $f(x) = \lim_{n\to\infty} f_n(x)$ $(x \in X)$ follows from the fact that the sequence of sets $\{X_n\}_1^\infty$ is increasing, $X_n \subset X_{n+1}$ $(n = 1, 2, \ldots)$, and $X = \bigcup_{n=1}^{\infty} X_n$. If $x \in X$ then there is n_0 such that $x \in X_n$ $(n \geq n_0)$ and $f_n(x) = f(x)$ $(n \geq n_0)$, therefore $f(x) = \lim_{n\to\infty} f_n(x)$.

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