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Časopis pro pěstování matematiky, roč. 94 (1969), Praha

ASYMPTOTIC FORMULAS FOR THE SOLUTIONS OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS $\mathbf{y}' = [\mathbf{A} + \mathbf{B}(x)]\,\mathbf{y}$

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Dedicated to Prof. OTAKAR BORŮVKA on the occasion of his scientific jubilee

Let $\mathbf{A}$ be a constant matrix with distinct characteristic roots $\lambda_1, \ldots, \lambda_n$. We can assume without loss of generality

(1) $$\operatorname{Re}\lambda_1 \leqq \ldots \leqq \operatorname{Re}\lambda_n .$$

If we denote by $\mathbf{h}^1, \ldots, \mathbf{h}^n$ eigen-vectors of $\mathbf{A}$ corresponding to the roots $\lambda_1, \ldots, \lambda_n$, then the fundamental matrix of solutions of the system

(2) $$\mathbf{z}' = \mathbf{A}\mathbf{z}$$

has the form

$$\mathbf{Z}(x) = (z_{ij}(x)) = (h_{ij} \exp \{\lambda_j x\}) .$$

Under various hypotheses on elements $b_{ij}(x)$ of a matrix $\mathbf{B}(x)$, many asymptotic formulas have been deduced for solutions of a system

(3) $$\mathbf{y}' = [\mathbf{A} + \mathbf{B}(x)]\,\mathbf{y}$$

in the form

(4) $$\mathbf{y} = \mathbf{Z}(x)\,[\mathbf{c} + \varepsilon(x)] ,$$

where $\mathbf{c}$ is a constant n-vector and $\varepsilon(x)$ is an n-vector, elements of which approach zero as $x \to \infty$. ([1], [2], [3].)

Using the result of the paper [4] we can immediately announce the following statement concerning the system (3):

Let the elements of the matrix $\mathbf{B}(x)$ *be continuous in* $J = \langle a, +\infty)$ *and let be*

$$\int_a^\infty \|\mathbf{Z}^{-1}(x)\,\mathbf{B}(x)\,\mathbf{Z}(x)\|\,\mathrm{d}x < \infty .^{1})$$

[1]) By the norm $\|\mathbf{A}\|$ of a matrix $\mathbf{A}$ it means the sum $\|\mathbf{A}\| = \sum_{i,j} |a_{ij}|$ of the absolute values of all its elements.

Then every solution of the system (3) *has the form*

$$\mathbf{y} = \mathbf{Z}(x) \sum_0^\infty (-1)^n \, \mathbf{R}^n(x) \, \mathbf{c} \, ,$$

where $\mathbf{c}$ *is a suitable constant vector and* $\mathbf{R}^n(x)$ *is a matrix defined by means of the formula*

$$\mathbf{R}^n(x) = \int_x^\infty \mathbf{Z}^{-1}(t) \, \mathbf{B}(t) \, \mathbf{Z}(t) \, \mathbf{R}^{n-1}(t) \, \mathrm{d}t \, , \quad \mathbf{R}^0(x) = \mathbf{E} \, ,$$

where $\mathbf{E}$ *is the unit matrix.*

Let us write

(4) $$\mathbf{y} = \mathbf{Z}(x) \left[\sum_0^p (-1)^n \, \mathbf{R}^n(x) \, \mathbf{c} + \mathbf{R}(x) \, \mathbf{c} \right] .$$

In the mentioned paper, the following estimate for the matrix $\mathbf{R}(x)$ is introduced

$$\| \mathbf{R}(x) \, \mathbf{c} \| \leqq \| \mathbf{c} \| \frac{\varepsilon^{p+1}(x)}{(p+1)!} \exp \{ \varepsilon(x) \} \, ,$$

where

$$\varepsilon(x) = \int_x^\infty \| \mathbf{Z}^{-1}(t) \, \mathbf{B}(t) \, \mathbf{Z}(t) \| \, \mathrm{d}t \, .$$

The purpose of this paper is to improve this estimate, to evaluate the second approximation for the fundamental matrix of solutions of the system (3), and to introduce an application to the nth order linear differential equation.

By h^{ij} we shall denote the algebraic complement of the term h_{ij} in the matrix $(\mathbf{h}^1, \ldots, \mathbf{h}^n)$. By $\Delta = \det (h_{ij})$ we shall denote, as usual, the determinant of the matrix (h_{ij}). We have obviously

$$\mathbf{Z}^{-1}(x) \, \mathbf{B}(x) = \left(\frac{1}{\Delta} \sum_{k=1}^n h^{ki} \exp \{ -\lambda_i x \} \, b_{kj}(x) \right) ,$$

$$\mathbf{Z}^{-1}(x) \, \mathbf{B}(x) \, \mathbf{Z}(x) = \left(\frac{1}{\Delta} \sum_{k,l=1}^n h^{ki} h_{lj} b_{kl}(x) \exp \{ (\lambda_j - \lambda_i) \, x \} \right) .$$

Now, a bound will be established for the elements $r^n_{ij}(x)$ of the matrix $\mathbf{R}^n(x)$. Let $\gamma(x)$ be a positive, continuous function in J satisfying in this intervall the inequality

(5) $$\left| \sum_{k,l=1}^n \frac{h^{ki} h_{lj}}{\Delta} \, b_{kl}(x) \right| \leqq \gamma(x)$$

$i, j = 1, \ldots, n$. It will be proved by induction that the condition

$$\int_a^\infty \gamma(x) \exp\{\mathrm{Re}\,(\lambda_n - \lambda_1)\, x\}\, \mathrm{d}x < \infty \tag{6}$$

implies

$$\left|r^p_{ij}(x)\right| \leqq n^{p-1} \exp\{\mathrm{Re}\,[p(\lambda_1 - \lambda_n) + \lambda_j - \lambda_i]\, x\} \frac{\varkappa^p(x)}{p!}, \tag{7}$$

where

$$\varkappa(x) = \int_x^\infty \gamma(t) \exp\{\mathrm{Re}\,(\lambda_n - \lambda_1)\, t\}\, \mathrm{d}t\,. \tag{8}$$

Indeed, it holds for $p = 1$

$$\left|r^1_{ij}(x)\right| = \left|\frac{1}{\Delta} \int_x^\infty \exp\{(\lambda_j - \lambda_i)\, t\} \sum_{k,l=1}^n h^{ki} h_{lj} b_{kl}(t)\, \mathrm{d}t\right| \leqq$$

$$\leqq \int_x^\infty \gamma(t) \exp\{\mathrm{Re}\,(\lambda_n - \lambda_1)\, t\} \exp\{\mathrm{Re}\,(\lambda_1 - \lambda_n + \lambda_j - \lambda_i)\, t\}\, \mathrm{d}t\,.$$

Then, since $\mathrm{Re}\,(\lambda_1 - \lambda_n + \lambda_j - \lambda_i) \leqq 0$ for $i, j = 1, \ldots, n$, the function $\exp\{\mathrm{Re}\,.\,(\lambda_1 - \lambda_n + \lambda_j - \lambda_i)\, x\}$ is non-increasing and we have

$$\left|r^1_{ij}(x)\right| \leqq \exp\{\mathrm{Re}\,(\lambda_1 - \lambda_n + \lambda_j - \lambda_i)\, x\}\, \varkappa(x)\,.$$

This proves the statement for $p = 1$. Now, using the induction hypothesis (7) we have

$$\left|r^{p+1}_{ij}(x)\right| = \left|\frac{1}{\Delta} \int_x^\infty \sum_{m=1}^n \sum_{k,l=1}^n h^{ki} h_{lm} b_{kl}(t) \exp\{(\lambda_m - \lambda_i)\, t\}\, r^p_{mj}(t)\, \mathrm{d}t\right| \leqq$$

$$\leqq \sum_{m=1}^n \int_x^\infty \gamma(t) \exp\{\mathrm{Re}\,(\lambda_m - \lambda_i)\, t\}\, n^{p-1} \exp\{\mathrm{Re}\,[p(\lambda_1 - \lambda_n) + \lambda_j - \lambda_m]\, t\} \frac{\varkappa^p(t)}{p!}\, \mathrm{d}t \leqq$$

$$\leqq n^p \exp\{\mathrm{Re}\,[(p+1)(\lambda_1 - \lambda_n) + \lambda_j - \lambda_i]\, x\} \int_x^\infty \varkappa'(t) \frac{\varkappa^p(t)}{p!}\, \mathrm{d}t =$$

$$= n^p \exp\{\mathrm{Re}\,[(p+1)(\lambda_1 - \lambda_n) + \lambda_j - \lambda_i]\, x\} \frac{\varkappa^{p+1}(x)}{(p+1)!}$$

and this completes the induction proof of (7).

In order to obtain the second approximation of the fundamental matrix of solution of (3) we shall consider the general solution of this system in the form (4) for $p = 1$. Using the inequality (7) we have the following estimate for the elements $(r_{ij}(x))$ of the matrix $\mathbf{R}(x)$

$$|r_{ij}(x)| = \left|\sum_{p=2}^{\infty} r_{ij}^p(x)\right| \leqq \sum_{p=2}^{\infty} |r_{ij}^p(x)| \leqq$$

$$\leqq \sum_{p=2}^{\infty} n^{p-1} \exp\{\operatorname{Re}[p(\lambda_1 - \lambda_n) + \lambda_j - \lambda_i]\, x\} \frac{\varkappa^p(x)}{p!} \leqq$$

$$\leqq \tfrac{1}{2} n \exp\{\operatorname{Re}[2(\lambda_1 - \lambda_n) + \lambda_j - \lambda_i]\, x\}\, \varkappa^2(x) \exp\{n\, \varkappa(x)\}\,.$$

From the fact that $\varkappa(x)$ approaches zero as $x \to \infty$, we deduce that there exists a constant K satisfying

$$\tfrac{1}{2} n \exp\{n\, \varkappa(x)\} \leqq K \tag{9}$$

for all $x \in J$. Thus

$$|r_{ij}(x)| \leqq K \exp\{\operatorname{Re}[2(\lambda_1 - \lambda_n) + \lambda_j - \lambda_i]\, x\}\, \varkappa^2(x)\,.$$

Now, a bound for the elements of the matrix $\mathbf{Z}(x)\, \mathbf{R}(x)$ can be easily derived. Denoting $\mathbf{Z}(x)\, \mathbf{R}(x) = (\varepsilon_{ij}(x))$, we have

$$|\varepsilon_{ij}(x)| = \left|\sum_{m=1}^{n} z_{im}(x)\, r_{mj}(x)\right| \leqq \sum_{m=1}^{n} |z_{im}(x)|\, |r_{mj}(x)| \leqq$$

$$\leqq \sum_{m=1}^{n} |h_{im}| \exp\{\operatorname{Re} \lambda_m x\}\, K \exp\{\operatorname{Re}[2(\lambda_1 - \lambda_n) + \lambda_j - \lambda_m]\, x\}\, \varkappa^2(x)\,,$$

therefore

$$|\varepsilon_{ij}(x)| \leqq K\, \varkappa^2(x) \exp\{\operatorname{Re}[2(\lambda_1 - \lambda_n) + \lambda_j]\, x\} \sum_{m=1}^{n} |h_{im}|\,. \tag{10}$$

Finally, if we denote $\mathbf{Z}(x)\, [\mathbf{R}_0(x) - \mathbf{R}_1(x)] = (\eta_{ij}(x))$, we find

$$\eta_{ij}(x) = h_{ij} \exp\{\lambda_j x\} - \frac{1}{\Delta} \sum_{m,k,l=1}^{n} h_{im} h_{lj} h^{km} \int_x^{\infty} b_{kl}(t) \exp\{\lambda_j t + \lambda_m(x - t)\}\, \mathrm{d}t$$

and the fundamental system of solutions has the form $y_{ij}(x) = \eta_{ij}(x) + \varepsilon_{ij}(x)$ and $\varepsilon_{ij}(x)$ satisfies (10).

Conclusion. *Let* $\mathbf{A}$ *be a constant matrix with distinct characteristic roots satisfying* (1). *Let* $\mathbf{Z}(x) = (h_{ij} \exp\{\lambda_j x\})$ *be a fundamental matrix of solutions of* (2) *and* h^{ij} *an algebraic complement of the term* h_{ij} *in the matrix* (h_{ij}). *Let the function* $\gamma(x)$ *be continuous and satisfy the hypotheses* (5) *and* (6). *Finally, let* $\varkappa(x)$ *be a function defined by means of* (8) *and* K *a suitable constant satisfying* (9).

Then, there exists a fundamental system of solutions of the form

$$y_{ij}(x) = h_{ij} \exp\{\lambda_j x\} - \frac{1}{\Delta} \sum_{m,k,l=1}^{n} h_{im} h_{lj} h^{km} \int_x^\infty b_{kl}(t) \exp\{\lambda_j t + \lambda_m(x-t)\}\, dt + \varepsilon_{ij}(x)$$

and $\varepsilon_{ij}(x)$ has a bound given by means of the formula (10).

Application to the nth order linear differential equation. *Suppose that the characteristic equation*

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0$$

has distinct roots and denote them so that $\operatorname{Re} \lambda_1 \leqq \ldots \leqq \operatorname{Re} \lambda_n$. *Let*

$$L = \max_{i=1,\ldots,n} |\lambda_i|, \quad M = \max_{i=1,\ldots,n} \prod_{\substack{l=1 \\ l \neq i}}^{n} (\lambda_i - \lambda_l)^{-1}, \quad \Delta = \prod_{\substack{i,j=1 \\ i>j}}^{n} (\lambda_i - \lambda_j).$$

If the functions $q_i(x)$ are continuous on J for $i = 1, \ldots, n$ and satisfy the condition

$$\int_a^\infty |q_i(x)| \exp\{\operatorname{Re}(\lambda_n - \lambda_1)\, x\}\, dx < \infty, \tag{11}$$

then the differential equation

$$y^{(n)} + [a_1 + q_1(x)]\, y^{(n-1)} + \ldots + [a_n + q_n(x)]\, y = 0 \tag{12}$$

has a fundamental system of solutions of the form

$$y_j^{(i-1)}(x) = \lambda_j^{i-1} \exp\{\lambda_j x\} +$$

$$+ \int_x^\infty \exp\{\lambda_j t\} \sum_{k=1}^{n} \lambda_j^{k-1} q_{n-k+1}(t) \sum_{m=1}^{n} \lambda_m^{i-1} \prod_{\substack{l=1 \\ l \neq m}}^{n} (\lambda_m - \lambda_l)^{-1} \exp\{\lambda_m(x-t)\}\, dt + \varepsilon_{ij}(x).$$

Here

$$\varkappa(x) = M \int_x^\infty \exp\{\operatorname{Re}(\lambda_n - \lambda_1)\, t\} \sum_{k=1}^{n} L^{k-1} |q_{n-k+1}(t)|\, dt,$$

$$|\varepsilon_{ij}(x)| \leqq K \varkappa^2(x) \exp\{\operatorname{Re}[2(\lambda_1 - \lambda_n) + \lambda_j]\, x\} \sum_{k=1}^{n} |\lambda_k^{i-1}|$$

and $K \geqslant 0$ is the upper bound of the function $\frac{1}{2} n \exp\{n\, \varkappa(x)\}$ in the intervall J.

Proof. Using notation of the above considerations we have

$$z_{ij}(x) = \lambda_j^{i-1} \exp\{\lambda_j x\}, \quad b_{ij}(x) = 0 \quad \text{for} \quad i \neq n, \quad b_{nj}(x) = -q_{n-j+1}(x),$$

$$\mathbf{Z}^{-1}(x)\,\mathbf{B}(x)\,\mathbf{Z}(x) = -\frac{1}{\Delta}\left(h^{ni} \exp\{(\lambda_j - \lambda_i)\,x\} \sum_{k=1}^{n} h_{kj} q_{n-k+1}(x)\right) =$$

$$= \left(-\prod_{\substack{l=1\\ l\neq i}}^{n} (\lambda_i - \lambda_l)^{-1} \exp\{(\lambda_j - \lambda_i)\,x\} \sum_{k=1}^{n} \lambda_j^{k-1} q_{n-k+1}(x)\right).$$

If we select a function $\gamma(x) = M \sum_{k=1}^{n} L^{k-1} |q_{n-k+1}(x)|$, the hypothesis (11) implies (6) and we can apply the above theorem to the equation (12). We obtain

$$\eta_{ij}(x) = \lambda_j^{i-1} \exp\{\lambda_j x\} +$$

$$+ \sum_{m=1}^{n} \lambda_m^{i-1} \exp\{\lambda_m x\} \prod_{\substack{l=1\\ l\neq m}}^{n} (\lambda_m - \lambda_l)^{-1} \int_x^{\infty} \exp\{(\lambda_j - \lambda_m)\,t\} \sum_{k=1}^{n} \lambda_j^{k-1}\, q_{n-k+1}(t)\,\mathrm{d}t =$$

$$= \lambda_j^{i-1} \exp\{\lambda_j x\} +$$

$$+ \int_x^{\infty} \exp\{\lambda_j t\} \sum_{k=1}^{n} \lambda_j^{k-1}\, q_{n-k+1}(t) \sum_{m=1}^{n} \lambda_m^{i-1} \prod_{\substack{l=1\\ l\neq m}}^{n} (\lambda_m - \lambda_l)^{-1} \exp\{\lambda_m(x - t)\}\,\mathrm{d}t$$

so that $y_j^{(i-1)}(x) = \eta_{ij}(x) + \varepsilon_{ij}(x)$ and the estimate (13) immediately follows from (10). This completes the proof.

Note that the case $n = 3$ was investigated by I. Res in [5].

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